NONDEGENERACY AND WEAK GLOBAL CONVERGENCE OF THE LLOYD ALGORITHM IN \mathbb{R}^{D*}

MARIA EMELIANENKO $^{\dagger},$ LILI JU $^{\ddagger},$ and Alexander Rand §

Abstract. The Lloyd algorithm originated in the context of optimal quantization and represents a fixed point iteration for computing an optimal quantizer. Reducing average distortion at every step, it constructs a Voronoi partition of the domain and replaces each generator with the centroid of the corresponding Voronoi cell. Optimal quantization is obtained in the case of a centroidal Voronoi tessellation (CVT), which is a special Voronoi tessellation of a domain $\Omega \in \mathbb{R}^d$ having the property that the generators of the Voronoi diagram are also the centers of mass, with respect to a given density function $\rho \geq 0$, of the corresponding Voronoi cells. The Lloyd iteration is currently the most popular and elegant algorithm for computing CVTs and optimal quantizers, but many questions remain about its convergence, especially in d-dimensional spaces (d > 1). In this paper, we prove that any limit point of the Lloyd iteration in any dimensional spaces is nondegenerate provided that Ω is a convex and bounded set and ρ belongs to $L^1(\Omega)$ and is positive almost everywhere. This ensures that the fixed point map remains closed and hence the standard theory of descent methods guarantees weak global convergence of the Lloyd iteration to the set of nondegenerate fixed point quantizers. While previously only conjectured, the convergence properties of the Lloyd iteration are rigorously justified under such minimal regularity assumptions on the density functional. The results presented in this paper go beyond existing convergence theories for CVTs and optimal quantization related algorithms and should be of interest to both the mathematical and engineering communities.

Key words. centroidal Voronoi tessellations, quantization, Lloyd algorithm, global convergence

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1. Introduction. Given a bounded domain $\Omega \in \mathbb{R}^d$ and a set of distinct points $\mathbf{Z} = {\mathbf{z}_i}_{i=1}^n \subset \Omega$, for each point \mathbf{z}_i , i = 1, ..., n, define the corresponding Voronoi region $V_i(\mathbf{Z})$, i = 1, ..., n, by

$$V_i(\mathbf{Z}) = \{ \mathbf{z} \in \Omega \mid |\mathbf{z} - \mathbf{z}_i| < |\mathbf{z} - \mathbf{z}_i| \text{ for } j = 1, \dots, n \text{ and } j \neq i \},\$$

where $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^d . Clearly $V_i(\mathbf{Z}) \cap V_j(\mathbf{Z}) = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^n \overline{V}_i(\mathbf{Z}) = \overline{\Omega}$ so that $\{V_i(\mathbf{Z})\}_{i=1}^n$ is a tessellation of Ω . We refer to $\mathbf{V}(\mathbf{Z}) = \{V_i(\mathbf{Z})\}_{i=1}^n$ as the Voronoi tessellation of Ω associated with the point set \mathbf{Z} . A point \mathbf{z}_i is called a generator; a subdomain $V_i(\mathbf{Z}) \subset \Omega$ is referred to as the Voronoi region corresponding to the generator \mathbf{z}_i . It is well known that the dual tessellation (in a graph-theoretical sense) to a Voronoi tessellation of Ω is the so-called Delaunay triangulation (DT). All Voronoi regions $V_i(\mathbf{Z})$'s are convex (especially convex polygons/polyhedra, except for their part on the boundary) if Ω is convex.

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Given a density function $\rho(\mathbf{z}) \geq 0$ defined on Ω , for any region $V \subset \Omega$, define \mathbf{c}_V , the mass center or centroid of V, by

$$\mathbf{c}_V = rac{\int_V \mathbf{y}
ho(\mathbf{y}) \, \mathrm{d}\mathbf{y}}{\int_V
ho(\mathbf{y}) \, \mathrm{d}\mathbf{y}}.$$

Then a special family of Voronoi tessellations can be defined in the following.

DEFINITION 1.1 (see [6]). We refer to a Voronoi tessellation $(\mathbf{Z}, \mathbf{V}(\mathbf{Z}))$ of Ω as a centroidal Voronoi tessellation (CVT) if and only if the points $\mathbf{Z} = \{\mathbf{z}_i\}_{i=1}^n$ which serve as the generators of the associated Voronoi regions $\mathbf{V}(\mathbf{Z}) = \{V_i(\mathbf{Z})\}_{i=1}^n$ are also the centroids of those regions, i.e., if and only if we have that

$$\mathbf{z}_i = \mathbf{c}_{V_i(\mathbf{Z})}$$
 for $i = 1, \ldots, n$.

General Voronoi tessellations do not satisfy the CVT property. It is worth noting that for a given domain and density, at least one CVT exists [1] but may not be unique [6]. The CVT concept can be generalized to very broad settings that range from abstract spaces and distance metrics to discrete point sets [6]. In the latter setting, it can be recognized as being closely related to the k-means and h-means algorithms in clustering and quantization applications [10]. Its extension to general surfaces and manifolds has also been studied in [7, 8]. CVTs are very useful in many applications, including but not limited to image and data analysis, vector quantization, resource optimization, design of experiments, optimal placement of sensors and actuators, cell biology, territorial behavior of animals, numerical partial differential equations, point sampling, meshless computing, mesh generation and optimization, reduced-order modeling, computer graphics, and mobile sensing networks.

CVTs possess an optimization property that can be used as a basis for an alternate definition. Given any set of points $\mathbf{Z} = {\mathbf{z}_i}_{i=1}^n$ in Ω and any tessellation $\mathbf{V} = {V_i}_{i=1}^n$ of Ω , define the *energy* by

(1.1)
$$\mathcal{E}(\mathbf{Z}, \mathbf{V}) = \sum_{i=1}^{n} \int_{V_i} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{z}_i|^2 \, \mathrm{d}\mathbf{y}.$$

Then it can be shown that \mathcal{E} is minimized only if (\mathbf{Z}, \mathbf{V}) forms a CVT of Ω [6]. Notice that (\mathbf{Z}, \mathbf{V}) still may not be a minimizer of \mathcal{E} although (\mathbf{Z}, \mathbf{V}) is a CVT. For some properties of the minimizers of the energy functional \mathcal{E} (i.e., CVTs or optimal quantizations), see discussions in [6, 2, 21]. Although the energy \mathcal{E} may not be directly identified with the energy of some physical system, it is often naturally associated with quantities such as *error distortion*, *variance*, and *cost* in many applications.

Currently the most popular and effective algorithm for construction of CVTs is the so-called Lloyd algorithm [18] that is a simple iteration between constructing Voronoi diagrams and mass centroids.

ALGORITHM 1.1 (Lloyd algorithm). We are given a domain Ω , a density function $\rho(\mathbf{x})$ defined on Ω , and a positive integer n.

- 0. Select an initial set of n distinct points $\mathbf{Z} = {\mathbf{z}_i}_{i=1}^n$ on Ω .
- 1. Construct the Voronoi regions $\mathbf{V}(\mathbf{Z}) = \{V_i(\mathbf{Z})\}_{i=1}^n$ of Ω associated with \mathbf{Z} .
- Determine the centroids of the Voronoi regions V(Z); these centroids form the new set of points Z.

3. If the new points meet some convergence criterion, return (**Z**, **V**(**Z**)) and terminate; otherwise, go to step 1.

The Lloyd algorithm also coincides with the simplest version of the k-means algorithm in clustering and quantization fields [10, 12, 18, 15]. A commonly used stopping criterion is to test whether $\|\mathbf{Z}^k - \mathbf{Z}^{k-1}\| \leq \epsilon$ holds for some predefined tolerance ϵ at some iteration k. It is worth noting that the energy $\mathcal{E}(\mathbf{Z}, \mathbf{V}(\mathbf{Z}))$ decreases monotonically along the Lloyd iteration. Some acceleration schemes based on multigrid techniques are proposed in [4]. A probabilistic version of the Lloyd method for computing CVTs and its parallel implementation have also been studied in [14]. One famous generalization in the vector quantization setting is the Linde–Buzo–Gray algorithm [17], widely used in statistics and data mining applications.

Since Lloyd's pioneering work, many studies have been done on convergence properties of the Lloyd iteration [6, 5, 13, 16, 19]. Despite the algorithm's long history and wide popularity in many applications, its convergence theory is far from being complete. Many researchers addressed the question of existence and uniqueness of an optimal quantizer. Most notably, in [9] and later in [24] and [16], it was shown that the Lloyd method is a local contraction (thus *locally* convergent) in the one-dimensional space when the density function is logarithmically concave. This result was reiterated in [6] with a much simpler proof. In [25], this result was extended to all continuous and positive densities in the one-dimensional setting of scalar quantization. A different approach was used in [5], where the fixed point map was shown to be closed and a weak global convergence result was presented for all continuous and positive density functionals along with some numerical studies and precise theoretic estimates of the local convergence rate.

In higher dimensions, convergence results are much more limited. For discrete distributions, the algorithm is shown to converge and, moreover, reach a CVT in a finite number of steps [23, 11]. In the case of continuous distributions, [11, 22] show convergence of the energy functional by defining the value of the Lloyd map on degenerate points. In order to ensure that the algorithm is closed, this requires that degenerate points be *defined* as fixed points in the definition of the algorithm. So, while standard theory of descent methods ensures that this extended Lloyd algorithm will converge to a fixed point, these limit points could be degenerate. Similarly, the multidimensional global convergence of a subsequence argument presented in [5] implicitly assumed nondegenerate points. This makes sense as it will be shown that points of degenerate points. This makes sense as it will be shown that points of degeneracy locally maximize the energy in some sense. Moreover, this is necessary in order to ensure that limit points are in fact CVTs.

In this paper we prove that for an arbitrary, positive density function under a mild L^1 regularity assumption, the fixed point mapping remains closed, which guarantees weak global convergence of the Lloyd iteration. Although we rely on the standard global convergence theory for descent algorithms as discussed in previous investigations (see, e.g., [22, 24]), we take a more direct approach of rigorously justifying the nondegeneracy of the limiting and intermediate iteration points, an approach that much simplifies the analysis of the method and that addresses a question that escaped rigorous treatment until now. These results ensure that Lloyd iterates do not approach degenerate points and that any limit point of the algorithm is a CVT with n distinct generators.

The rest of the paper is organized as follows. In section 2, we introduce some

necessary notation and give an overview of some relevant results and properties. In subsection 3.1, we derive precise estimates on the distance from a centroid to the boundary of the domain and obtain an upper bound on the difference of two Voronoi regions. In subsection 3.2 we present the proof of nondegeneracy of the limit points of the Lloyd iteration, followed in section 3.3 by the main result of the weak global convergence of Lloyd algorithm. Conclusions and some open questions are given in section 4.

2. Preliminaries. For any set $V \subset \Omega$, let us denote by m(V) the measure of V, i.e.,

$$m(V) := \int_V \mathrm{d}\mathbf{y},$$

and by M(V) the mass of V, i.e.,

$$M(V) := \int_V \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

The fact that ρ is positive almost everywhere means that M(V) > 0 if m(V) > 0. We also define $\hat{\Omega}^n$ as follows:

$$\hat{\Omega}^n := \{ \mathbf{Z} = (\mathbf{z}_1, \dots, \mathbf{z}_n) \in \overline{\Omega}^n \mid \mathbf{z}_i \neq \mathbf{z}_j \ \forall i \neq j \}.$$

The so-called Lloyd map is the function $\mathbf{T} : \hat{\Omega}^n \to \hat{\Omega}^n$ that takes a tuple of generators into the tuple of the centroids of its Voronoi regions (exactly the iteration used in the Lloyd algorithm). More precisely, let $\mathbf{T} = (\mathbf{T}_1, \dots, \mathbf{T}_n)$; then, for any $\mathbf{Z} \in \hat{\Omega}^n$,

$$\mathbf{T}_{i}(\mathbf{Z}) = \frac{\int_{V_{i}(\mathbf{Z})} \mathbf{y} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y}}{\int_{V_{i}(\mathbf{Z})} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y}}$$

for i = 1, ..., n. The Lloyd map is continuous from $\hat{\Omega}^n$ to $\hat{\Omega}^n$ [6, 5] but not defined on $\bar{\Omega}^n \setminus \hat{\Omega}^n$. The Lloyd algorithm is then the process generating a sequence of point sets:

$$\mathbf{Z}, \mathbf{T}(\mathbf{Z}), \mathbf{T}^2(\mathbf{Z}), \dots, \mathbf{T}^k(\mathbf{Z}), \dots$$

Clearly, the resulting Lloyd iterates are bounded.

For any $\mathbf{Z} \in \hat{\Omega}^n$, the quantization energy is defined as

$$\mathcal{G}(\mathbf{Z}) = \mathcal{E}(\mathbf{Z}, \mathbf{V}(\mathbf{Z})) = \sum_{i=1}^{n} \int_{V_i(\mathbf{Z})} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{z}_i|^2 \, \mathrm{d}\mathbf{y}.$$

The quantization energy function is also continuous on $\hat{\Omega}^n$. Furthermore, this energy can be extended continuously to $\overline{\Omega}^n$. When $\mathbf{Z} \notin \hat{\Omega}^n$, the value of \mathcal{G} is given by the same formula except we consider \mathbf{Z} without duplicate components. For example, in the case of n = 2, if $\mathbf{Z} = (\mathbf{z}_1, \mathbf{z}_2)$ and $\mathbf{z}_1 \neq \mathbf{z}_2$, then

$$\mathcal{G}(\mathbf{Z}) = \int_{V_1(\mathbf{Z})} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{z}_1|^2 \, \mathrm{d}\mathbf{y} + \int_{V_2(\mathbf{Z})} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{z}_2|^2 \, \mathrm{d}\mathbf{y}.$$

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In the degenerate case, we have $\mathbf{Z} = (\mathbf{z}_1, \mathbf{z}_1)$, so that

$$\mathcal{G}(\mathbf{Z}) = \mathcal{G}((\mathbf{z}_1)) = \int_{\Omega} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{z}_1|^2 \, \mathrm{d}\mathbf{y}.$$

The derivative of \mathcal{G} in $\hat{\Omega}^n$ is given [6] by

(2.1)
$$\frac{\partial \mathcal{G}}{\partial \mathbf{z}_i}(\mathbf{Z}) = 2M(V_i(\mathbf{Z}))(\mathbf{Z}_i - \mathbf{T}_i(\mathbf{Z})),$$

and thus the set of critical points of \mathcal{G} in $\hat{\Omega}^n$ is exactly the set of fixed points of the Lloyd map and thus is the set of CVTs.

For convenience of discussion, a more general energy \mathcal{H} is defined in [5] as follows: for any $(\mathbf{Y}, \mathbf{Z}) \in (\hat{\Omega}^n, \hat{\Omega}^n)$,

$$\mathcal{H}(\mathbf{Y}, \mathbf{Z}) = \mathcal{E}(\mathbf{Z}, \mathbf{V}(\mathbf{Y})) = \sum_{i=1}^{n} \int_{V_{i}(\mathbf{Y})} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{z}_{i}|^{2} \, \mathrm{d}\mathbf{y}.$$

Notice that \mathcal{H} cannot be continuously extended to $(\overline{\Omega}^n, \overline{\Omega}^n)$ like the function \mathcal{G} . The Lloyd algorithm is in fact a dual minimization process [5]: for any $\mathbf{Z} \in \hat{\Omega}^n$,

$$egin{aligned} \mathcal{H}(\mathbf{Z},\mathbf{T}(\mathbf{Z})) &= \min_{\mathbf{Y}\in\hat{\Omega}^n}\mathcal{H}(\mathbf{Z},\mathbf{Y}), \ \mathcal{H}(\mathbf{Z},\mathbf{Z}) &= \min_{\mathbf{Y}\in\hat{\Omega}^n}\mathcal{H}(\mathbf{Y},\mathbf{Z}). \end{aligned}$$

As long as ρ is positive almost everywhere, the above minimization problems on the right-hand side both have a unique solution. Thus we have that

(2.2)
$$\mathcal{G}(\mathbf{T}(\mathbf{Z})) = \mathcal{H}(\mathbf{T}(\mathbf{Z}), \mathbf{T}(\mathbf{Z})) \le \mathcal{H}(\mathbf{Z}, \mathbf{T}(\mathbf{Z})) \le \mathcal{H}(\mathbf{Z}, \mathbf{Z}) = \mathcal{G}(\mathbf{Z}).$$

Uniqueness of the centroid implied by the positive density in a convex set ensures that if $(\mathbf{Z}, \mathbf{V}(\mathbf{Z}))$ is not a CVT, then $\mathcal{G}(\mathbf{T}(\mathbf{Z})) < \mathcal{G}(\mathbf{Z})$.

The following definition of weak convergence will be used throughout the paper.

DEFINITION 2.1. Denote by $\sigma(x, S)$ a distance between a point $x \in \Omega$ and a set Sand by Γ the set of fixed points of \mathbf{T} . We will say that the sequence of Lloyd iterates $\{\mathbf{Z}^i\}_{i=0}^{\infty}$ is weakly convergent if $\lim_{i\to+\infty} \nabla \mathcal{G}(\mathbf{Z}^i) = \mathbf{0}$ and $\lim_{i\to+\infty} \sigma(\mathbf{Z}^i, \Gamma) = 0$.

In other words, the sequence of Lloyd iterates converges in a weak sense if the energy values asymptotically approach some limit and the sequence of Lloyd iteration points approaches the set of fixed points of \mathbf{T} .

In order to analyze *global* convergence of the Lloyd algorithm in general dimensional spaces, we recall the following result.

THEOREM 2.1 (see [5, 20]). If the iterates in the Lloyd algorithm stay in a compact set where the Lloyd map **T** is continuous, then the algorithm is weakly globally convergent, i.e., $\lim_{i\to+\infty} \nabla \mathcal{G}(\mathbf{Z}^i) = \mathbf{0}$ for the Lloyd iterates $\{\mathbf{Z}^i\}_{i=0}^{\infty}$ starting from any initial guess (in other words, any limit point of $\{\mathbf{Z}^i\}_{i=0}^{\infty}$ is a critical point of \mathcal{G}).

Theorem 2.1 was proved in [5] using the properties of \mathcal{H} and the (weak) global convergence theorem stated in [20]. It was also shown in [5] and [25] that the Lloyd algorithm has to converge to the set of critical points of the energy in one-dimensional space provided that ρ is strictly positive and smooth, but for higher dimensional cases, this question is still open, as discussed earlier in the introduction. It has also never been rigorously justified that the set of limit points cannot contain more than one point, except for the case of log-concave density in the onedimensional case, restricting the convergence in the aforementioned result to the "weak" sense as given in Definition 2.1. Although we conjecture that a stronger convergence result can be obtained in most practical cases, this question requires a separate rigorous treatment, while at present we focus our attention on extending Theorem 2.1 to higher dimensions.

More precisely, the goal of this paper is to present a proof for the weak global convergence of the Lloyd algorithm in any dimensional space. We do this by showing that limit points of the Lloyd method in \mathbb{R}^d are nondegenerate, and hence the algorithm converges in the sense of Definition 2.1 for any L^1 density functions in any dimension.

3. Weak global convergence of Lloyd iteration. In the following part of this paper, we assume that the following conditions hold.

Assumption 3.1. (1) The domain $\Omega \in \mathbb{R}^d$ is a convex and bounded set with the diameter

diam(
$$\Omega$$
) := sup $|\mathbf{z} - \mathbf{y}| = R_{\Omega} < +\infty$.

(2) The density function ρ belongs to $L^1(\Omega)$ and is positive almost everywhere. Consequently, we have that

$$0 < M(\Omega) = \|\rho\|_{L^1(\Omega)} = \int_{\Omega} \rho(\mathbf{y}) \mathrm{d}\mathbf{y} < +\infty.$$

For the proof of weak global convergence of the Lloyd algorithm, we will show that any limit point of the algorithm is nondegenerate (in $\hat{\Omega}^n$) and thus the iterates stay inside a compact set in $\hat{\Omega}^n$. The weak global convergence of the algorithm will then follow from Theorem 2.1. This ensures that any limit point of the algorithm does in fact correspond to a CVT. Before these results can be shown, a number of technical intermediate lemmas must be shown. The goal of these lemmas is to estimate how the quantization energy changes between iterations of the Lloyd map when iterates approach the boundary of $\hat{\Omega}^n$. As iterates approach the boundary of $\hat{\Omega}^n$, some generator approaches the boundary of its Voronoi region. As all Voronoi regions are convex sets, it becomes important to bound the distance between the boundary and the centroid of a generic convex set in Ω with a prescribed mass.

3.1. Some lemmas. First, let us present a classic result from measure theory.

LEMMA 3.1. Let Assumption 3.1 be satisfied. Then for any $\epsilon > 0$, there exists a constant $\eta > 0$ such that

(3.1)
$$M(V) = \int_{V} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} \le \epsilon$$

for any set $V \subset \Omega$ with $m(V) \leq \eta$.

The proof of this lemma is given in the appendix. For any $\epsilon \geq 0$, define

$$E_{\epsilon} = \{ \eta \in [0, m(\Omega)] \mid M(V) \le \epsilon \text{ if } V \subset \Omega \text{ with } m(V) \le \eta \},\$$

and Lemma 3.1 tells us that E_{ϵ} is a nonempty set, so that

(3.2)
$$\eta_{\epsilon} := \sup_{\eta \in E_{\epsilon}} \eta$$

is well defined and satisfies that $\eta_{\epsilon_1} \leq \eta_{\epsilon_2}$ if $\epsilon_1 \leq \epsilon_2$. It is also obvious that $\eta_{\epsilon} > 0$ if $\epsilon > 0$. For a given $\epsilon \geq 0$, η_{ϵ} is often difficult to compute, but here we assert only its existence.

Now we estimate the distance from the boundary to the centroid of a general convex set. The important feature of the estimate below is that the lower bound on the distance between the centroid and the boundary of a general convex set depends only on the set through the mass of the set.

For any $V \in \Omega$, let $R_V := \operatorname{diam}(V)$ denote the diameter of V (clearly $R_V \leq R_\Omega$), and define the V-dependent constant δ_V by

$$\delta_V := \frac{\eta_{M(V)/4}^2}{64R_V^{2d-1}},$$

where $\eta_{M(V)/4}$ is the constant defined in (3.2) with $\epsilon = \frac{M(V)}{4}$. Then the following result holds.

LEMMA 3.2. Let Assumption 3.1 be satisfied. Then for any convex set $V \subset \Omega$, it holds that

(3.3)
$$\operatorname{dist}(\mathbf{c}_V, \partial V) \ge \delta_V,$$

where dist($\mathbf{c}_V, \partial V$) denotes the shortest distance from the centroid of V denoted by \mathbf{c}_V to the boundary of V.

Proof. Without loss of generality, we assume that the origin is the nearest boundary point to the centroid \mathbf{c}_V , and we let \mathbf{c}_V have the coordinate $(\mathbf{0}, x)$ for some x > 0, where $\mathbf{0}$ denotes the zero vector in \mathbb{R}^{d-1} . Since V is convex and $\sup_{\mathbf{z},\mathbf{y}\in\Omega} |\mathbf{z}-\mathbf{y}| = R_V$, we must have $R_V \ge y_d \ge 0$ for any $\mathbf{y} = (y_1, \ldots, y_d) \in V$. Further, the *d*th coordinate of \mathbf{c}_V exactly equals dist $(\mathbf{c}_V, \partial V)$. Then, from the definition of the centroid, we have that

(3.4)
$$dist(\mathbf{c}_{V}, \partial V) = \frac{\int_{V} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y}}{\int_{V} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y}}$$
$$= \frac{\int_{0}^{R_{V}} \int_{A(y_{d})} \rho(\tilde{\mathbf{y}}, y_{d}) y_{d} \, \mathrm{d}\tilde{\mathbf{y}} \mathrm{d}y_{d}}{M(V)}$$
$$= \frac{\int_{0}^{R_{V}} y_{d}[\int_{A(y_{d})} \rho(\tilde{\mathbf{y}}, y_{d}) \, \mathrm{d}\tilde{\mathbf{y}}] \, \mathrm{d}y_{d}}{M(V)},$$

where $\tilde{\mathbf{y}} = (y_1, \dots, y_{d-1})$ and $A(y_d) = \{ \tilde{\mathbf{y}} \in \mathbb{R}^{d-1} \mid (\tilde{\mathbf{y}}, y_d) \in V \}$. Let us define

$$f_{\rho}(y_d) := \int_{A(y_d)} \rho(\tilde{\mathbf{y}}, y_d) \,\mathrm{d}\tilde{\mathbf{y}}.$$

The representation above has essentially reduced the problem to a similar onedimensional problem. To bound (3.4) from below, follow the following steps. In the first step, we find a set such that $\int_S f_{\rho}(y_d) dy_d$ is sufficiently large and f_{ρ} is bounded below away from 0. Second, a lower bound on this Lebesgue measure of this set is found. Finally, the third step is to combine these estimates to get a lower bound on (3.4) in terms of the mass of V.

Step 1. For any number $\mu > 0$, define $S_{\mu} = \{y_d \mid f_{\rho}(y_d) > \mu R_V^{d-1}\}$ and let $L(S_{\mu})$ denote the measure of S_{μ} . First, we have

$$\begin{split} M(V) &= \int_{V} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} \\ &= \int_{0}^{R_{V}} f_{\rho}(y_{d}) \, \mathrm{d}y_{d} \\ &= \int_{S_{\mu}} f_{\rho}(y_{d}) \, \mathrm{d}y_{d} + \int_{[0,R_{V}]-S_{\mu}} f_{\rho}(y_{d}) \, \mathrm{d}y_{d} \\ &\leq \int_{S_{\mu}} f_{\rho}(y_{d}) \, \mathrm{d}y_{d} + \mu R_{V}^{d}. \end{split}$$

Set $\mu_1 = \frac{M(V)}{2R_V^d}$; then the above inequality gives us

(3.5)
$$\int_{S_{\mu_1}} f_{\rho}(y_d) \, \mathrm{d}y_d \ge M(V) - \mu_1 R_V^d = M(V) - \frac{M(V)}{2} = \frac{M(V)}{2}.$$

Step 2. From the fact that

$$M(V) \ge \int_{S_{\mu}} f_{\rho}(y_d) \, \mathrm{d}y_d \ge L(S_{\mu})\mu R_V^{d-1},$$

it is easy to deduce that

$$L(S_{\mu}) \le \frac{M(V)}{\mu R_V^{d-1}},$$

and so that

$$m(V_{\mu}) \le L(S_{\mu})R_V^{d-1} \le \frac{M(V)}{\mu},$$

where $V_{\mu} = \{ (\tilde{\mathbf{y}}, y_d) \in V \mid y_d \in S_{\mu}, \ \tilde{\mathbf{y}} \in A(y_d) \}$. Setting $\mu_2 = \frac{M(V)}{\eta_{M(V)/4}}$, we have

$$m(V_{\mu_2}) \le \frac{M(V)}{\mu_2} = \eta_{M(V)/4}.$$

Then, by Lemma 3.1, we obtain that

(3.6)
$$\int_{S_{\mu_2}} f_{\rho}(y_d) \, \mathrm{d}y_d = \int_{S_{\mu_2}} \int_{A(y_d)} \rho(\tilde{\mathbf{y}}, y_d) \, \mathrm{d}\tilde{\mathbf{y}} \mathrm{d}y_d = \int_{V_{\mu_2}} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} \le \frac{M(V)}{4}$$

According to (3.5) and (3.6), we get

(3.7)
$$\int_{S_{\mu_1}-S_{\mu_2}} f_{\rho}(y_d) \, \mathrm{d}y_d \ge \frac{M(V)}{2} - \frac{M(V)}{4} \ge \frac{M(V)}{4}.$$

Notice that we also have

(3.8)
$$\int_{S_{\mu_1}-S_{\mu_2}} f_{\rho}(y_d) \, \mathrm{d}y_d \le \mu_2 R_V^{d-1} L(S_{\mu_1}).$$

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The combination of (3.7) and (3.8) gives us

(3.9)
$$L(S_{\mu_1}) \ge \frac{M(V)}{4\mu_2 R_V^{d-1}}.$$

Step 3. Using (3.4) and (3.9), we get

$$dist(\mathbf{c}_{V}, \partial V) = \frac{\int_{0}^{R_{V}} y_{d} f_{\rho}(y_{d}) \, \mathrm{d}y_{d}}{M(V)}$$

$$\geq \frac{\int_{S_{\mu_{1}}} y_{d} \mu_{1} R_{V}^{d-1} \, \mathrm{d}y_{d}}{M(V)}$$

$$\geq \frac{\mu_{1} R_{V}^{d-1} \int_{0}^{L(S_{\mu_{1}})} y_{d} \, \mathrm{d}y_{d}}{M(V)}$$

$$\geq \frac{\mu_{1} R_{V}^{d-1} [L(S_{\mu_{1}})]^{2}}{2M(V)}$$

$$\geq \frac{\mu_{1} M(V)}{32\mu_{2}^{2} R_{V}^{d-1}}$$

$$\geq \frac{\eta_{M(V)/4}}{64R_{V}^{2d-1}},$$

which completes the proof. $\hfill \Box$

Remark 3.1. If $\rho \in L^q(\Omega)$ for some q > 1, then δ_V can be selected such that it does not depend on the constant $\eta_{M(V)/4}$ such as

$$\delta_V := \frac{M(V)^{2q/(q-1)}}{4^{(5q-3)/(q-1)} \|\rho\|_{L^q(V)}^{2q/(q-1)} R_V^{2d-1}}.$$

This follows from that fact that applying the Hölder inequality to $\int_{S_{\mu}} f_{\rho}(y_d) \, dy_d$ gives us that

$$\int_{S_{\mu}} f_{\rho}(y_d) \, \mathrm{d}y_d \le \frac{\|\rho\|_{L^q(V)} [M(V)]^{(q-1)/q}}{\mu^{(q-1)/q}},$$

and so that

$$\int_{S_{\mu_2}} f_{\rho}(y_d) \, \mathrm{d}y_d \le \frac{M(V)}{4},$$

where $\mu_2 = \frac{(4\|\rho\|_{L^q(V)})^{q/(q-1)}}{[M(V)]^{1/(q-1)}}.$

The next lemma is a local estimate on the difference in the quantization energy over a particular set with respect to different generators. For our purposes, it is important to compare the energy with respect to the centroid to the energy generated by a point near the boundary. To prove this, the previous lemma is essential; it ensures that the centroid is sufficiently far away from the boundary. This again leads to an estimate that depends on the particular convex set V through M(V).

For any convex set $V \subset \Omega$, let us define the V-dependent constant γ_V by

$$\gamma_V := \frac{M(V)\delta_V^2}{4}.$$

LEMMA 3.3. Let Assumption 3.1 be satisfied and $V \subset \Omega$ be a convex set. Then for any $\mathbf{z} \in V$ with $\operatorname{dist}(\mathbf{z}, \partial V) \leq \frac{\delta_V}{2}$, it holds that

(3.10)
$$\int_{V} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{c}_{V}|^{2} \, \mathrm{d}\mathbf{y} \leq \int_{V} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{z}|^{2} \, \mathrm{d}\mathbf{y} - \gamma_{V},$$

where \mathbf{c}_V is the centroid of V.

•

Proof. First, from Lemma 3.2, it is obvious that

$$|\mathbf{z} - \mathbf{c}_V| \ge \operatorname{dist}(\mathbf{c}_V, \partial V) - \operatorname{dist}(\mathbf{z}, \partial V) \ge \frac{\delta_V}{2}.$$

We have

$$\begin{split} \int_{V} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{z}|^{2} \, \mathrm{d}\mathbf{y} &- \int_{V} \rho(\mathbf{y}) |\mathbf{y} - c_{V}|^{2} \, \mathrm{d}\mathbf{y} \\ &= \int_{V} \rho(\mathbf{y}) \langle \mathbf{y} - \mathbf{z}, \mathbf{y} - \mathbf{z} \rangle \mathrm{d}\mathbf{y} - \int_{V} \rho(\mathbf{y}) \langle \mathbf{y} - \mathbf{c}_{V}, \mathbf{y} - \mathbf{c}_{V} \rangle \mathrm{d}\mathbf{y} \\ &= \int_{V} \rho(\mathbf{y}) (|\mathbf{y}|^{2} + |\mathbf{z}|^{2} - 2 \langle \mathbf{y}, \mathbf{z} \rangle) \mathrm{d}\mathbf{y} \\ &- \int_{V} \rho(\mathbf{y}) (|\mathbf{y}|^{2} + |\mathbf{c}_{V}|^{2} - 2 \langle \mathbf{y}, \mathbf{c}_{V} \rangle \rangle \mathrm{d}\mathbf{y} \\ &= \int_{V} \rho(\mathbf{y}) (|\mathbf{z}|^{2} - |\mathbf{c}_{V}|^{2}) \, \mathrm{d}\mathbf{y} + 2 \int_{V} \rho(\mathbf{y}) \langle \mathbf{y}, \mathbf{c}_{V} - \mathbf{z} \rangle \mathrm{d}\mathbf{y}. \end{split}$$

Notice that

(3.11)
$$\int_{V} \rho(\mathbf{y}) (|\mathbf{z}|^{2} - |\mathbf{c}_{V}|^{2}) \, \mathrm{d}\mathbf{y} = M(V) (|\mathbf{z}|^{2} - |\mathbf{c}_{V}|^{2})$$

and

(3.12)

$$\int_{V} \rho(\mathbf{y}) \langle \mathbf{y}, \mathbf{c}_{V} - \mathbf{z} \rangle \mathrm{d}\mathbf{y} = \left\langle \int_{V} \mathbf{y} \rho(\mathbf{y}) \mathrm{d}\mathbf{y}, \mathbf{c}_{V} - \mathbf{z} \right\rangle$$
$$= \langle M(V) \mathbf{c}_{V}, \mathbf{c}_{V} - \mathbf{z} \rangle$$
$$= M(V) (|\mathbf{c}_{V}|^{2} - \langle \mathbf{c}_{V}, \mathbf{z} \rangle).$$

Using (3.11) and (3.12), we go further and get

$$\int_{V} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{z}|^{2} \, \mathrm{d}\mathbf{y} - \int_{V} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{c}_{V}|^{2} \, \mathrm{d}\mathbf{y}$$
$$= M(V)(|\mathbf{z}|^{2} + |\mathbf{c}_{V}|^{2} - 2\langle \mathbf{c}_{V}, \mathbf{z} \rangle)$$
$$= M(V)|\mathbf{z} - \mathbf{c}_{V}|^{2}$$
$$\geq \frac{M(V)\delta_{V}^{2}}{4},$$

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which deduces (3.10) and completes the proof.

Before presenting the next lemma, we need to define some notation. For any point set $\mathbf{Z} \in \Omega^n$, denote by \mathbf{Z}_i the *i*th point of \mathbf{Z} .

DEFINITION 3.1. Let $k, n \in \mathbb{N}$. For any points $\mathbf{Z} \in \hat{\Omega}^k$ and $\mathbf{Y} \in \hat{\Omega}^n$, define the d_* -distance from \mathbf{Z} to \mathbf{Y} by

$$d_*(\mathbf{Y}, \mathbf{Z}) = \max_{1 \le j \le k} \min_{1 \le i \le n} |\mathbf{Z}_j - \mathbf{Y}_i|.$$

Then the natural definition of the distance between \mathbf{Z} and \mathbf{Y} is given by

$$d(\mathbf{Y}, \mathbf{Z}) = \max(d_*(\mathbf{Y}, \mathbf{Z}), d_*(\mathbf{Z}, \mathbf{Y})).$$

This is a generalization of the regular Euclidean difference between two elements of Ω^n to the case of vectors of a different length. This definition will become useful for us when we consider a degenerate set of generators in the proof of our main theorem. Provided with this definition, let us consider the set theoretic difference of the two Voronoi tessellations generated by the vectors \mathbf{Z} and \mathbf{Y} as

$$\sum_{i=1}^{k} M\left(V_i(\mathbf{Z}) - \bigcup_{j \in S_i} V_j(\mathbf{Y})\right).$$

This definition is illustrated in Figure 3.1 below.



FIG. 3.1. An illustration of the definition of $d(\mathbf{Y}, \mathbf{Z})$ and the difference of Voronoi tessellations generated by \mathbf{Z} and \mathbf{Y} , where $\mathbf{Z} \in \hat{\Omega}_3$ and $\mathbf{Y} \in \hat{\Omega}_4$ (shaded).

The next lemma for quantifies the fact that if two possibly degenerate points in Ω^n are close together in the sense of Definition 3.1, then the corresponding Voronoi regions of these points are also close together in some sense. The appropriate measurement of how far apart these Voronoi regions are is the total mass of the difference of corresponding Voronoi regions.

LEMMA 3.4. Let Assumption 3.1 be satisfied. Fix $\mathbf{Z} \in \hat{\Omega}^k$ for $k \leq n$. For any $\epsilon > 0$, there exists a $\delta_{\epsilon} > 0$ such that if $\mathbf{Y} \in \hat{\Omega}^n$ and $d(\mathbf{Z}, \mathbf{Y}) \leq \delta_{\epsilon}$, then

$$\sum_{i=1}^{k} M\left(V_i(\mathbf{Z}) - \bigcup_{j \in S_i} V_j(\mathbf{Y})\right) \le \epsilon,$$

where $\{S_i\}_{i=1}^k$ is any partition of $\{1, \ldots, n\}$ satisfying $S_i \in \{j \mid |\mathbf{Z}_i - \mathbf{Y}_j| \leq |\mathbf{Z}_m - \mathbf{Y}_j|$ for all $m \in \{1, \ldots, k\}\}$.

Proof. Let us suppose that $\mathbf{Y} \in \hat{\Omega}^n$ and $d(\mathbf{Z}, \mathbf{Y}) < \delta$ for some $\delta > 0$. Since $\mathbf{Z} \in \hat{\Omega}^k$ is fixed, let us define the constant $d_{\mathbf{Z}}$ by

$$d_{\mathbf{Z}} = \min_{i \neq j} \frac{|\mathbf{Z}_i - \mathbf{Z}_j|}{2}.$$

We let δ be small enough such that $\delta \leq \frac{d\mathbf{z}}{2}$; then it is not difficult to verify that S_i is a nonempty set for each $1 \leq i \leq k$. From Definition 3.1, it is also obvious that for any $1 \leq i \leq k$,

$$|\mathbf{y} - \mathbf{Z}_i| - \delta \le |\mathbf{y} - \mathbf{Y}_i| \le |\mathbf{y} - \mathbf{Z}_i| + \delta \quad \forall \mathbf{y} \in \Omega, \ j \in S_i.$$

For any \mathbf{y} , if $\mathbf{y} \in V_i(\mathbf{Z}) - \bigcup_{j \in S_i} V_j(\mathbf{Y})$ (i.e., $\mathbf{y} \in V_i(\mathbf{Z})$ and $\mathbf{y} \notin \bigcup_{j \in S_i} V_j(\mathbf{Y})$), then we will argue by contradiction that $\operatorname{dist}(\mathbf{y}, \partial V_i(\mathbf{Z}))$, the shortest distance from \mathbf{y} to the boundary of $V_i(\mathbf{Z})$, satisfies

(3.13)
$$\operatorname{dist}(\mathbf{y}, \partial V_i(\mathbf{Z})) < \sigma = \frac{3\delta}{\alpha_{\mathbf{Z}}},$$

where $\alpha_{\mathbf{Z}} = \frac{2}{\sqrt{(1+R_{\Omega}/d_{\mathbf{Z}})^2 + (R_{\Omega}/d_{\mathbf{Z}})^2}}$ is a constant depending only on \mathbf{Z} and R_{Ω} . Let us first assume that $\operatorname{dist}(\mathbf{y}, \partial V_i(\mathbf{Z})) \geq \sigma$. We will show that $\mathbf{y} \in \bigcup_{j \in S_i} V_j(\mathbf{Y})$. Take any $1 \leq j \leq k$ with $j \neq i$; without loss of generality, assume that $V_i(\mathbf{Z})$ and $V_j(\mathbf{Z})$ are neighbor Voronoi cells. Notice that $V_i(\mathbf{Z})$ and $V_j(\mathbf{Z})$ are both convex polygons/polyhedra except for the part belonging to the boundary of Ω ; see Figure 3.2 for an illustration of two-dimensional cases.



FIG. 3.2. Two neighboring Voronoi regions.

One can easily verify that

$$\begin{aligned} |\mathbf{y} - \mathbf{Z}_{j}| - |\mathbf{y} - \mathbf{Z}_{i}| &= \sqrt{(d_{ij} + w_{\mathbf{y}})^{2} + h_{\mathbf{y}}^{2}} - \sqrt{(d_{ij} - w_{\mathbf{y}})^{2} + h_{\mathbf{y}}^{2}} \\ &\geq \frac{2d_{ij}w_{\mathbf{y}}}{\sqrt{(d_{ij} + w_{\mathbf{y}})^{2} + h_{\mathbf{y}}^{2}}} \\ &= \frac{2w_{\mathbf{y}}}{\sqrt{(1 + w_{\mathbf{y}}/d_{ij})^{2} + (h_{\mathbf{y}}/d_{ij})^{2}}}. \end{aligned}$$

Since $d_{ij} \ge d_{\mathbf{Z}}, R_{\Omega} \ge w_{\mathbf{y}} > \sigma$, and $h_{\mathbf{y}} \le R_{\Omega}$, we obtain

$$|\mathbf{y} - \mathbf{Z}_j| - |\mathbf{y} - \mathbf{Z}_i| \ge \frac{2\sigma}{\sqrt{(1 + R_\Omega/d_\mathbf{Z})^2 + (R_\Omega/d_\mathbf{Z})^2}}$$

= 3\delta.

Then, for any $j \in S_i$ and any $m \in S_l$ with $l \neq i$, it holds that

$$\begin{aligned} |\mathbf{y} - \mathbf{Y}_j| &\leq |\mathbf{y} - \mathbf{Z}_i| + \delta \\ &\leq |\mathbf{y} - \mathbf{Z}_l| - 3\delta + \delta \\ &\leq |\mathbf{y} - \mathbf{Y}_m| + \delta - 2\delta \\ &\leq |\mathbf{y} - \mathbf{Y}_m| - \delta \\ &< |\mathbf{y} - \mathbf{Y}_m|. \end{aligned}$$

This means that $\mathbf{y} \in \bigcup_{j \in S_i} V_j(\mathbf{Y})$ and gives us a contradiction with $\mathbf{y} \notin \bigcup_{j \in S_i} V_j(\mathbf{Y})$, so that the inequality (3.13) holds. Since $V_i(\mathbf{Z}) \subset \Omega$ is convex and diam $(\Omega) = R_{\Omega} < +\infty$, it holds that

(3.14)
$$m(\partial V_i(\mathbf{Z})) \le S_d \left(\frac{R_\Omega}{2}\right)^{d-1},$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ denotes the hypersurface area of an *d*-sphere of unit radius; see [3] for details. Then the inequality (3.13) with (3.14) implies that

$$m(V_i(\mathbf{Z}) - \bigcup_{j \in S_i} V_j(\mathbf{Y})) \le \frac{\sigma S_d R_{\Omega}^{d-1}}{2^{d-1}}.$$

Then selecting

$$\delta = \delta_{\epsilon} := \min\left(\frac{d\mathbf{z}}{2}, \ \frac{2^{d-1}\eta_{\epsilon/k}\alpha_{\mathbf{Z}}}{3S_d R_{\Omega}^{d-1}}\right)$$

ensures that $m(V_i(\mathbf{Z}) - \bigcup_{j \in S_i} V_j(\mathbf{Y})) \leq \eta_{\epsilon/k}$. Due to Lemma 3.1, this means that

$$\sum_{i=1}^{k} M\left(V_{i}(\mathbf{Z}) - \bigcup_{j \in S_{i}} V_{j}(\mathbf{Y})\right) = \sum_{i=1}^{k} \int_{V_{i}(\mathbf{Z}) - \bigcup_{j \in S_{i}} V_{j}(\mathbf{Y})} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y}$$
$$\leq \sum_{i=1}^{k} \frac{\epsilon}{k}$$
$$\leq \epsilon,$$

and thus the result holds. $\hfill \Box$

Remark 3.2. In the case that $\rho \in L^q(\Omega)$ for some q > 1, the Hölder inequality again can be used to find a δ_{ϵ} in the above lemma which does not depend on the constant $\eta_{\epsilon/k}$ such as

$$\delta = \delta_{\epsilon} := \min\left(\frac{d\mathbf{z}}{2}, \frac{2^{d-1}\alpha_{\mathbf{Z}}}{3S_d R_{\Omega}^{d-1}} \left(\frac{\epsilon}{k\|\rho\|_{L^q(\Omega)}}\right)^{q/(q-1)}\right).$$

3.2. Nondegeneracy of limit points. Now, let us present our first important result.

THEOREM 3.5. Let Assumption 3.1 be satisfied. Given $n \in \mathbb{N}$ and any initial point $\mathbf{Z}^0 \in \hat{\Omega}^n$, let $\{\mathbf{Z}^i\}_{i=0}^{\infty}$ be the iterates of the Lloyd algorithm starting with \mathbf{Z}^0 . Then any limit point \mathbf{Z} of these iterates is nondegenerate (i.e., $\mathbf{Z} \in \hat{\Omega}^n$).

Proof. Let $\mathbf{Z} \in \overline{\Omega}^n$ be any limit point of these iterates $\{\mathbf{Z}^i\}_{i=0}^{\infty}$. We will argue by contradiction that $\mathbf{Z} \in \hat{\Omega}^n$. Let us first assume that $\mathbf{Z} = (\mathbf{Z}_1, \dots, \mathbf{Z}_n)$ is degenerate, i.e., in the sense that several of its components are equal. Without loss of generality, assume that $\mathbf{Z}_1 = \cdots = \mathbf{Z}_r$ for some $1 < r \leq n$.

Since \mathbf{Z}^{i} 's are Lloyd iterates, by the continuity of \mathcal{G} on $\overline{\Omega}^{n}$ and the monotone decrease of \mathcal{G} along \mathbf{Z}^{i} 's due to (2.2), we know that

(3.15)
$$\mathcal{G}(\mathbf{Z}^i) \searrow_{i \to +\infty} \mathcal{G}(\mathbf{Z});$$

i.e., $\mathcal{G}(\mathbf{Z}^i)$ monotonically decreases and converges to $\mathcal{G}(\mathbf{Z})$. Let $\tilde{\mathbf{Z}} \in \hat{\Omega}^k$ (k = n - r + 1 < n) be the set of points in \mathbf{Z} without duplication; i.e.,

$$\tilde{\mathbf{Z}} = (\tilde{\mathbf{Z}}_1, \dots, \tilde{\mathbf{Z}}_k) = (\mathbf{Z}_1, \mathbf{Z}_{r+1}, \dots, \mathbf{Z}_n)$$

It is obvious that $\mathcal{G}(\tilde{\mathbf{Z}}) = \mathcal{G}(\mathbf{Z})$. From $\{\mathbf{Z}^i\}_{i=0}^{\infty}$, select a subsequence $\{\mathbf{Z}^{i_j}\}_{j=0}^{\infty}$ such that $\mathbf{Z}^{i_j} \to \mathbf{Z}$ as $j \to +\infty$.

Now define

$$\alpha = \frac{M(V_1(\tilde{\mathbf{Z}}))}{2r}, \quad \beta = \frac{\eta_{\alpha/4}^2}{64R_{\Omega}^{2d-1}}, \quad \text{and} \quad \gamma = \frac{\alpha\beta^2}{4}.$$

Obviously we have $\alpha, \beta, \gamma > 0$ since $(V_1(\tilde{\mathbf{Z}}))$ is a convex set with positive measure.

Now let us select a j^* large enough such that the following inequalities hold:

(3.16)
$$M(\cup_{m=1}^{r} V_m(\mathbf{Z}^{i_{j^*}})) > \frac{M(V_1(\mathbf{Z}))}{2},$$

(3.17) $\mathcal{G}(\mathbf{Z}^{i_{j^*}}) - \mathcal{G}(\tilde{\mathbf{Z}}) < \frac{\gamma}{2},$

(3.18)
$$|\mathbf{Z}_m^{i_{j^*}} - \mathbf{Z}_m| < \frac{\beta}{4}, \quad m = 1, 2, \dots, n.$$

Inequality (3.16) holds for large j^* by Lemma 3.4 using $\epsilon = \frac{M(V_1(\tilde{\mathbf{Z}}))}{2}$. Further, that also means that for some $m \in \{1, \ldots, r\}$, $M(V_m(\mathbf{Z}^{i_{j^*}})) > \alpha$. Without loss of generality, assume this m is 1, i.e., $M(V_1(\mathbf{Z}^{i_{j^*}})) > \alpha$. That gives us

(3.19)
$$\delta_{V_1(\mathbf{Z}^{i_j*})} \ge \beta, \qquad \gamma_{V_1(\mathbf{Z}^{i_j*})} \ge \gamma.$$

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The inequalities (3.18) and (3.19) imply

$$\operatorname{dist}(\mathbf{Z}_{1}^{i_{j^{*}}}, \partial V_{1}(\mathbf{Z}^{i_{j^{*}}})) \leq |\mathbf{Z}_{1}^{i_{j^{*}}} - \mathbf{Z}_{1}| + |\mathbf{Z}_{2}^{i_{j^{*}}} - \mathbf{Z}_{1}| < \frac{\beta}{2} \leq \frac{\delta_{V_{1}(\mathbf{Z}^{i_{j^{*}}})}}{2}.$$

Then by Lemma 3.3 and (3.19), we immediately obtain

(3.20)

$$\begin{aligned}
\int_{V_{1}(\mathbf{Z}^{i_{j^{*}}})} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{c}_{V_{1}(\mathbf{Z}^{i_{j^{*}}})}|^{2} \, \mathrm{d}\mathbf{y} \\
&\leq \int_{V_{1}(\mathbf{Z}^{i_{j^{*}}})} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{Z}_{1}^{i_{j^{*}}}|^{2} \, \mathrm{d}\mathbf{y} - \gamma_{V_{1}(\mathbf{Z}^{i_{j^{*}}})} \\
&\leq \int_{V_{1}(\mathbf{Z}^{i_{j^{*}}})} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{Z}_{1}^{i_{j^{*}}}|^{2} \, \mathrm{d}\mathbf{y} - \gamma.
\end{aligned}$$

Now, let us estimate the quantization energy $\mathcal{G}(\mathbf{T}(\mathbf{Z}^{i_{j^*}}))$. Using the dual minimization property, we get

$$egin{aligned} \mathcal{G}(\mathbf{T}(\mathbf{Z}^{i_{j^*}})) &= \mathcal{H}(\mathbf{T}(\mathbf{Z}^{i_{j^*}}),\mathbf{T}(\mathbf{Z}^{i_{j^*}})) \ &\leq \mathcal{H}(\mathbf{Z}^{i_{j^*}},\mathbf{T}(\mathbf{Z}^{i_{j^*}})) \ &= \int_{V_1(\mathbf{Z}^{i_{j^*}})}
ho(\mathbf{y})|\mathbf{y}-\mathbf{T}_1(\mathbf{Z}^{i_{j^*}})|^2 \ \mathrm{d}\mathbf{y} \ &+ \sum_{k=2}^n \int_{V_k(\mathbf{Z}^{i_{j^*}})}
ho(\mathbf{y})|\mathbf{y}-\mathbf{T}_k(\mathbf{Z}^{i_{j^*}})|^2 \ \mathrm{d}\mathbf{y}. \end{aligned}$$

It is easy to see that

(3.21)
$$\sum_{k=2}^{n} \int_{V_{k}(\mathbf{Z}^{i_{j^{*}}})} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{T}_{k}(\mathbf{Z}^{i_{j^{*}}})|^{2} \, \mathrm{d}\mathbf{y} \leq \sum_{k=2}^{n} \int_{V_{k}(\mathbf{Z}^{i_{j^{*}}})} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{Z}_{k}^{i_{j^{*}}}|^{2} \, \mathrm{d}\mathbf{y}$$

since $\mathbf{T}_k(\mathbf{Z}^{i_{j^*}})$ is the centroid of $V_k(\mathbf{Z}^{i_{j^*}})$. With (3.17), (3.20), and (3.21), we have

$$\begin{split} \mathcal{G}(\mathbf{T}(\mathbf{Z}^{i_{j^*}})) &\leq \int_{V_1(\mathbf{Z}^{i_{j^*}})} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{Z}_1^{i_{j^*}}|^2 \, \mathrm{d}\mathbf{y} - \gamma \\ &+ \sum_{k=2}^n \int_{V_k(\mathbf{Z}^{i_{j^*}})} \rho(\mathbf{y}) |\mathbf{y} - \mathbf{Z}_k^{i_{j^*}}|^2 \, \mathrm{d}\mathbf{y} \\ &= \mathcal{G}(\mathbf{Z}^{i_{j^*}}) - \gamma \\ &< \mathcal{G}(\tilde{\mathbf{Z}}) + \frac{\gamma}{2} - \gamma \\ &= \mathcal{G}(\tilde{\mathbf{Z}}) - \frac{\gamma}{2}, \end{split}$$

which contradicts the fact (3.15). We conclude that $\mathbf{Z} \in \hat{\Omega}^n$; i.e., any limit point of the Lloyd iteration cannot be a degenerate point. \Box

THEOREM 3.6. Let Assumption 3.1 be satisfied. Given $n \in \mathbb{N}$ and any initial point $\mathbf{Z}^0 \in \hat{\Omega}^n$, let $\{\mathbf{Z}^i\}_{i=0}^{\infty}$ be the iterates of the Lloyd algorithm starting with \mathbf{Z}^0 . Then there exists a constant $D_{\mathbf{Z}^0} > 0$ depending only on n, Ω , ρ , and \mathbf{Z}^0 such that for any $i \geq 0$,

$$|\mathbf{Z}_{i}^{i}-\mathbf{Z}_{k}^{i}|>D_{\mathbf{Z}^{0}}$$

for all $j, k \in \{1, \ldots, n\}$ and $j \neq k$.

Proof. We will use the contradiction argument again. Let us assume that the theorem is not true, then for all $m \in \mathbb{N}$, there always exist an index i_m , some $j_m, k_m \in \{1, \ldots, n\}$, and $j_m \neq k_m$ such that

$$|\mathbf{Z}_{j_m}^{i_m} - \mathbf{Z}_{k_m}^{i_m}| < \frac{1}{m}$$

Let us consider the subsequence $\{\mathbf{Z}^{i_m}\}_{m=1}^{\infty}$. Without loss of generality, assume the index set $I = \{i_m\}_{m=1}^{\infty}$ is a monotonically increasing sequence (otherwise we can do that by adding some restriction such as $i_{k+1} > i_k$ during choosing $\{\mathbf{Z}^{i_m}\}_{m=1}^{\infty}$). Notice that the set $JK = \{(j_m, k_m)\}_{m=1}^{\infty}$ has at most n(n-1) different elements, so there exists a pair $\{(j^*, k^*)\} \in JK$ that appears infinitely many times along the increasing of m, and let us record their appearances by the increasing sequence $\{m_1, m_2, \ldots\} \subset \mathbb{N}$. This means that

(3.22)
$$|\mathbf{Z}_{j^*}^{i_{m_p}} - \mathbf{Z}_{k^*}^{i_{m_p}}| < \frac{1}{m_p} \quad \forall \ p \in \mathbb{N}.$$

Since $\{\mathbf{Z}^{i_{m_p}}\}\$ are bounded, there always exists a convergent subsequence of $\{\mathbf{Z}^{i_{m_p}}\}\$, and assume \mathbf{Z} is the corresponding limit point. Due to (3.22), we must have $\mathbf{Z}_{j^*} = \mathbf{Z}_{k^*}$, which means that \mathbf{Z} is a degenerate point. However, \mathbf{Z} is also a limit point of the original Lloyd iterates $\{\mathbf{Z}^i\}_{i=0}^{\infty}$, and according to Theorem 3.5, \mathbf{Z} could not be a degenerate point. We then get a contradiction, and so the theorem holds. \Box

Remark 3.3. In Theorem 3.6, we are only able to show existence of the lower bound of the distances between any pair of generators of all Lloyd iterates starting with \mathbf{Z}^{0} . For the one-dimensional case, an explicit estimate was derived in [5] when the density function is strictly positive and smooth.

A direct consequence of Theorem 3.6 is the following corollary.

COROLLARY 3.7. Let Assumption 3.1 be satisfied. Given $n \in \mathbb{N}$ and any initial point $\mathbf{Z}^0 \in \hat{\Omega}^n$, let $\{\mathbf{Z}^i\}_{i=0}^{\infty}$ be the iterates of the Lloyd algorithm starting with \mathbf{Z}^0 . Then the Lloyd map is continuous at any of the iterates $\{\mathbf{Z}^i\}_{i=0}^{\infty}$ and $m(V_j(\mathbf{Z}^i)) \geq (\frac{D_{\mathbf{Z}^0}}{2})^d$ for any $1 \leq j \leq n$ and $i \geq 0$.

3.3. Main result. Finally, let us present our main result in the following theorem.

THEOREM 3.8 (weak global convergence). Let Assumption 3.1 be satisfied. Given $n \in \mathbb{N}$ and any initial point $\mathbf{Z}^0 \in \hat{\Omega}^n$, let $\{\mathbf{Z}^i\}_{i=0}^{\infty}$ be the iterates of the Lloyd algorithm starting with \mathbf{Z}^0 . Then the following hold:

- (1) $\{\mathbf{Z}^i\}_{i=0}^{\infty}$ is weakly convergent (i.e., $\lim_{i\to+\infty} \nabla \mathcal{G}(\mathbf{Z}^i) = 0$) and any limit point of $\{\mathbf{Z}^i\}_{i=0}^{\infty}$ is also a nondegenerate critical point of the quantization energy \mathcal{G} (and thus a CVT).
- (2) Moreover, it also holds that $\lim_{i\to+\infty} \|\mathbf{Z}^{i+1} \mathbf{Z}^i\| = 0$.

Proof. Using the result of Corollary 3.7, we see that we can define a compact set (away from the degenerate points) such that for any initial point $\mathbf{Z}^0 \in \hat{\Omega}^n$, the Lloyd

iterates (the images of the Lloyd maps) will stay in such a compact set. Thus, we can apply Theorem 2.1 and Corollary 3.7 to deduce the weak global convergence of the Lloyd iterations. The first part of the theorem is then complete with the help of Theorem 3.5.

From Corollary 3.7, we also know that for any $1 \le k \le n$ and $i \ge 0$,

$$m(V_k(\mathbf{Z}^i)) \ge \left(\frac{D_{\mathbf{Z}^0}}{2}\right)^d > 0.$$

Together with Lemma 3.1 and the assumption that ρ is positive almost everywhere, we know that there exists some constant $M^* > 0$ such that

$$(3.23) M(V_k(\mathbf{Z}^i)) \ge M^* > 0.$$

Using (2.1), we then get

$$\begin{aligned} \|\nabla \mathcal{G}(\mathbf{Z}^{i})\| &= \left(\sum_{k=1}^{n} \left|\frac{\partial \mathcal{G}}{\partial \mathbf{z}_{k}}(\mathbf{Z}^{i})\right|^{2}\right)^{1/2} \\ &= \left(\sum_{k=1}^{n} |2M(V_{k}(\mathbf{Z}^{i}))(\mathbf{Z}_{k}^{i} - \mathbf{T}_{k}(\mathbf{Z}^{i}))|^{2}\right)^{1/2} \\ &\geq 2M^{*} \left(\sum_{k=1}^{n} |\mathbf{Z}_{k}^{i} - \mathbf{T}_{k}(\mathbf{Z}^{i})|^{2}\right)^{1/2} \\ &= 2M^{*} \|\mathbf{Z}^{i} - \mathbf{Z}^{i+1}\|. \end{aligned}$$

Consequently, we get

(3.24)

$$0 \le \lim_{i \to +\infty} \|\mathbf{Z}^{i+1} - \mathbf{Z}^{i}\| \le \frac{1}{2M^*} \lim_{i \to +\infty} \|\nabla \mathcal{G}(\mathbf{Z}^{i})\| = 0,$$

which deduces the second part of this theorem. \Box

Remark 3.4. It is a consequence of part (1) of Theorem 3.8 that the generators of any limit point configuration of the Lloyd algorithm cannot lie on the boundary of the convex domain, since the limit point is a CVT and the density function ρ is positive almost everywhere.

Remark 3.5. Part (2) of Theorem 3.8 in fact tells us that the Lloyd method always terminates under general practical stopping criteria.

4. Conclusions and open questions. In this paper, we prove that any limit point of the Lloyd iteration is nondegenerate provided that Ω is a convex and bounded set and ρ belongs to $L^1(\Omega)$ and is positive almost everywhere in any dimensional space. It follows that, for every initial point set, the Lloyd iteration always approaches the set of nondegenerate critical points of the so-called quantization energy (which is exactly the set of CVTs). These results go beyond those presented in earlier papers; they enlarge the class of densities in the one-dimensional case and assign a stronger meaning to the statement of global convergence of the Lloyd iteration by eliminating the possibility of degeneracy. Although the proofs in their generality hold for all L^1 density functions, we have been able to provide a more constructive characterization

for the constants δ_V and δ_{ϵ} in the case of $L^q, q > 1$, which possess their own merit from a practical point of view.

There are still some open questions. Throughout our proof, the convexity of Ω is required in many places. The first question is whether this condition is really necessary or whether Ω can be a nonconvex set with some constraints. Another question is under which conditions weak convergence transforms into a strong single limit-point convergence. Also, it remains to explore whether the proof can be extended to replace the density function with a more general measure; ideally, it would be extended to a set of measures more general than those considered in [22]. Finally, it is also very meaningful and interesting to study the convergence properties of the Lloyd algorithm when the CVT is defined under some other error weighting functions, such as the very commonly used l^1 and l^{∞} that arise in many engineering applications. The geometric proof provided here clearly provides room for such generalizations; however, the detailed analysis would complicate current presentation and hence is left to a separate treatment. These questions and generalizations are the focus of current research and will be addressed in subsequent publications.

Appendix. Proof of Lemma 3.1.

Proof. Suppose that the claim is false. Then there exists an $\epsilon_0 > 0$ such that for all $k \in \mathbb{N}$, $\exists A_k \subset \Omega$ with $m(A_k) \leq (1/2)^{k+1}$, it holds that $\int_{A_k} \rho(\mathbf{y}) \, d\mathbf{y} > \epsilon_0$. Let $B_k = \bigcup_{i=k}^{\infty} A_i$; clearly $m(B_k) \leq \sum_{k=1}^{\infty} m(A_i) \leq (1/2)^k$ and $\int_{B_k} \rho(\mathbf{y}) \, d\mathbf{y} > \epsilon_0$ for all $k \in \mathbb{N}$. Since $\{\int_{B_k} \rho(\mathbf{y}) \, d\mathbf{y}\}_{k=1}^{\infty}$ is a positive nonincreasing sequence, it is easy to see that

$$\lim_{k \to +\infty} \int_{B_k} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} \ge \epsilon_0 > 0.$$

On the other hand, notice that $B_1 \supset B_2 \supset \cdots$. By the dominated convergence theorem in measure theory, we have

$$\lim_{k \to +\infty} \int_{B^*} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} = \lim_{k \to +\infty} \int_{\Omega} (\mathbf{1}_{B_k}(\mathbf{y})\rho(\mathbf{y})) \, \mathrm{d}\mathbf{y}$$
$$= \int_{\Omega} \lim_{k \to +\infty} (\mathbf{1}_{B_k}(\mathbf{y})\rho(\mathbf{y})) \, \mathrm{d}\mathbf{y}$$
$$= 0,$$

which gives us a contradiction, and so this lemma holds. $\hfill \Box$

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