# Notes for IHES minicourse: Poisson–Voronoi tessellations and fixed price in higher rank

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This minicourse is about the paper of the same title, which is joint work with Mikołaj Frączyk and Sam Mellick [FMW23]. In the first lecture, we state the main theorem and overview our method of proof. We motivate fixed price and the ideal Poisson–Voronoi tessellation and define the Poisson point process.

In the second, third, and fourth lectures (by Sam Mellick) we discuss some of the results from cost and fixed price theory in [AM22] behind our main theorem.

In the fifth lecture, we construct the "boundary" of a semisimple Lie group on which the Poisson point process inducing the ideal Poisson–Voronoi tessellation lives and prove almost every pair of cells in the tessellation shares an unbounded border in higher rank.

## The main theorem

Main Theorem [FMW23]: Let $G$ be a higher rank (the maximal dimension of a $\mathbb{R}$-split torus in $G$ is at least 2) semisimple Lie group or a product of automorphism groups of trees (trees are regular with bounded degree at least 3 and products are finite with at least 2 factors), and let $X$ be its symmetric space $G/K$ for some maximal compact subgroup $K$.

Good examples to keep in mind for $G$:

- $\mathrm{SL}_n(\mathbb{R}), n \geq 3$, then $X = G/\mathrm{SO}_n$ is the space of positive definite symmetric matrices in $\mathrm{SL}_n$
- $\mathrm{SL}_2 \times \mathrm{SL}_2$, then $X = \mathbb{H}_2 \times \mathbb{H}_2$
- the automorphism group of $\mathbb{T}_3 \times \mathbb{T}_3$ (where $\mathbb{T}_3$ is the tree with regular degree 3), then $X = \mathbb{T}_3 \times \mathbb{T}_3$

Let $\{\Gamma_i\}_{i\in\mathbb{N}}$ be a sequence of torsion-free lattices in $G$ (lattice: discrete subgroup such that $G/\Gamma_i$ has a finite $G$-invariant measure; torsion-free: no elements of finite order so $\Gamma_i\backslash X$ is a finite volume manifold or CW complex; note $\Gamma_i$ is the fundamental group of $\Gamma_i\backslash X$) such that $\Gamma_i\backslash X$ Benjamini–Schramm (BS) converges to $X$.

BS convergence: The injectivity radius around a typical point in $\Gamma_i\backslash X$ goes to infinity; when $G$ is simple, this is equivalent to $\mathrm{vol}(\Gamma_i\backslash G)) \to \infty$ (see [ABB+17] for a precise definition). In general an example is a sequence $\Gamma_i$ of normal finite index subgroups of a lattice $\Gamma$ such that $\cap\Gamma_i = \{1\}$.

The sequence is not necessarily contained in a common lattice and lattices are not necessarily irreducible (an example of a reducible lattice in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$).

Then

$$\lim_{i\to\infty} \frac{d(\Gamma_i) - 1}{\mathrm{vol}(\Gamma_i\backslash X)} = 0.$$

This limit is called the *rank gradient* of $G$. When it is 0 we say $G$ has *vanishing rank gradient.*

This means the minimal number of generators of $\Gamma_i$ grows sublinearly in the volume of its manifold $\Gamma_i \backslash X$. The conclusion is false without the higher rank assumption (consider for example hyperbolic surfaces).

Some prior results:

Abért, Gelander, and Nikolov [AGN17] proved vanishing rank gradient for sequences of subgroups of “right angled lattices” in a higher rank simple Lie group and conjectured vanishing rank gradient for the general lattice case as in our main theorem.

Lubotzky and Slutsky [LS22] proved the rank of a congruence subgroup in a nonuniform lattice in a higher rank simple Lie group is logarithmically bounded (stronger than vanishing rank gradient) by its covolume in the group (and prove a sharper, asymptotically optimal bound for the subclass of “2-generic” groups).

We have a corollary regarding the first mod-$p$ homology group of $\Gamma_i$, since $d(\Gamma_i)$ bounds its dimension.

Corollary to the main theorem: Let $G, X$, and $\{\Gamma_i\}$ be as in the main theorem. Let $p$ be prime. Then

$$\lim_{i \to \infty} \frac{\dim_{\mathbb{F}_p} H_1(\Gamma_i, \mathbb{F}_p)}{\mathrm{vol}(\Gamma_i \backslash X)} = 0.$$

Frączyk proved the corollary for $p = 2$ using entirely different methods in [Fra22].

## Method of proof

We do not consider lattices in the proof of the main theorem. We apply the following two results instead.

- Abért, Mellick [AM22], independently Carderi 2023 [Car23]: For $G, X$, and $\{\Gamma_i\}$ as in the main theorem, if $G$ has “fixed price 1” then $G$ has vanishing rank gradient.
- Abért, Mellick [AM22]: the Poisson point process has maximal “cost.”

In subsequent lectures, we'll discuss these theorems. For now, we give a brief overview of cost and fixed price theory, and then define the Poisson point process and discuss its role.

The cost of a group (acting on a standard probability space) is a measure-theoretic analogue to the minimal number of generators of a group. Gaboriau developed the theory of cost in the 2000's ([Gab98, Gab00], also see [Gab10]). The text [KM04] provides an introduction to the theory.

Let $\Gamma$ be a countable group. We consider measure-preserving and essentially free (the set of fixed points has zero measure) actions of $\Gamma$ on any standard probability space $(X, \mu)$. An action $\Gamma \curvearrowright (X, \mu)$ induces an orbit equivalence relation on $X$. Informally (we will provide a precise definition in a later lecture) the *cost of an action* measures, inside an orbit, the minimal average number of edges (with respect to $\mu$, and normalized by 1/2) required at each vertex to "wire together" its orbit in a $\Gamma$-equivariant and $\mu$-measurable way.

First example: any Bernoulli shift on $\mathbb{Z}$ has cost 1. To be explicit, consider the full shift on $(1/2, 1/2)$: $\mathbb{Z} \curvearrowright (\{0,1\}^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ where $\mu(0) = \mu(1) = 1/2$. It helps to sketch an orbit equivalence class.

A group has *fixed price* if ALL of its essentially free, measure preserving actions on ANY standard probability space have the same cost. It is an open question of Gaboriau's whether there exists a group without fixed price. The theory and question extend to nondiscrete locally compact second countable (lcsc) groups as well.

Motivating examples:

- the free group on $k$ generators has fixed price $k$ [Gab98]
- countable, infinite amenable groups have fixed price 1 [OW80]
- countable, infinite property (T) groups have cost 1 [HP20] (see [GJSM25] for a generalization to nondiscrete groups)
- real higher rank, irreducible, nonuniform lattices have fixed price 1 [Gab00]
- $\mathrm{SL}_2 \mathbb{Z}$ has fixed price 13/12 [Gab00] (and $\mathrm{SL}_2 \mathbb{R}$ has fixed price greater than 1 [CGMT21])

Big picture idea: relations between generators of a group (or the geometry of a manifold) may drive down the cost of the group.

Our route to proving the main theorem is by proving:

Theorem [FMW23]: Let $G$ be as in the main theorem. The Poisson point process on $G$ has cost 1 (implying $G$ has fixed price 1, which implies: vanishing rank gradient, and any lattice in $G$ has fixed price 1).

In many cases in the literature on cost, proving cost 1 relies on manipulating some $\mathbb{Z}$-structure in the relevant group. In our case, we use the "ideal Poisson–Voronoi tessellation" (IPVT) to do so. The IPVT is a new random object (introduced in [BCP22], developed independently in [DCE+23, FMW23]) which has drastically different behavior when considered on Euclidean, hyperbolic, and higher rank spaces.

The IPVT can be defined as a limit of a sequence of Poisson point processes, as illustrated below in Figure 1, from [DCE+23] (which contains many helpful and beautiful illustrations). Before examining the IPVT further, we define and motivate the Poisson point process.

## The Poisson point process

Informally, a Poisson point process (sometimes we use the abbreviation Ppp) on a nondiscrete space $X$ is a random scattering of discrete points in $X$. Any action on $X$ also acts on the Poisson point process. It is the nondiscrete analogue of a Bernoulli shift.

An intuitive way to think about a Poisson random variable is that it models an experiment with some average rate of discrete successes over continuous time (like waiting or arrival times, including phone calls and particle emission in radioactive decay; see [Sti00] for an interesting history).

In a Bernoulli shift, we can think of 1's as successes and 0's as failures. To move to continuous time, we can take a limit of a collection of Bernoulli random variables (yielding a binomial), and we end up with a Poisson random variable. This leads to the definition of a Poisson point process.

Formally, a Poisson point process $\Pi$ with intensity $\alpha$ (a fixed, non-negative real constant) on a locally compact second countable space $X$ with Haar measure vol (from the isometry group $G$ of $X$, for example; good examples of spaces $X$ to have in mind are $\mathbb{R}^2, \mathbb{H}^2$, and if you are feeling ambitious, $\mathbb{H}^2 \times \mathbb{H}^2$) is a random variable taking values on the space

$$\mathbb{M}(X) := \{\text{locally finite } \sum_{i \in \mathbb{N}} \delta(x_i) \mid x_i \in X\}$$

(locally finite means in a finite ball, there are only finitely many "turned on" points) characterized by the conditions:

- For any measurable $A \subseteq X$, the number of points of $\Pi$ in $A$, $\Pi(A)$, is given by a Poisson random variable with parameter equal to $\alpha \operatorname{vol}(A)$.
- For any measurable and disjoint $A, B \subseteq X$, the random variables $\Pi(A), \Pi(B)$ are independent.

These conditions imply points of $\Pi$ are uniformly distributed in any measurable $A \subseteq X$.

For our purposes, Ppps are measure-preserving dynamical systems $G \curvearrowright (\mathbb{M}(X), \mathbb{P}_\alpha)$ where $\mathbb{P}_\alpha$ is the probability law of a Ppp with intensity $\alpha$, and the action is given by $g\Pi(A) = \Pi(g^{-1}A)$ for all $g \in G$ and measurable $A \subseteq X$.

Some notes on the various probabilities occurring in the definition:

- $\Pi$ is a random variable (function) from vol-measurable subsets of $X$ to $\mathbb{N}$
- for measurable $A \subseteq X$, the restriction of $\Pi$ to $A$ is the "$A$-window" of $\Pi$, with output of the form
$$\Pi_A := \sum_{0 \le i \le \Pi(A)}^{\Pi(A)} \delta(x_i)$$
for some $x_i \in A$ (independently and uniformly distributed)
- $\mathbb{P}_\alpha$ is a probability function from measurable events in $\mathbb{M}(X)$ to $[0,1]$
    - with $\mathbb{P}_\alpha$ one can measure, for example, the probability that a Poisson point process with intensity $\alpha$ restricted to the unit ball centered at the origin in $X$ contains 1 point at any location within distance $\epsilon$ from the origin
    - for more details on this type of measure, see for example [DVJ03, Appendix A2]

A natural question to ask is when two Ppps of different intensities on the same space are isomorphic as measure-preserving dynamical systems.

Theorem [Wil23]: Any two Poisson point processes over a locally compact second countable (lcsc) group are isomorphic.

That is, up to isomorphism of measure-preserving dynamical systems, Poisson point processes of different intensities on the same group are indistinguishable (in contrast to the case for Bernoulli shifts). This extends prior results in [OW80, SW19].

It is remarkable that taking a limit of these random processes tells us something about the algebraic-geometric relationship between a fundamental group and its manifold. To motivate the connection between the Poisson point process and the main theorem, we present a (non-standard) history of the Poisson point process, as told by the author (an ergodic theorist).

We begin with physics and classical mechanics around 1900; in particular the idea that to study a physical system, one can consider the state space of all possible outcomes, like the space of temperature values at each point in a room over time. The state space then is a manifold.

Around 1895, Poincaré introduced the fundamental group of a manifold, as an attempt towards classification of them.

To study evolution of a state space over time, one should try to understand geodesic flow. In the 1920's, Artin, Birkhoff, Morse, and other mathematicians introduced the field of symbolic dynamics in this context: one partitions a manifold into finitely or countably many pieces and records the location of a point along a geodesic over (integer-valued) time only up to the partition [Hed39].

Going back to the fundamental group, it determines which substrings of values are forbidden (in symbolic dynamics, this is called a shift of finite type).

On the other hand, one can consider a more abstract version of these strings, putting no restrictions on the values, so strings need no longer model geodesic flow (BUT an important perspective is, perhaps in some sense they still do); instead, the occurrence of each symbol is random, independent and identically distributed. Then you end up with a Bernoulli shift. And as we have seen, the Poisson point process is its nondiscrete brother.

## The ideal Poisson–Voronoi tessellation

There are two independent constructions of the IPVT (so far) [DCE+23, FMW23]. It is an open question (an exercise for the minicourse, and Question 7.1 in [DCE+23]) whether they are equivalent. We describe the perspective in [DCE+23] and then construct the IPVT as in [FMW23].

As in the last section, we let $X$ be a lcsc space. But to "see" what is going on, you should think of $\mathbb{H}^2$. A Poisson point process on $X$ induces a *Voronoi tessellation* on $X$: each point in the process is the "site" of a cell in the tessellation, and each cell consists of points in $X$ which are closer to its site than any other site. Some points of $X$ will lie on boundaries between cells; if this is a problem for our application, we can assign them to a particular cell with an equivariant and measurable function (see any of [AM22, Wil23, FMW23] for details).

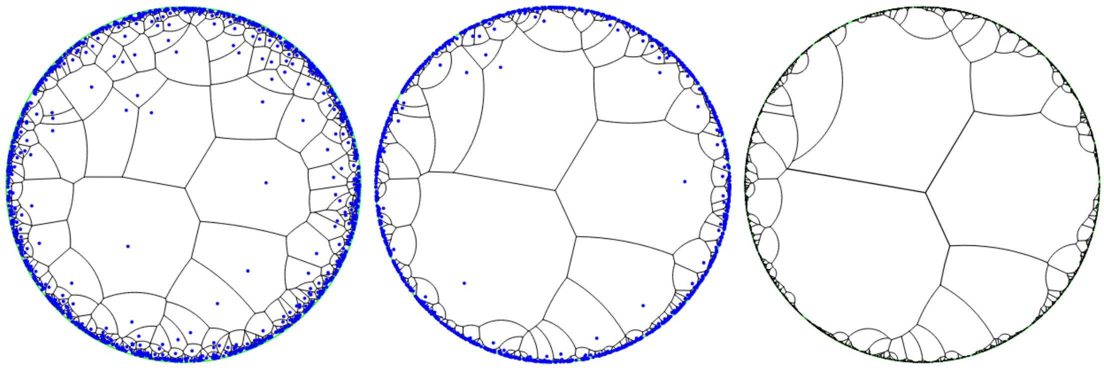

Figure 1: Poisson–Voronoi tessellations of the hyperbolic plane (in the unit disk model) with decreasing intensity. Their limit is the ideal Poisson–Voronoi tessellation of the hyperbolic plane. From [DCE+23].

Voronoi tessellations are useful in many applications in pure and applied mathematics (as evidenced on their Wikipedia page). But we are most interested in a limit of them—fixing $X$, taking a sequence of Poisson–Voronoi tessellations, and sending the intensity $\alpha$ of the Ppps to 0. A nontrivial object, called the *ideal Poisson–Voronoi tessellation* (IPVT) emerges whenever balls in $X$ grow quickly enough to build a nontrivial boundary (that is, when $X$ has superpolynomial growth [DCE+23, Section 3]). It is an exercise to show the limit on $\mathbb{R}^n$ produces a trivial object.

In [DCE+23], the authors prove the described sequence of tessellations on $\mathbb{H}^n$ converges to a tessellation whose sites are elements of $\partial\mathbb{H}^n \times \mathbb{R}$, where $\partial\mathbb{H}^n$ is the Gromov boundary of $\mathbb{H}^n$, consisting of endpoints of geodesic rays (see [BH99, Section 8] for boundary definitions; regarding convergence of tessellations, D'Achille et al. use the Fell topology).

D'Achille et al. describe the $\mathbb{R}$-component as a time delay, and this is a good way to think about it. Imagine a point traveling in $\mathbb{H}^n$ to the Gromov boundary along a geodesic. It must start traveling somewhere, and wherever it begins has a fixed distance to the origin.

Tracking the time delay allows us to measure finite "distances" between boundary points and points in $X$ in such a way so that the boundary, as a set, is invariant under the action of $X$ (or the action of its isometry group $G$).

We do not prove convergence of tessellations in [FMW23]. We construct a boundary of $X$, for any nonamenable lcsc space $X$, with the goal of obtaining a $G$-invariant measure on it (non-amenability is not necessary, but is sufficient as it implies exponential growth). We view the IPVT as the induced tessellation from a Poisson point process on the boundary. Whenever $X$ is the symmetric space of a semisimple real Lie group, we prove the constructed boundary is in fact a homogeneous space isomorphic to $B \times \mathbb{R}$, where $B$ is the Furstenberg boundary (equivalently, in this context, the Poisson boundary). This is enough to prove the main theorem.

## The boundary

Let $X$ be a nonamenable lcsc space, and let $G$ be the isometry group of $X$ (but think of $X$ as $\mathbb{H}^2, \mathbb{H}^2 \times \mathbb{H}^2$, or the symmetric space for $\mathrm{SL}_3\,\mathbb{R}$). Let $d$ be a $G$-invariant metric on $X$ (existence follows from the lcsc and isometric conditions) and vol a $G$-invariant measure on $X$. Consider the embedding map $\iota : X \to \mathcal{C}(X)$, where $\mathcal{C}(X)$ is the space of continuous functions on $X$, given by $\iota(x) := d_x$ for $x \in X$, with $d_x(y) := d(x, y)$ for $y \in X$. This embedding map gives:

- A left-invariant measure on $\iota(X)$ by taking the pushforward measure $\iota_*(\mathrm{vol})$.
- A natural compactification of $\iota(X)$ with respect to the standard compact-open topology on $\mathcal{C}(X)$.

However, we want a left-invariant measure on some sort of boundary of $X$. So consider, for $t \in \mathbb{R}$, the map $\iota_t : X \to \mathcal{C}(X)$ defined by $\iota_t(x) := \tilde{d}(x)$ with $\tilde{d}(y) := d(x, y) - t$. The value $t$ acts as a time delay—it generalizes $d(o, x)$, where $o$ is a fixed origin in $X$. We will take a limit.

Set $D := \mathrm{cl}\{\iota_t(X) \mid t \in \mathbb{R}\} \subseteq \mathcal{C}(X)$. By compactness, a subsequential limit exists for $\{\iota_{t_*}(\mathrm{vol})/\operatorname{vol}(B(o,t))\}$; call it $\mu$. It is $G$-invariant. We normalize by $\operatorname{vol}(B(o,t))^{-1}$ to obtain a boundary measure. As a lemma, we prove $\mu$ is nontrivial (using exponential growth of $\operatorname{vol}(B(o,t))$). We call the space $(D, \mu)$ the *corona space* of $G$ (this naming first appeared in [DCE+23]).

Proposition [FMW23]: The corona space for a semisimple real Lie group $G$ (equivalently, its symmetric space $X$) is isomorphic as a $G$-space to $(G/\tilde{U}, \mathrm{vol})$, where $\tilde{U}$ is the maximal unimodular subgroup of a minimal parabolic subgroup $P$ in $G$ and vol is the induced $G$-invariant measure on $G/\tilde{U}$ from $G$.

When $G = \mathrm{SL}_n(\mathbb{R})$, one can choose $P$ to be the subgroup of upper triangular matrices. See [FMW23] for details on $\tilde{U}$ (which appears as $U$ in the paper). Note in higher rank it is strictly larger than the subgroup of unipotent matrices in $P$. It is helpful to think of the corona space as $G/\tilde{U}$ for dynamical purposes we will encounter shortly. But to "see" the corona space, it is perhaps more helpful to think of the isomorphic $G/P \times \mathbb{R}$. Indeed, this perspective gives a good description of the $G$-invariant measure on $G/\tilde{U}$: it is a product of the harmonic measure on $G/P$ and the associated Radon-Nikodym derivative on $\mathbb{R}$ (there is only one $G$-invariant measure on $G/\tilde{U}$ up to scaling).

Elements in $G/\tilde{U}$ can be represented as functions on $X$ and have the form

$$\beta(x) = c + \lim_{n\to\infty} d(\gamma(n), x) - d(\gamma(n), o)$$

for a "good" escaping sequence $\gamma(n)$ (not all escaping sequences work; for an example, find a bad one in $\mathbb{H}_2 \times \mathbb{H}_2$) and $c \in \mathbb{R}$. One can think of them as generalized Busemann functions.

In [FMW23], we define the IPVT to be the Voronoi tessellation on $X$ induced from the Ppp on $G/\tilde{U}$. This Ppp is a random variable with realization of the form

$$\sum_{i\in\mathbb{N}} \delta(\beta_i)$$

for some locally finite collection $B := \{\beta_i\} \subset G/\tilde{U}$. By our construction, for $x \in X$, we have $\beta_i(x) \in \mathbb{R}$. Additionally, one can show:

- $\mu(\beta \in G/\tilde{U} \mid \beta(o) \leq 0) = 1$
- for all $c \in \mathbb{R}$, $\mu(\beta \in G/\tilde{U} \mid \beta(o) \leq c) < \infty$

The IPVT cell for $\beta_i$ is

$$\text{cell}(\beta_i) := \{x \in X \mid \beta_i(x) \leq \beta_j(x) \text{ for all } \beta_j \in B\}.$$

Ignoring details regarding where to send boundary points of cells, the collection $\{\text{cell}(\beta_i)\}$ is a tessellation on $X$; it is the IPVT.

We are ready to state (and prove) the last piece necessary in order to obtain the main theorem.

## Unbounded borders in higher rank

We claimed earlier the IPVT behaves very differently in hyperbolic space versus higher rank. The following theorem captures this behavior. We focus on the Lie group case and leave out the tree case for simplicity (in fact, the $G/\tilde{U}$ perspective is not helpful for products of trees, and we stick to the $G/P \times \mathbb{R}$ perspective in the paper for this case).

Theorem [FMW23]: Let $G$ be a higher rank semisimple real Lie group. Almost every pair of cells in the IPVT on $G$ shares an unbounded border almost surely.

We will sketch the proof, but first, remember our goal is to prove the Ppp on $G$ has cost 1, meaning, we want to construct a $G$-equivariant and measurable connected graph, using points of the Ppp as vertices, that looks, on average, like $\mathbb{Z}$ (perhaps only in a limit). Assuming the above theorem, the proof of cost 1 goes as follows:

- Coupling the Ppp on $G$ with an independent IPVT on $G$ does not change the cost of the Ppp [FMW23, Section 2].
- The Ppp on $G$ restricted to a cell of an independent IPVT has cost 1 [FMW23, Section 7].
- Since almost every pair of cells shares an unbounded border almost surely, we can simply connect points that are within some $r$-radius of the border between two cells and within some $r'$-radius of each other (there are infinitely many such points by Borel–Cantelli), run an $\epsilon$-percolation on these added edges, and send $\epsilon$ to 0.

Sketch of the proof of unbounded borders:

Fix $\beta_1, \beta_2 \in G/\tilde{U}$ and let $x \in X$ such that $\beta_1(x) = \beta_2(x)$, so $x$ is "equidistant" from $\beta_1$ and $\beta_2$. Suppose $\beta_1 = g_1\tilde{U}, \beta_2 = g_2\tilde{U}$ for some $g_1, g_2 \in G$. The $G$-stabilizer of $\{\beta_1, \beta_2\}$ is $S = g_1\tilde{U}g_1^{-1} \cap g_2\tilde{U}g_2^{-1}$. One can prove, using the Bruhat decomposition, that

$$S \cong \mathbb{R}^{(\text{real rank of } G)-1} \times M$$

where $M$ is a compact subgroup in $G$ [FMW23, Lemma 4.1].

In particular, the stabilizer $S$ is non-compact if and only if $G$ is higher rank. We will show an unbounded subset of $Sx$ belongs to both cell$(\beta_1)$ and cell$(\beta_2)$ for any Ppp on $G/\tilde{U}$ containing $\beta_1$ and $\beta_2$ ($\mathbb{P}$-almost surely). Let $\Pi$ be such a Ppp (we use the Palm measure for the Ppp to legally make such a claim; see [LP18, Section 9] for a definition and details). We may assume it has intensity 1 (all Ppps on $G/\tilde{U}$ are isomorphic [FMW23, Remark 3.7]).

Set $c := \beta_1(x) = \beta_2(x)$ and $A := \{\beta \in G/\tilde{U} \mid \beta(x) < c\}$. We noted before that $\mu(A) < \infty$. This implies $\mathbb{P}(\Pi(A) = 0) = e^{-\mu(A)} > 0$. Let $\{s_j\}_{j\in\mathbb{N}} \subseteq S$ be an escaping sequence. We have

$$s_jA = \{s_j\beta \in G/\tilde{U} \mid \beta(x) < c\} = \{\beta \in G \mid \beta(s_jx) < c\}.$$

A few remarks:

- the action of $G$ on $G/\tilde{U}$ is given by $g\beta(x) = \beta(g^{-1}x)$
- $c = s_j\beta_1(x) = s_j\beta_2(x)$ for all $j$
- $\mathbb{P}(\Pi(s_jA) = 0) = \mathbb{P}(\Pi(A) = 0)$ for all $j$

Let $E_j$ be the event that $\Pi(s_jA) = 0$. Whenever $E_j$ occurs, the point $s_jx \in \text{cell}(\beta_1) \cap \text{cell}\,\beta_2$.

If we have a subcollection of $\{E_j\}$ of "independent enough" events, we can apply Borel–Cantelli to conclude events in the subcollection occur infinitely often, implying the statement of the theorem. It is sufficient to invoke the Howe–Moore theorem (but not necessary, see [FMW23, Section 7]), which tells us

$$\lim_{j,j'\to\infty} \mu(s_jA \cap s_{j'}A) = 0.$$

Recall for measurable and disjoint sets $A', A''$ the random variables $\Pi(A'), \Pi(A'')$ are independent. So there exists a subsequence $\{s_{j_k}\}$ such that the events $\{E_{j_k}\}$ are indeed independent enough (the Kochen–Stone Lemma makes this precise; see [FMW23, Section 6]). Thus almost surely (with respect to the law $\mathbb{P}$ for the Ppp) we have an unbounded border between almost every pair (with respect to $\mu^2$; this assumption is necessary at the step where we referenced the Palm measure) of boundary points.

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