

Exercises for IHES minicourse: Poisson–Voronoi tessellations and fixed price in higher rank

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Definition 1 (Poisson point process). A Poisson point process Π with intensity $\alpha \in \mathbb{R}_{>0}$ on a locally compact second countable (lcsc) space X with Haar measure μ is a random variable taking values on the space

$$\mathbb{M}(X) := \{\text{locally finite } \sum_{i \in \mathbb{N}} \delta(x_i) \mid x_i \in X\}$$

characterized by the conditions:

- For any measurable $A \subseteq X$, the number of points of Π in A , $\Pi(A)$, is given by a Poisson random variable with parameter equal to $\alpha\mu(A)$.
- For any measurable and disjoint $A, B \subseteq X$, the random variables $\Pi(A), \Pi(B)$ are independent.

Exercise 1. Prove the 2 above conditions imply that for any measurable $A \subseteq X$, the points of Π in A (denoted $\Pi|_A$) are uniformly distributed in A .

Exercise 2. In fact, the first condition above implies the second. Prove this.

Exercise 3. Construct a Poisson point process on the unit square in \mathbb{R}^2 , given a uniform random variable on $[0, 1]$. Construction here means that you should be able to write some code in, for example, python or R, that outputs a Poisson point process on the unit square in \mathbb{R}^2 .

Exercise 4. Apply the properties of the Poisson point process and Exercise 3 to describe how to construct a Poisson point process on \mathbb{R} (or if you prefer, any lcsc space), given a uniform random variable on $[0, 1]$.

Exercise 5. Prove the Poisson point process on a compact space, for example the circle, is not a standard probability space (meaning, it is an atomic probability space—find the atom).

Definition 2 (Ergodicity). An action of a group G on a probability space (M, \mathbb{P}) is said to be measure-preserving if for all $g \in G$ and measurable $B \subseteq M$, we have $\mathbb{P}(g^{-1}B) = \mathbb{P}(B)$. In this case we also say \mathbb{P} is G -invariant.

If \mathbb{P} is G -invariant and for all measurable $B \subseteq M$ such that $g^{-1}(B) = B$ for all $g \in G$ implies $\mathbb{P}(B) \in \{0, 1\}$, we say \mathbb{P} is ergodic.

Exercise 6. Prove the Poisson point process on \mathbb{R} (or any lcsc space) is ergodic.

The notes [LP18] are a nice introduction to the Poisson point process, as is the book [Kin93].

Definition 3 (Rank gradient). Let G be a semisimple real Lie group, and let X be its symmetric space G/K for some maximal compact subgroup K . Let $\{\Gamma_i\}_{i \in \mathbb{N}}$ be a sequence of torsion-free lattices in G such that $\Gamma_i \backslash X$ Benjamini–Schramm (BS) converges to X .

BS convergence: The injectivity radius around a typical point in $\Gamma_i \backslash X$ goes to infinity; when G is simple, this is equivalent to $\text{vol}(\Gamma_i \backslash G) \rightarrow \infty$.

The limit

$$\lim_{i \rightarrow \infty} \frac{d(\Gamma_i) - 1}{\text{vol}(\Gamma_i \backslash X)}$$

is the rank gradient of G .

Exercise 7. Prove the rank gradient of $\text{SL}_2(\mathbb{R})$ does not vanish.

Definition 4 (Busemann function). Let X be a metric space and let $c : [0, \infty) \rightarrow X$ be a geodesic ray. The function $\beta_c : X \rightarrow \mathbb{R}$ defined by

$$\beta_c(x) = \lim_{t \rightarrow \infty} (d(x, c(t)) - t)$$

is the Busemann function associated to the geodesic ray c .

Exercise 8. Identify the Busemann functions on \mathbb{R}^n . Then identify them on \mathbb{H}^n .

The book [BH99] is a great reference for boundary behavior of CAT(0) spaces, in particular semisimple Lie groups.

Exercise 9. Using the construction in [DCE⁺23] or [FMW23, Section 3], or both, prove the ideal Poisson–Voronoi tessellation (IPVT) for \mathbb{R}^n is trivial. Provide a condition in terms of the growth rate of a space which guarantees the existence of a non-trivial IPVT.

Exercise 10 (part of Question 7.1 in [DCE⁺23]). Prove the constructions in [DCE⁺23] and [FMW23] of the IPVT for \mathbb{H}^n are equivalent.

References

- [DCE⁺23] Matteo D’Achille, Nicolas Curien, Nathanaël Enriquez, Russell Lyons, and Meltem Ünel. Ideal Poisson–Voronoi tessellations on hyperbolic spaces. *Preprint*, arXiv:2303.16831, 2023.
- [BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.
- [FMW23] Mikolaj Fraczyk, Sam Mellick, and Amanda Wilkens. Poisson–Voronoi tessellations and fixed price in higher rank. *Preprint*, arXiv:2307.01194, 2023.
- [Kin93] J. F. C. Kingman. *Poisson processes*, volume 3 of *Oxford Studies in Probability*. The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.
- [LP18] Günter Last and Mathew Penrose. *Lectures on the Poisson process*, volume 7 of *Institute of Mathematical Statistics Textbooks*. Cambridge University Press, Cambridge, 2018.