MATH 259 – FIRST UNIT TEST

Tuesday, February 24, 2009.

NAME: SOLUTIONS

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A	C	D	G	H
B	F	E		

Instructions:

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	20	
2	12	
3	22	
4	26	
5	20	
Total	100	

SOLUTIONS

1. 20 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

You should not use your calculator on this problem for anything except evaluating functions or arithmetic. In particular, you should not use your calculator to evaluate integrals or find anti-derivatives.

You may use the following trigonometric identity without having to justify it:

$$\sqrt{2\cdot(1-\cos(\theta))} = 2\cdot\sin\left(\frac{\theta}{2}\right).$$

(a) **(8 points)** Find the volume of the parallelpiped with adjacent edges *PQ*, *PR* and *PS*, where *P*, *Q*, *R* and *S* are the points:

$P = (3, 0, 1)$ $\quad Q = (-1, 2, 5)$ $\quad R = (5, 1, -1)$ $\quad S = (0, 4, 2)$.

Clearly indicate your final answer.

First, form vectors to represent the edges of the parallelpiped:

$\vec{a} = \langle -4, 2, 4 \rangle \quad \vec{b} = \langle 2, 1, -2 \rangle \quad \vec{c} = \langle -3, 4, 1 \rangle$

The volume of the parallelpiped is given by:

$$V = |\vec{a} \bullet (\vec{b} \times \vec{c})|.$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = \langle 9, 4, 11 \rangle$$

so that:

$$V = |\vec{a} \bullet (\vec{b} \times \vec{c})| = |-36 + 8 + 44| = 16.$$

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You should not use your calculator on this problem for anything except evaluating functions or arithmetic. In particular, you should not use your calculator to evaluate integrals or find anti-derivatives.

You may use the following trigonometric identity without having to justify it:

$$\sqrt{2\cdot(1-\cos(\theta))} = 2\cdot\sin\left(\frac{\theta}{2}\right).$$

(b) **(12 points)** Find the **exact** length of one arch of the parametric curve:

$$x(t) = r\cdot(t-\sin(t)) \qquad \text{and} \qquad y(t) = r\cdot(1-\cos(t)).$$

Your final answer may include the positive constant $r > 0$.

The start and end of one arch are given by consecutive solutions of $y(t) = 0$, or $\cos(t) = 1$. An arch starts at $t = 0$ and ends at $t = 2\pi$. These are the limits of integration.

$$x'(t) = r\cdot(1-\cos(t)) \qquad y'(t) = r\cdot\sin(t)$$

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{r^2(1 - 2\cos(t) + \sin^2(t) + \cos^2(t))}$$

$$= 2r\cdot\sin(t/2)$$ using identity given above.

$$\text{Arc length} = \int_0^{2\pi} 2r\cdot\sin(t/2)\,dt$$

$$= \left[-4r\cdot\cos(t/2)\right]_0^{2\pi}$$

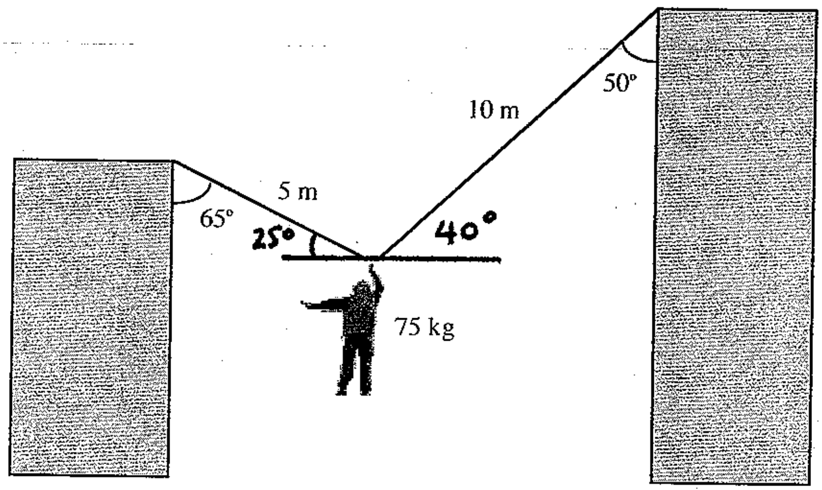
$$= 8r.$$

2. **12 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.**

The Great Brunetti is a tightrope walker. The Great Brunetti captivates his audiences by using a specially constructed balancing pole that makes it appear to audience members that he is off balance and might fall.

Unfortunately for the Great Brunetti, occasionally he does fall. The diagram below shows such an accident where the Great Brunetti was attempting to walk a tightrope between two skyscrapers. As he fell, the Great Brunetti managed to grab the rope.

One part of the rope is 5 meters in length; the other part of the rope is 10 meters in length. The Great Brunetti has a mass of 75 kg. Find the magnitude of the tension in each length of rope. Clearly indicate your final answers and give appropriate units.



Let T_5 = magnitude of tension in 5m rope (N).

T_{10} = magnitude of tension in 10 m rope (N).

Assuming a static equilibrium we have:

Horizontal: $-T_5 \cdot \cos(25°) + T_{10} \cdot \cos(40°) = 0$

Vertical: $T_5 \cdot \sin(25°) + T_{10} \cdot \sin(40°) = (75)(9.8)$

Substituting values for the trigonometric functions and solving the resulting 2×2 system of linear equations gives:

$T_{10} = 735$ N $\qquad T_5 \approx 621.2188$ N

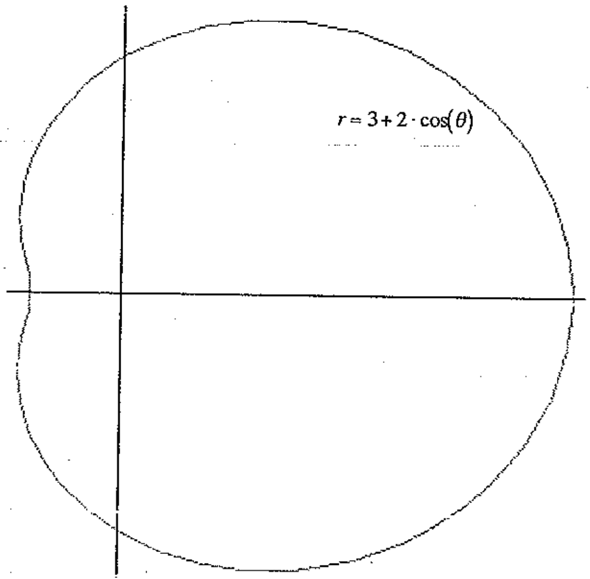
SOLUTIONS

3. **22 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.**

Consider the curve defined by the polar equation:

$$r = 3 + 2\cdot\cos(\theta).$$

(a) **(14 points)** Find the coordinates (x and y) of all points where the tangent line to the polar curve is **vertical**. Find the **exact** coordinates (x and y) of **all** points where the tangent line is vertical. Show your work and record your results in the table at the bottom of the page. No work = no credit.



$$x(\theta) = 3\cos(\theta) + 2\cos^2(\theta)$$

$$x'(\theta) = -3\sin(\theta)\cdot\left[1 + \frac{4}{3}\cdot\cos(\theta)\right].$$

Solutions of $x'(\theta) = 0$ for $0 \le \theta \le 2\pi$:

$\underline{\sin(\theta) = 0}$

$\theta = 0, \pi, 2\pi$

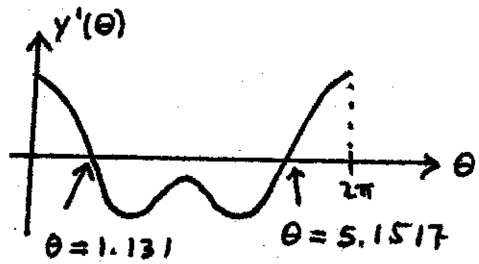
$\underline{1 + \frac{4}{3}\cos(\theta) = 0}$

$\theta \doteq 2.419$

$\theta \doteq 3.864$

$$y(\theta) = 3\sin(\theta) + 2\sin(\theta)\cdot\cos(\theta)$$

$$y'(\theta) = 3\cos(\theta) + 2\cos^2(\theta) - 2\sin^2(\theta)$$



There are no $\dfrac{y'(\theta)}{x'(\theta)} = \dfrac{0}{0}$ situations to resolve.

If you give your answers in decimal form, include **at least four (4) decimal places.**

x	y
5	0
−1	0
−1.125	0.9921567416
−1.125	−0.9921567416

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SOLUTIONS

Consider the curve defined by the parametric equations:

$$x(t) = e^{2t} - e^{-2t} \quad \text{and} \quad y(t) = 3e^{2t} + e^{-2t}.$$

(b) **(8 points)** Find the coordinates (x and y) of **all** points where the tangent line to the parametric curve is **horizontal**. Find the **exact** coordinates (x and y) of **all** points where the tangent line is horizontal. Show your work and record your results in the table below. No work = no credit.

Want points where $x'(t) \neq 0$ and $y'(t) = 0$.

$$y'(t) = 6e^{2t} - 2e^{-2t} = 0$$

So: $6e^{2t} = 2e^{-2t}$

$$e^{4t} = 1/3$$

$$t = 1/4 \cdot \ln(1/3).$$

$x'(t) = 2e^{2t} + 2e^{-2t} = 0$ has no solutions.

If you give your answers in decimal form, include **at least four (4) decimal places.**

x	y
-1.1547005 38	3.464101615

SOLUTIONS

4. 26 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

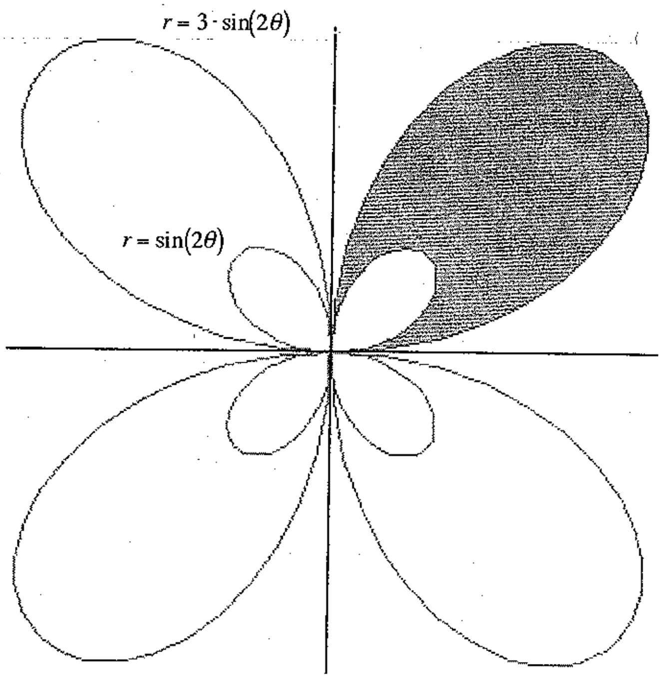
Find the **exact** area of each shaded region shown below. Show your work – no work = no credit.

You should not use your calculator on this problem for anything except evaluating functions or arithmetic. In particular, you should not use your calculator to evaluate integrals or find anti-derivatives.

You may use the following trigonometric identities without having to verify them:

$$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x)) \qquad \cos^2(x) = \tfrac{1}{2}(1+\cos(2x)).$$

(a) (12 points)



The limits of integration are $\theta = 0$ and $\theta = \pi/2$. (These are consecutive solutions of $3 \cdot \sin(2\theta) = 0$.)

Area =

$$\frac{1}{2}\int_0^{\pi/2} 9 \cdot \sin^2(2\theta)\,d\theta - \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 8\sin^2(2\theta)\,d\theta$$

$$= 4\int_0^{\pi/2} \sin^2(2\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} 1 - \cos(4\theta)\,d\theta$$

$$= 2\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\pi/2}$$

$$= \pi.$$

Continued on the next page.

SOLUTIONS

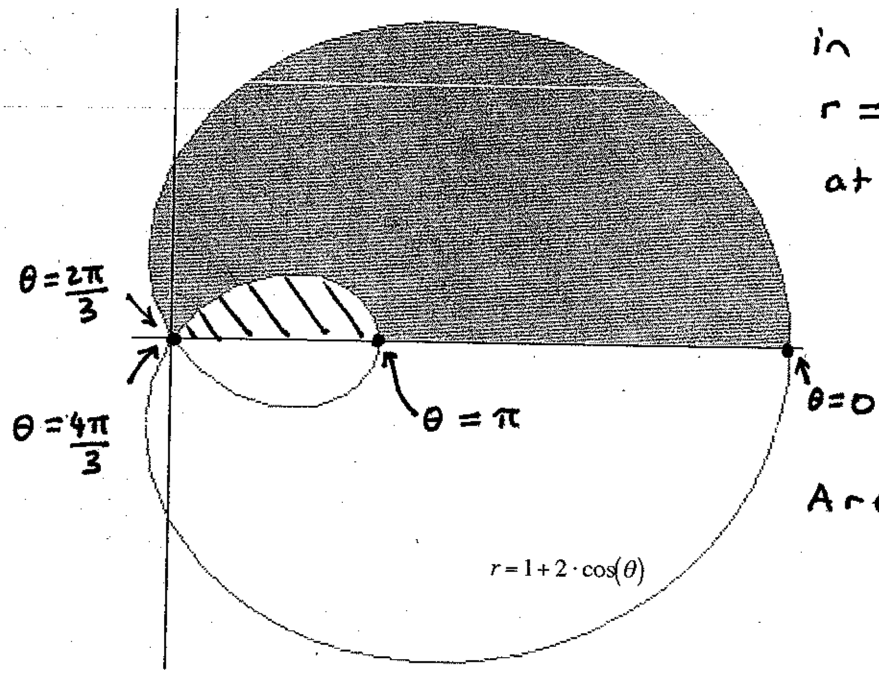
Find the area of each shaded region shown below. Show your work – no work = no credit.

You should not use your calculator on this problem for anything except evaluating functions or arithmetic. In particular, you should not use your calculator to evaluate integrals or find anti-derivatives.

You may use the following trigonometric identities without having to verify them:

$$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x)) \qquad \cos^2(x) = \tfrac{1}{2}(1+\cos(2x)).$$

(b) (14 points)



The points we are interested in are the x-intercepts of $r = 1 + 2\cdot\cos(\theta)$. These occur at: $\theta = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$.

The gray area is given by:

$$\text{Area} = \frac{1}{2}\int_0^{2\pi/3} (1+2\cdot\cos(\theta))^2 \cdot d\theta - \frac{1}{2}\int_{\pi}^{4\pi/3} (1+2\cdot\cos(\theta))^2 \cdot d\theta$$

Now, $\int (1+2\cos(\theta))^2\, d\theta = \int (1 + 4\cos(\theta) + 4\cos^2(\theta))\, d\theta$

$$= \int (1 + 4\cos(\theta) + 2\cdot(1+\cos(2\theta)))\, d\theta$$

$$= 3\theta + 4\cdot\sin(\theta) + \sin(2\theta) + C.$$

So:

$$\text{Area} = \frac{1}{2}\Big[3\theta + 4\cdot\sin(\theta) + \sin(2\theta)\Big]_0^{2\pi/3} - \frac{1}{2}\Big[3\theta + 4\cdot\sin(\theta) + \sin(2\theta)\Big]_{\pi}^{4\pi/3}$$

$$= \frac{\pi}{2} + \frac{3\sqrt{3}}{2} \approx 4.16887258$$

SOLUTIONS

5. **20 Points. NO PARTIAL CREDIT WITHOUT WORK.**

(a) **(8 points)** Find symmetric equations for the line of intersection formed by the planes:

$$x + y + z = 2 \qquad \text{and} \qquad x - 2y + 3z = 6.$$

(Many solutions possible.)

A point on the line of intersection is: $(0, 0, 2)$.

The direction vector of the line is:

$$\langle 1, 1, 1 \rangle \times \langle 1, -2, 3 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 5, -2, -3 \rangle$$

Vector equation for line: $\langle x, y, z \rangle = \langle 0, 0, 2 \rangle + t \cdot \langle 5, -2, -3 \rangle$.

Symmetric equations for line: $\dfrac{x}{5} = \dfrac{y}{-2} = \dfrac{z-2}{-3}$

(b) **(6 points)** Find the equation of the plane that contains all of the following points:

$$(1, 3, 2) \qquad (3, -1, 6) \qquad (5, 2, 0).$$

Form vectors that run between the points:

$\vec{a} = \langle 2, -4, 4 \rangle \qquad \vec{b} = \langle 4, -1, -2 \rangle$.

Normal vector of plane:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \langle 12, 20, 14 \rangle.$$

Equation of plane:

$$12(x-1) + 20(y-3) + 14(z-2) = 0.$$

(Many solutions possible.)

Continued on the next page.

(c) **(6 points)** Find the area of the triangle formed by the following points:

$$(1, 2, 3) \qquad (3, 0, -1) \qquad (2, 2, 2).$$

Find vectors for two sides of the triangle:

$\vec{a} = \langle 2, -2, -4 \rangle$ $\qquad$ $\vec{b} = \langle 1, 0, -1 \rangle$.

Then Area $= \frac{1}{2} |\vec{a} \times \vec{b}|$. Now:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -4 \\ 1 & 0 & -1 \end{vmatrix} = \langle 2, 2, 2 \rangle.$$

$$\text{Area} = \frac{1}{2}\sqrt{2^2 + 2^2 + 2^2} = \sqrt{3}.$$