Quadratic hedging in affine stochastic volatility models

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Pittsburgh, February 20, 2006

(based on joint work with **F. Hubalek, L. Krawczyk, A. Pauwels**)

Hedging problem

$S_t = S_0 \exp(X_t)$ asset price process

$H = f(S_T)$ option (contingent claim)

How to hedge the risk from selling the claim?

Hedging error: $v + \int_0^T \varphi_t dS_t - H$

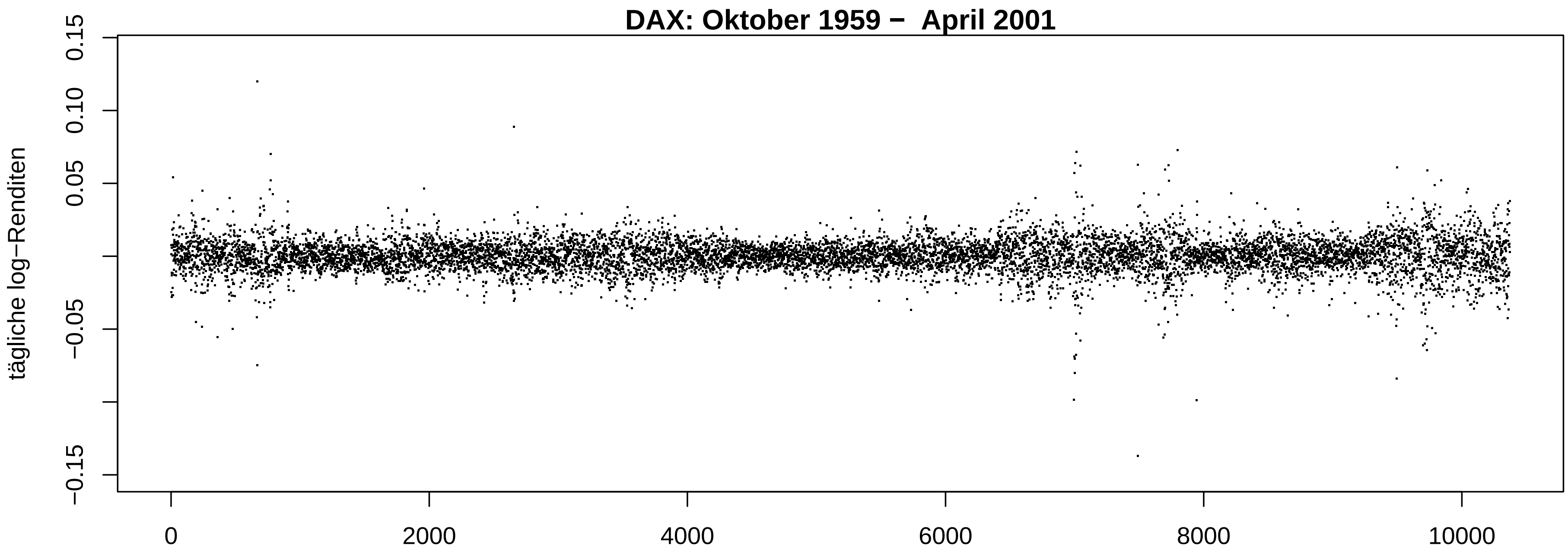
Stock price process

$$S_t = S_0 \exp(X_t)$$

- standard market model: X Brownian motion with drift
  d. h. X continuous, homogeneous in time
  $\longrightarrow$ perfect hedge exists
  (Black & Scholes, Merton 1973)

- empirical literature:
  1. large daily price changes anormally often *(heavy tails)*
  2. clustering of large daily price changes *(volatility clustering)*

- alternative models:
  1. Lévy processes with jumps
  2. processes with stochastic volatility


DAX: Oktober 1959 − April 2001
tägliche log−Renditen
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0.10
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2000
4000
6000
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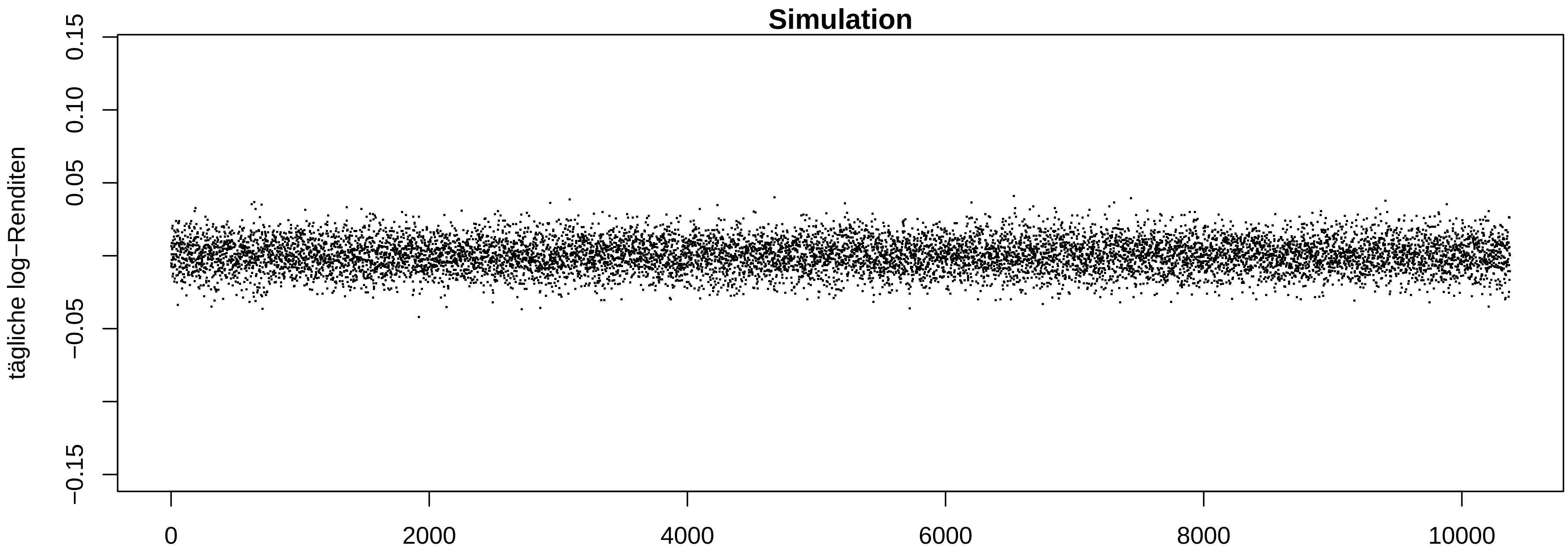

Simulation
tägliche log−Renditen
0.15
0.10
0.05
−0.05
−0.15
0
2000
4000
6000
8000
10000

Quadratic hedging in affine models

$S_t = S_0 \exp(X_t)$ *martingale*

X_t component of an *affine process*

$H = f(S_T)$ option (contingent claim)

$$\min_{v,\varphi} E\left(\left(v + \int_0^T \varphi_t dS_t - H\right)^2\right)$$

$v^\star$ variance-optimal initial capital

$\varphi^\star$ variance-optimal hedging strategy

Some affine stochastic volatility models

Stein & Stein (1991)

$$
\begin{aligned}
dX_t &= (\mu + \delta \sigma_t^2)dt + \sigma_t dW_t, \\
d\sigma_t &= (\kappa - \lambda \sigma_t)dt + \alpha dZ_t
\end{aligned}
$$

Heston (1993)

$$
\begin{aligned}
dX_t &= (\mu + \delta v_t)dt + \sqrt{v_t} dW_t, \\
dv_t &= (\kappa - \lambda v_t)dt + \sigma \sqrt{v_t} dZ_t.
\end{aligned}
$$

Barndorff-Nielsen & Shephard (2001)

$$
\begin{aligned}
dX_t &= (\mu + \delta v_{t-})dt + \sqrt{v_{t-}}dW_t + \varrho dZ_t, \\
dv_t &= -\lambda v_{t-} dt + dZ_t.
\end{aligned}
$$

$$
\begin{aligned}
dX_t &= (\mu + \delta v_{t-})dt + \sqrt{v_{t-}}dW_t + \sum_{k=1}^{\nu} \varrho_k dZ_t^k, \\
v_t &= \sum_{k=1}^{\nu} \alpha_k v_t^{(k)}, \\
dv_t^{(k)} &= -\lambda_k v_{t-}^{(k)} dt + dZ_t^k.
\end{aligned}
$$

Carr, Geman, Madan, Yor (2003)

$$\begin{aligned} X_t &= X_0 + \mu t + L_{V_t} + \varrho(v_t - v_0), \\ dV_t &= v_t dt, \\ dv_t &= (\kappa - \lambda v_t)dt + \sigma\sqrt{v_t}dZ_t, \end{aligned}$$

$$\begin{aligned} X_t &= X_0 + \mu t + L_{V_t} + \varrho Z_t, \\ dV_t &= v_{t-}dt, \\ dv_t &= -\lambda v_{t-}dt + dZ_t. \end{aligned}$$

Carr, Wu (2003)

$$\begin{aligned} dX_t &= \mu dt + v_t^{1/\alpha} dL_t, \\ dv_t &= (\kappa - \lambda v_t)dt + \sigma\sqrt{v_t}dZ_t. \end{aligned}$$

Carr, Wu (2004)

$$\begin{aligned} X_t &= X_0 + \mu t + L_{V_t}, \\ dV_t &= v_{t-}dt, \\ v_t &= v_0 + \kappa t + Z_{V_t} \end{aligned}$$

Affine semimartingales

Characteristics of semimartingale X in $\mathbb{R}^d$:

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu([0,t] \times G) = \int_0^t F_s(G) ds \quad \forall G \in \mathscr{B}^d$$

$$\begin{aligned} b_t &= b_{(0)} + \sum_{j=1}^d X^j_{t-} b_{(j)} \\ c_t &= c_{(0)} + \sum_{j=1}^d X^j_{t-} c_{(j)} \\ F_t(G) &= F_{(0)}(G) + \sum_{j=1}^d X^j_{t-} F_{(j)}(G) \end{aligned}$$

with given Lévy-Khintchine triplets $(b_{(j)}, c_{(j)}, F_{(j)})$, $j = 0, \ldots, d$ on $\mathbb{R}^d$

Characterization by Duffie, Filipovic, Schachermayer (2003)

$$E\left(e^{i\lambda^\top X_{s+t}}\middle|\mathscr{F}_s\right) = \exp\left(\Psi^0(t,i\lambda) + \Psi^{(1,\dots,d)}(t,i\lambda)^\top X_s\right), \quad \lambda \in \mathbb{R}^d,$$

with

$$\Psi^{(1,\dots,d)} = (\Psi^1,\dots,\Psi^d) : \mathbb{R}_+ \times (\mathbb{C}^m_- \times i\mathbb{R}^n) \to (\mathbb{C}^m_- \times i\mathbb{R}^n),$$
$$\Psi^0 : \mathbb{R}_+ \times (\mathbb{C}^m_- \times i\mathbb{R}^n) \to \mathbb{C}$$

solving the following system of *generalized Riccati equations*:

$$\Psi^0(0,u) = 0, \quad \Psi^{(1,\dots,d)}(0,u) = u,$$
$$\frac{d}{dt}\Psi^j(t,u) = -\psi_j(\Psi^{(1,\dots,d)}(t,u)), \quad j = 0,\dots,d$$

and ψ_j denoting the *Lévy exponent* of $(b_{(j)}, c_{(j)}, F_{(j)})$:

$$\psi_j(u) = u^\top b_{(j)} + \frac{1}{2}u^\top c_{(j)}u + \int (e^{u^\top x} - 1 - u^\top h(x))F_{(j)}(dx)$$

General structure of the variance-optimal hedge

Cf. Föllmer & Sondermann (1986)

Galtchouk-Kunita-Watanabe decomposition:

$$H = V_0 + \int_0^T \xi_t dS_t + R_T,$$

where R martingale, orthogonal to S (i.e. RS martingale)

Mean value process of the option: $V_t := E(H|\mathscr{F}_t)$

Variance-optimal hedge: $v^\star = V_0, \quad \varphi_t^\star = \xi_t = \frac{d\langle V,S\rangle_t}{d\langle S,S\rangle_t}$

Hedging error:

$$E\left(\left(v^\star + \int_0^T \varphi_t^\star dS_t - H\right)^2\right) = E\left(\left\langle V - \int_0^\cdot \varphi_t^\star dS_t, V - \int_0^\cdot \varphi_t^\star dS_t\right\rangle_T\right)$$

Problem: How to compute V_t, ξ_t?

Integral representation of options

Cf. Hubalek & Krawczyk (1998), Carr & Madan (1999), Raible (2000)

- Assumption: option of the form

$$H = \int_{R-i\infty}^{R+i\infty} S_T^u \, p(u) du,$$

  with some function $p(u)$,

- Example: European call

$$H = (S_T - K)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} S_T^u \frac{K^{1-u}}{u(u-1)} du$$

  with arbitrary $R > 1$.

Integral representation of several options

call: $$(S_T - K)^+ = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} S_T^u \frac{K^{1-u}}{u(u-1)}du \qquad (R > 1)$$

put: $$(K - S_T)^+ = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} S_T^u \frac{K^{1-u}}{u(u-1)}du \qquad (R < 0)$$

power call: $$((S_T - K)^+)^2 = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} S_T^u \frac{2K^{1-u}}{u(u-1)(u-2)}du \qquad (R > 2)$$

self-quanto call: $$(S_T - K)^+ S_T = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} S_T^u \frac{K^{1-u}}{(u-1)(u-2)}du \qquad (R > 2)$$

digital option: $$1_{\{S_T > K\}} = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} S_T^u \frac{K^{-u}}{u}du \qquad (R > 0)$$

log contract: $$\log(S_T) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} S_T^u \frac{1}{u^2}du - \frac{1}{2\pi i}\int_{R'-i\infty}^{R'+i\infty} S_T^u \frac{1}{u^2}du$$

$(R' < 0, R > 0)$

The variance-optimal hedging strategy

$$\begin{aligned} v^\star &= E(H) = \int_{R-i\infty}^{R+i\infty} V(u)_0\, p(u)\, du, \\ \varphi_t^\star &= \int_{R-i\infty}^{R+i\infty} \frac{V(u)_{t-}}{S_{t-}} \frac{\varphi_1(t,u)v_{t-} + \varphi_2(t,u)}{\vartheta_1 v_{t-} + \vartheta_2} p(u)du, \end{aligned}$$

where

- $\psi_1(q) = iq'\beta - \frac{1}{2}q'\alpha q + \int \left(e^{iq'x} - 1 - iq'h(x)\right) \mu(dx)$,
- $\psi_2(q) = iq'b - \frac{1}{2}q'aq + \int \left(e^{iq'x} - 1 - iq'h(x)\right) m(dx)$,
- $V(u)_t := E(\exp(u \ln S_T)|\mathscr{F}_t) = \exp\left(u \ln S_t + \Phi_1(T-t,0,u)v_t + \Phi_2(T-t,0,u)\right)$,
- Φ_1, Φ_2 are solutions of generalized Riccati equations, related with ψ_1 and ψ_2,
- $\varphi_j(t,u) = \psi_j(-i\Phi_1(T-t,0,u), -i(u+1)) - \psi_j(-i\Phi_1(T-t,0,u), -iu) - \psi_j(0,-i)$,
- $\vartheta_j = \psi_j(0,-2i) - 2\psi_j(0,-i)$.

The expected squared hedging error

$$E\left(\left(v^\star + \int_0^T \varphi_t^\star dS_t - H\right)^2\right)$$

$$= \int_{R-i\infty}^{R+i\infty}\int_{R-i\infty}^{R+i\infty}\int_0^T e^{\gamma_2+(u_1+u_2)\ln S_0}\Big(\exp(\Phi_1(t,\gamma_1,u_1+u_2)v_0+\Phi_2(t,\gamma_1,u_1+u_2))\Big(\frac{l_2}{\vartheta_1}(D_2\Phi_1(t,\gamma_1,u_1+u_2)v_0$$

$$+D_2\Phi_2(t,\gamma_1,u_1+u_2))+\frac{l_1\vartheta_1-l_2\vartheta_2}{\vartheta_1^2}\Big)+\frac{l_0\vartheta_1^2-l_1\vartheta_1\vartheta_2+l_2\vartheta_2^2}{\vartheta_1^3}\exp\Big(-\frac{\vartheta_2}{\vartheta_1}\gamma_1\Big)\int_0^1\Big(\frac{\vartheta_1}{\vartheta_2}+\gamma_1 s\Big)\exp\Big(\frac{\vartheta_2}{\vartheta_1}\gamma_1 s$$

$$+\Phi_1\Big(t,\frac{\vartheta_1}{\vartheta_2}\ln s+\gamma_1 s,u_1+u_2\Big)v_0+\Phi_2\Big(t,\frac{\vartheta_1}{\vartheta_2}\ln s+\gamma_1 s,u_1+u_2\Big)\Big)\,ds\Big)\,p(u_1)p(u_2)dtdu_1du_2,$$

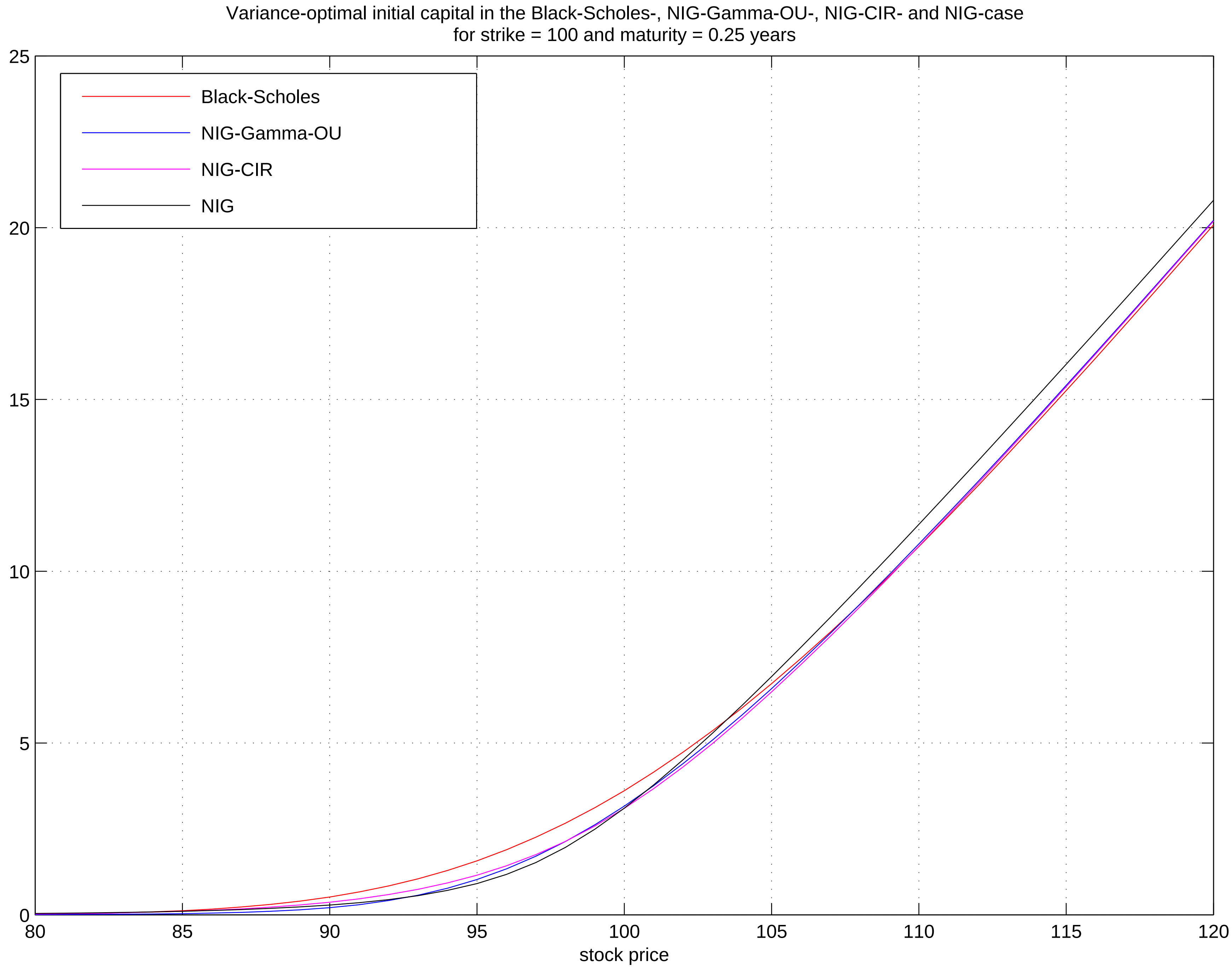
where

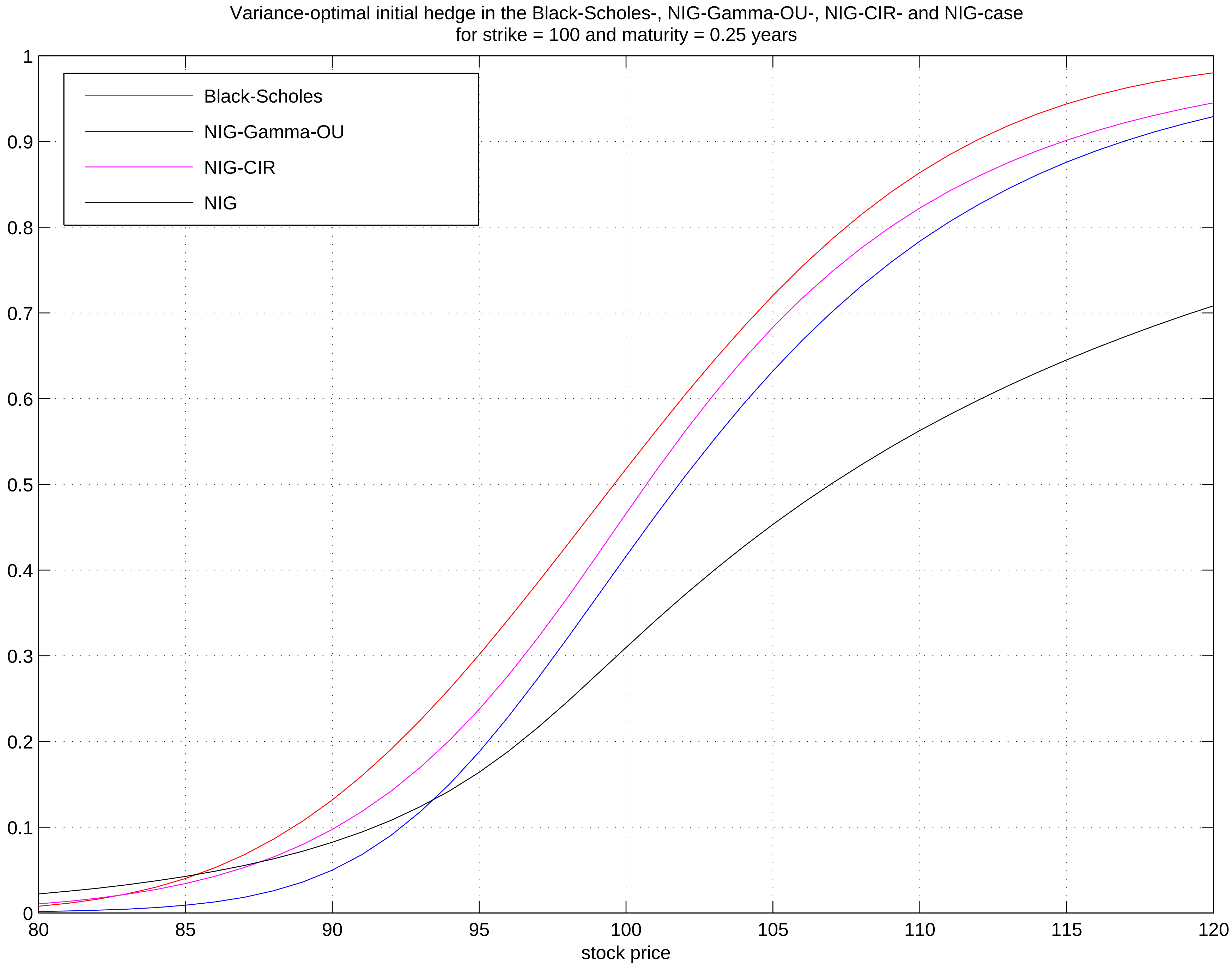
$$\begin{aligned}
l_0 &= l_0(t,u_1,u_2)=\vartheta_2\lambda_2(t,u_1,u_2)-\varphi_2(t,u_1)\varphi_2(t,u_2),\\
l_1 &= l_1(t,u_1,u_2)=\vartheta_2\lambda_1(t,u_1,u_2)+\vartheta_1\lambda_2(t,u_1,u_2)-\varphi_1(t,u_1)\varphi_2(t,u_2)-\varphi_1(t,u_2)\varphi_2(t,u_1),\\
l_2 &= l_2(t,u_1,u_2)=\vartheta_1\lambda_1(t,u_1,u_2)-\varphi_1(t,u_1)\varphi_1(t,u_2),\\
\gamma_j &= \gamma_j(t,u_1,u_2)=\Phi_j(T-t,0,u_1)+\Phi_j(T-t,0,u_2),\\
\varphi_j(t,u) &= \psi_j(-i\Phi_1(T-t,0,u),-i(u+1))-\psi_j(-i\Phi_1(T-t,0,u),-iu)-\psi_j(0,-i),\\
\lambda_j(t,u_1,u_2) &= \psi_j(-i\gamma_1(t,u_1,u_2),-i(u_1+u_2))-\psi_j(-i\Phi_1(T-t,0,u_1),-iu_1)-\psi_j(-i\Phi_1(T-t,0,u_2),-iu_2),\\
\vartheta_j &= \psi_j(0,-2i)-2\psi_j(0,-i).
\end{aligned}$$

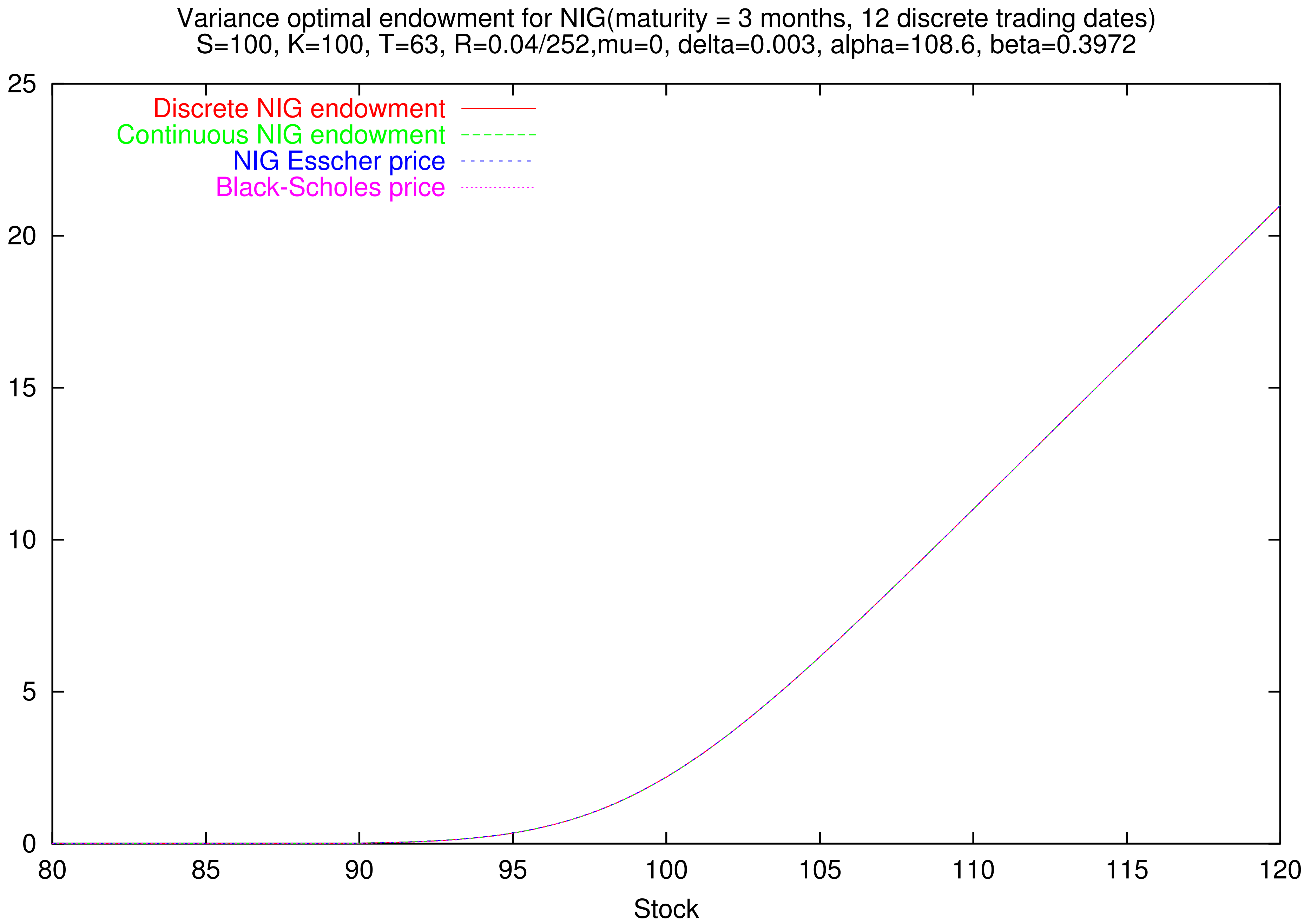
Numerical illustration

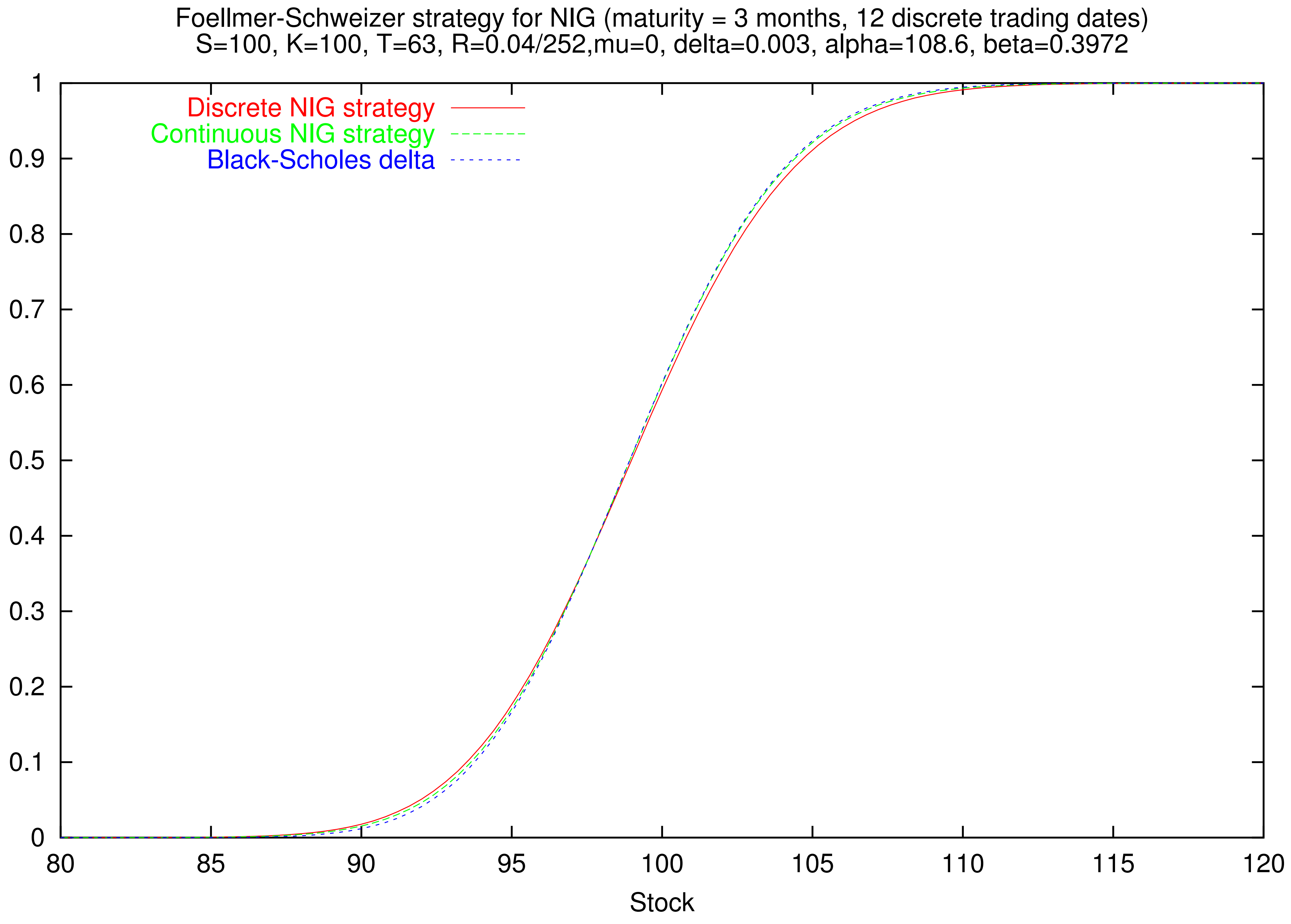
expected squared hedging error for an at-the-money call, $T = 0.25$ years				
model	parameters	option price	variance of the hedging error: optimal hedge	variance of the hedging error: no hedge
Black-Scholes	$\sigma = 0.1812$	3.61	0	$31.13 \approx (5.58)^2$
NIG	$\alpha = 6.1882$ $\beta = -3.8941$ $\delta = 0.1622$	3.10	$9.82 \approx (3.13)^2$	$21.20 \approx (4.60)^2$
NIG-Γ-OU	$a = 0.4239$ $b = 1.1757$ $\lambda = 0.6252$ $\alpha = 29.4722$ $\beta = -15.9048$ $\delta = 1$ $y_0 = 0.5071$	3.16	$2.26 \approx (1.50)^2$	$19.50 \approx (4.42)^2$
NIG-CIR	$\kappa = 0.5391$ $\eta = 0.7377$ $\sigma = 1.2849$ $\alpha = 18.4815$ $\beta = -4.8412$ $\delta = 1$ $y_0 = 0.4685$	3.10	$3.69 \approx (1.92)^2$	$24.86 \approx (4.99)^2$

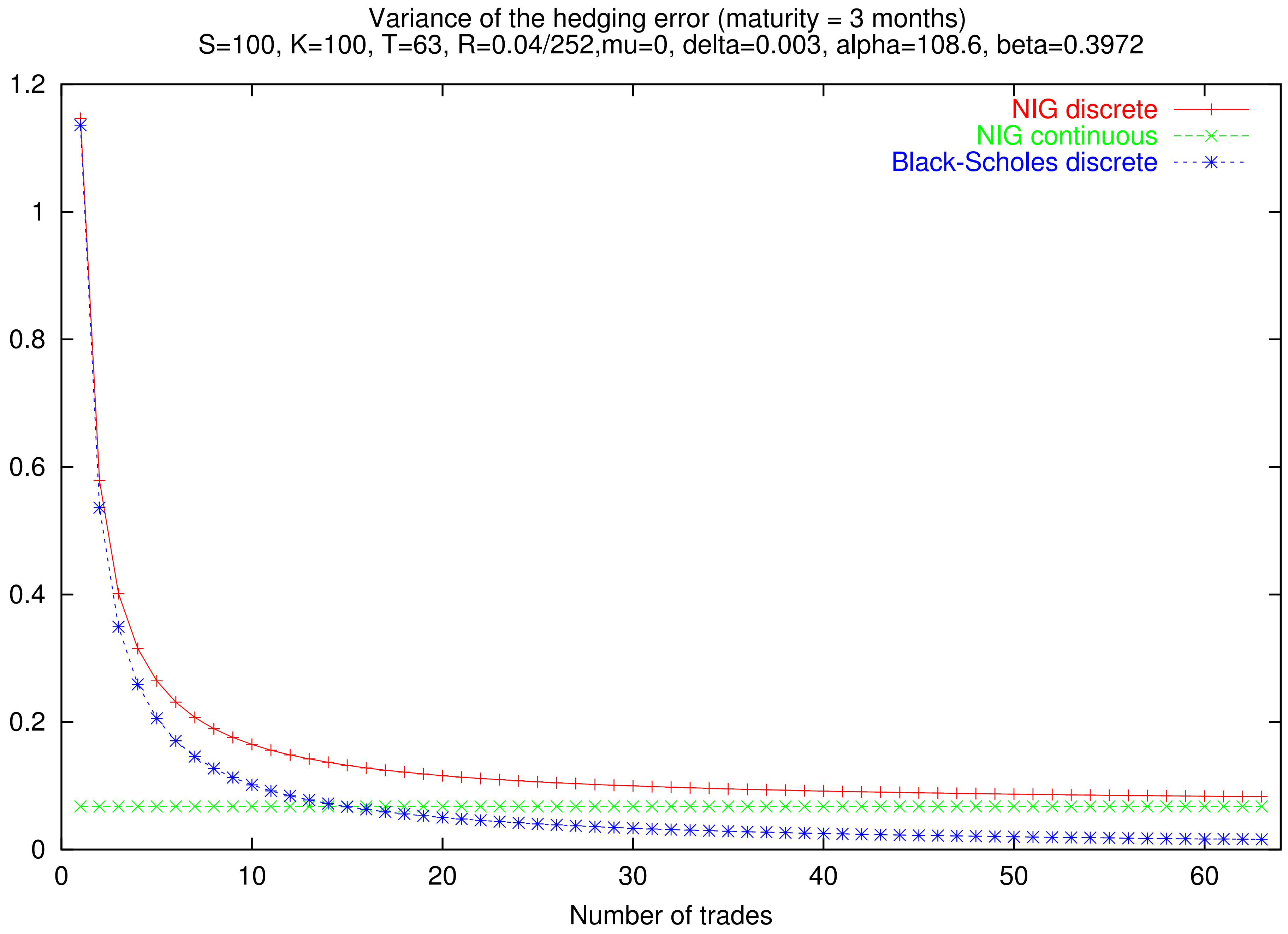
Parameters obtained via calibration by Schoutens (2003)


Variance-optimal initial capital in the Black-Scholes-, NIG-Gamma-OU-, NIG-CIR- and NIG-case
for strike = 100 and maturity = 0.25 years
Black-Scholes
NIG-Gamma-OU
NIG-CIR
NIG
0
5
10
15
20
25
80
85
90
95
100
105
110
115
120
stock price


Variance-optimal initial hedge in the Black-Scholes-, NIG-Gamma-OU-, NIG-CIR- and NIG-case
for strike = 100 and maturity = 0.25 years
Black-Scholes
NIG-Gamma-OU
NIG-CIR
NIG
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
80
85
90
95
100
105
110
115
120
stock price


Variance optimal endowment for NIG(maturity = 3 months, 12 discrete trading dates)
S=100, K=100, T=63, R=0.04/252,mu=0, delta=0.003, alpha=108.6, beta=0.3972
Discrete NIG endowment
Continuous NIG endowment
NIG Esscher price
Black-Scholes price
0
5
10
15
20
25
80
85
90
95
100
105
110
115
120
Stock


Foellmer-Schweizer strategy for NIG (maturity = 3 months, 12 discrete trading dates)
S=100, K=100, T=63, R=0.04/252,mu=0, delta=0.003, alpha=108.6, beta=0.3972
Discrete NIG strategy
Continuous NIG strategy
Black-Scholes delta
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
80
85
90
95
100
105
110
115
120
Stock


Variance of the hedging error (maturity = 3 months)
S=100, K=100, T=63, R=0.04/252,mu=0, delta=0.003, alpha=108.6, beta=0.3972
NIG discrete
NIG continuous
Black-Scholes discrete
1.2
1
0.8
0.6
0.4
0.2
0
10
20
30
40
50
60
Number of trades