Exercises for IHES minicourse: Poisson–Voronoi tessellations and fixed price in higher rank

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Definition 1 (Poisson point process)**.** *A Poisson point process* Π *with intensity* $\alpha \in \mathbb{R}_{>0}$ *on a locally compact second countable (lcsc) space* X *with Haar measure* μ *is a random variable taking values on the space*

$$\mathbb{M}(X) := \{ \text{ locally finite } \sum_{i \in \mathbb{N}} \delta(x_i) \mid x_i \in X \}$$

characterized by the conditions:

- *For any measurable* $A \subseteq X$*, the number of points of* Π *in* A*,* $\Pi(A)$*, is given by a Poisson random variable with parameter equal to* $\alpha\mu(A)$*.*
- *For any measurable and disjoint* $A, B \subseteq X$*, the random variables* $\Pi(A), \Pi(B)$ *are independent.*

Exercise 1. Prove the 2 above conditions imply that for any measurable $A \subseteq X$, the points of Π in A (denoted $\Pi|_A$) are uniformly distributed in A.

Exercise 2. In fact, the first condition above implies the second. Prove this.

Exercise 3. Construct a Poisson point process on the unit square in $\mathbb{R}^2$, given a uniform random variable on $[0, 1]$. Construction here means that you should be able to write some code in, for example, python or R, that outputs a Poisson point process on the unit square in $\mathbb{R}^2$.

Exercise 4. Apply the properties of the Poisson point process and Exercise 3 to describe how to construct a Poisson point process on $\mathbb{R}$ (or if you prefer, any lcsc space), given a uniform random variable on $[0, 1]$.

Exercise 5. Prove the Poisson point process on a compact space, for example the circle, is not a standard probability space (meaning, it is an atomic probability space—find the atom).

Definition 2 (Ergodicity)**.** *An action of a group* G *on a probability space* $(M, \mathbb{P})$ *is said to be measure-preserving if for all* $g \in G$ *and measurable* $B \subseteq M$*, we have* $\mathbb{P}(g^{-1}B) = \mathbb{P}(B)$*. In this case we also say* $\mathbb{P}$ *is* G*-invariant.*

If $\mathbb{P}$ *is* G*-invariant and for all measurable* $B \subseteq M$ *such that* $g^{-1}(B) = B$ *for all* $g \in G$ *implies* $\mathbb{P}(B) \in \{0, 1\}$*, we say* $\mathbb{P}$ *is ergodic.*

Exercise 6. Prove the Poisson point process on $\mathbb{R}$ (or any lcsc space) is ergodic.

The notes [LP18] are a nice introduction to the Poisson point process, as is the book [Kin93].

Definition 3 (Rank gradient)**.** *Let* G *be a semisimple real Lie group, and let* X *be its symmetric space* G/K *for some maximal compact subgroup* K*. Let* $\{\Gamma_i\}_{i \in \mathbb{N}}$ *be a sequence of torsion-free lattices in* G *such that* $\Gamma_i \backslash X$ *Benjamini–Schramm (BS) converges to* X*.*

BS convergence: The injectivity radius around a typical point in $\Gamma_i \backslash X$ goes to infinity; when G is simple, this is equivalent to $\mathrm{vol}(\Gamma_i \backslash G)) \to \infty$.

The limit

$$\lim_{i \to \infty} \frac{d(\Gamma_i) - 1}{\mathrm{vol}(\Gamma_i \backslash X)}$$

is the rank gradient of G.

Exercise 7. Prove the rank gradient of $\mathrm{SL}_2(\mathbb{R})$ does not vanish.

Definition 4 (Busemann function)**.** *Let X be a metric space and let $c : [0, \infty) \to X$ be a geodesic ray. The function $\beta_c : X \to \mathbb{R}$ defined by*

$$\beta_c(x) = \lim_{t \to \infty} (d(x, c(t)) - t)$$

is the Busemann function associated to the geodesic ray c.

Exercise 8. Identify the Busemann functions on $\mathbb{R}^n$. Then identify them on $\mathbb{H}^n$.

The book [BH99] is a great reference for boundary behavior of CAT(0) spaces, in particular semisimple Lie groups.

Exercise 9. Using the construction in [DCE+23] or [FMW23, Section 3], or both, prove the ideal Poisson–Voronoi tessellation (IPVT) for $\mathbb{R}^n$ is trivial. Provide a condition in terms of the growth rate of a space which guarantees the existence of a non-trivial IPVT.

Exercise 10 (Open question)**.** Prove the constructions in [DCE+23] and [FMW23] of the IPVT for $\mathbb{H}^n$ are equivalent.

References

[DCE+23] Matteo D'Achille, Nicolas Curien, Nathanaël Enriquez, Russell Lyons, and Meltem Ünel. Ideal Poisson-Voronoi tessellations on hyperbolic spaces. *Preprint*, arXiv:2303.16831, 2023.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[FMW23] Mikolaj Fraczyk, Sam Mellick, and Amanda Wilkens. Poisson-Voronoi tessellations and fixed price in higher rank. *Preprint*, arXiv:2307.01194, 2023.

[Kin93] J. F. C. Kingman. *Poisson processes*, volume 3 of *Oxford Studies in Probability*. The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.

[LP18] Günter Last and Mathew Penrose. *Lectures on the Poisson process*, volume 7 of *Institute of Mathematical Statistics Textbooks*. Cambridge University Press, Cambridge, 2018.