Leray numbers of Tolerance complexes

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Copenhagen-Jerusalem Combinatorics Seminar
Simplicial complexes

\[ V = \text{finite set} \]
Simplicial complexes

- $V$ is a finite set
- $k \leq 2^V$ is called a simplicial complex if:
Simplicial complexes

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- $K \subseteq 2^V$ is called a simplicial complex if:

$$A \in K \Rightarrow \forall B \subseteq A, B \in K$$
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  \[
  A \in K \implies \forall B \subseteq A, B \in K
  \]

- A set $A \in K$ is called a simplex of $K$
Simplicial complexes

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  \[
  A \in K \implies \forall B \subseteq A, \quad B \in K
  \]
- A set \( A \in K \) is called a (face) simplex of \( K \)
Simplicial complexes

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- $K \subseteq 2^V$ is called a simplicial complex if:
  \[ A \in K \implies \forall B \subseteq A, \quad B \in K \]

- A set $A \in K$ is called a (face) simplex of $K$.
- The dimension of a simplex $A$ is $|A| - 1$. 
Simplicial complexes

- $V$ finite set
- $K \subseteq 2^V$ is called a simplicial complex if:
  \[ A \in K \Rightarrow \forall B \subseteq A, B \in K \]
- A set $A \in \mathcal{K}$ is called a (face) simplex of $K$.
- The dimension of a simplex $A$ is $|A|-1$.
- Dimension of $K = \max$ dimension of a simplex.
Simplicial complexes

- We can view simp. complexes as geometric objects:
Simplicial complexes

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\[ K = 1 \rightarrow 2 \rightarrow 3 \]

\[ K = 1 \rightarrow 4 \rightarrow 3 \]
Simplicial complexes

- We can view simp. complexes as geometric objects:

\[
K = \begin{array}{c}
    1 \\
    2 \\
    3 \\
    4 \\
\end{array} \quad \iff \quad K = \left\{ \{1, 2, 3\}, \{1, 4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \emptyset \right\}
\]
Simplicial complexes

- We can view simp. complexes as geometric objects:

\[ K = \begin{array}{ccc}
2 & 1 & 3 \\
4 & & \\
& & \\
\end{array} \iff K = \begin{array}{c}
\{1,2,3\}, \{1,4\}, \\
\{1,2,3\}, \{2,3\}, \\
\{1,3\}, \{2,3\}, \{3,4\}, \\
\emptyset
\end{array} \]

- We can study the topology of a complex.
Homology

\[ \tilde{H}_i(K) = i \text{-dimensional reduced homology group of } K \text{ (with real coefficients)} \]
Homology

- $\tilde{H}_i(K) = i$-dimensional reduced homology group of $K$ (with real coefficients)

- Informally: $\tilde{H}_i(K)$ counts "$i$-dimensional holes" in $K"
Homology

- \( \tilde{H}_i(K) \) is \( i \)-dimensional reduced homology group of \( K \) (with real coefficients).

Informally: \( \tilde{H}_i(K) \) counts "\( i \)-dimensional holes" in \( K \).

E.g.

\[ K = \begin{array}{c}
\text{pentagon}
\end{array} \]
Homology

- $\tilde{H}_i(K) =$ $i$-dimensional reduced homology group of $K$ (with real coefficients)

- Informally: $\tilde{H}_i(K)$ counts "$i$-dimensional holes" in $K$.

E.g.

$K = \begin{array}{c}
\begin{array}{c}
\text{pentagon}
\end{array}
\end{array}$

$\tilde{H}_i(K) = \begin{cases} 
\mathbb{R} & \text{if } i=1 \\
0 & \text{otherwise}
\end{cases}$
Homology

- $\tilde{H}_i(K) = \text{i-dimensional reduced homology group of } K \text{ (with real coefficients)}$

- Informally: $\tilde{H}_i(K)$ counts "i-dimensional holes" in $K$.

E.g.

$K =$ [Diagram of a 3-dimensional polyhedron]
Homology

- $\tilde{H}_i(K) = \text{i-dimensional reduced homology group of } K \text{ (with real coefficients)}$

- Informally: $\tilde{H}_i(K)$ counts "i-dimensional holes" in $K$.

E.g.

$K = \quad \tilde{H}_i(K) = \begin{cases} \mathbb{R} & \text{ if } i = 2 \\ 0 & \text{ otherwise} \end{cases}$
Homology

- $\tilde{H}_i(K) = \text{i-dimensional reduced homology group of } K \text{ (with real coefficients)}$

- Informally: $\tilde{H}_i(K)$ counts "i-dimensional holes" in $K$.

E.g.

- d-dimensional $K = \text{sphere}$
Homology

- \( \tilde{H}_i(K) \) = \( i \)-dimensional reduced homology group of \( K \) (with real coefficients)

- Informally: \( \tilde{H}_i(K) \) counts "\( i \)-dimensional holes" in \( K \).

E.g.

- \( d \)-dimensional \( K = \text{sphere} \)

\[
\tilde{H}_i(K) = \begin{cases} \mathbb{R} & \text{if } i=d \\ \{0\} & \text{otherwise} \end{cases}
\]
Leray numbers

$K = \text{simp. complex on vertex set } V$
Leray numbers

- $K = \text{simp. complex on vertex set } V$
- For $U \subseteq V$,

$$K[U] = \{ \sigma \subseteq U : \sigma \in K \}$$
Leray numbers

- $K = \text{simp. complex on vertex set } V$
- For $U \subseteq V$, 
  $K[U] = \{\sigma \subseteq U : \sigma \in K\}$
  (subcomplex induced by $U$)
Leray numbers

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Leray numbers

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For \( U \subseteq V \),

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(subcomplex induced by \( U \)
Leray numbers

- $K = \text{simp. complex on vertex set } V$
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Leray numbers

- $K$ is simp. complex on vertex set $V$
- For $U \subseteq V$, $K[U] = \{ \sigma \subseteq U : \sigma \in K \}$ (subcomplex induced by $U$)
- $K$ is $d$-Leray if $H_i(K[U]) = 0$ for all $i \geq d$ and $U \subseteq V$. 

$K[U] = L$
Leray numbers

- \( K = \text{simp. complex on vertex set } V \)
- For \( U \subseteq V \),
  \[ K[U] = \{ \sigma \subseteq U : \sigma \in K \} \]
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Leray numbers

- $K = \text{simp. complex on } V$
- For $U \subseteq V$, $K[U] = \{ \sigma \subseteq U : \sigma \in K \}$ (subcomplex induced by $U$)
- $K$ is $d$-Leray if $\tilde{H}_i(K[U]) = 0$ for all $i \geq d$ and $U \subseteq V$. 

"Homological dimension of $K" 

Leray number of $K = \text{minimal } d \text{ s.t. } K \text{ is } d\text{-Leray."}
Collapsibility

Let \( \sigma \in \mathcal{K} \) s.t. \( 10 \leq d \)
Collapsibility

Let $o \in K$ s.t. $1 \leq d$ and $o$ is contained in unique maximal face $t \in K$
Collapsibility

Let \( o \in K \) st. \( 1 \leq d \) and \( o \) is contained in unique maximal face \( t \in K \).
Collapsibility

Let $v \in K$ s.t. $1 \leq d$ and $v$ is contained in unique maximal face $teK$

Elementary $d$-collapse:
Collapsibility

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Elementary $d$-collapse:
Collapsibility

Let \( \sigma \in K \) s.t. \( 10 \leq d \) and \( \sigma \) is contained in unique maximal face \( t \in K \)

Elementary \( d \)-collapse:
Collapsibility

Let \( o \in K \) st. \( 10 \leq d \) and \( o \) is contained in unique maximal face \( t \in K \)

Elementary \( d \)-collapse:

1. If \( E \) sequence of elem. \( d \)-coll.
2. From \( K \) to \( \emptyset \): \( K \) is \( d \)-collapsible
Collapsibility

E.g.

\[ K = \]
Collapsibility

E.g.

\[ K = \]

\[ K \text{ is not } 1\text{-collapsible} \]
Collapsibility

E.g.

\[ K = \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = \sigma_i \]

we will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = \sigma_i \tau_i \]

we will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K_1 = \]

we will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = K_1 \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K_1 = \]

we will show that K is 2-collapsible.
Collapsibility

E.g.

\[ K_2 = \]

we will show that

\( K \) is 2-collapsible.
Collapsibility

E.g.

$K = K_2$

we will show that $K$ is 2-collapsible.
Collapsibility

E.g.,

\[ K = \frac{\tau_3}{\sigma_3} \]

we will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

$K_3$

we will show that $K$ is 2-collapsible.
Collapsibility

E.g.

$K = \sigma_3$

We will show that $K$ is 2-collapsible.
Collapsibility

E.g.

\[ K = 3 \]

we will show that 

\[ K \text{ is 2-collapsible}. \]
Collapsibility

E.g.

\[ K = \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = \sigma_s \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = \]

we will show that

\( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K_5 \]

we will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = \mathcal{K}_5 \]

\[ \sigma_2 = \emptyset \]

we will show that

\[ K \] is 2-collapsible.
Collapsibility

E.g.

\[ K_5 \]

\[ \sigma_6 = \emptyset \]

\[ \tau_6 \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K_6 \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = \]

\[ K \text{ is 2-collapsible.} \]
Collapsibility

E.g.

\[ K = \]

\[ K \text{ is } 2\text{-collapsible}. \]

• Collapsibility of \( K = \)

  minimal \( d \) s.t. \( K \) is \( d \)-collapsible.
Representability

$F = \{ F_1, F_2, \ldots, F_n \}$ family of sets.
Representability

- $F = \{F_1, F_2, \ldots, F_n\}$ family of sets.
- $N(F) =$ nerve of $F$.
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- $N(F) =$ nerve of $F$:
  - A vertex is assigned to each set in the family.
Representability

- $F = \{F_1, F_2, \ldots, F_n\}$ family of sets.
- $N(F) =$ nerve of $F$:
  - A vertex is assigned to each set in the family.
  - Simplices correspond to subfamilies with non empty intersection.
Representability

E.g.

\[ F = \]
Representability

E.g.

\[ F = \quad N(F) = \]
Representability

E.g.

\[ F = \quad N(F) = \]
Representability

E.g.

\[ F = \quad N(F) = \]

\[ \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{example.png}}
\end{array} \]
Representability

E.g.

\[ F = \quad N(F) = \]

\[ \quad \]

\[ \quad \]

\[ \quad \]
Representability

E.g.

\[ F = \text{[diagram]} \quad N(F) = \text{[diagram]} \]

- d-representable complex = nerve of a family of convex sets in \( \mathbb{R}^d \)
Representability

E.g.

\[ F = \]

\[ N(F) = \]

- \( d \)-representable complex = nerve of a family of convex sets in \( \mathbb{R}^d \)
k is d-representable
$K$ is \textit{d-collapsible}

$K$ is \textit{d-representable}
\( K \) is \( d \)-collapsible \( \Rightarrow \) \( K \) is \( d \)-representable \( \Rightarrow \) \( K \) is \( d \)-Leray
Wegner's Theorem (75)

\[ K \text{ is } d\text{-collapsible} \]

\[ K \text{ is } d\text{-representable} \]

\[ K \text{ is } d\text{-Leray} \]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets.
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common,
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i=1, \ldots, m$).
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ $(i = 1, \ldots, m)$.

E.g.

\[ d=1 \quad \begin{array}{ccc}
    \text{green line} & \text{blue line} & \text{red line}
\end{array} \quad \mathbb{R}^1 \]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i = 1, \ldots, m$).

E.g.

\[ d=1 \]

\[ \mathbb{R} \]
Helly's Theorem

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Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ $(i=1, \ldots, m)$.

E.g.

\[ d=1 \]

\[ \mathbb{R} \]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i = 1, \ldots, m$).

E.g.

\[ d=1 \]

[Diagram showing three lines intersecting at a point in the plane.]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i=1, \ldots, m$).

E.g.

\begin{align*}
  d=1 & \quad \begin{array}{|c|}
  \hline
  1 \\
  \hline
\end{array} & d=2 & \quad \begin{array}{|c|c|}
  \hline
  1 & \hline
  \hline
  2 & \hline
\end{array}
\end{align*}
Helly's Theorem

Let \( C_1, \ldots, C_m \subseteq \mathbb{R}^d \) be convex sets. If every \( d+1 \) of these sets have a point in common, then there is a point in common to all \( C_i \) (\( i = 1, \ldots, m \)).

E.g.

\[
\begin{align*}
\text{d=1} & \quad \text{d=2} \\
\end{align*}
\]
Helly’s Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i=1, \ldots, m$).

E.g.

$d=1$

\[ \mathbb{R} \]

$d=2$

\[ \text{Circle and Rectangle} \]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i=1, \ldots, m$).

E.g.

$$d=1 \quad \text{and} \quad d=2$$
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i=1, \ldots, m$).

E.g.

\begin{align*}
\text{d=1} & \quad \text{d=2} \\
\text{\hspace{1cm}} & \quad \text{\hspace{1cm}}
\end{align*}
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i=1, \ldots, m$).

E.g.

\begin{align*}
\text{d=1} & \hspace{2cm} \text{d=2} \\
\end{align*}
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i=1, \ldots, m$).

E.g.

$d=1$

$\mathbb{R}^1$

$d=2$

not convex!
Missing faces

\[ K = \text{ simp. complex on vertex set } V. \]
Missing faces

- $K = \text{simp. complex on vertex set } V.$
- A missing face is a set $F \subseteq V$

s.t.: 

Missing faces

- \( K = \text{simp. complex on vertex set } V \).
- A missing face is a set \( \tau \subseteq V \) s.t: (1) \( \tau \notin K \).
Missing faces

- \( K = \text{simp. complex on vertex set } V \).
- A missing face is a set \( \tau \subseteq V \) such that:
  1. \( \tau \notin K \),
  2. \( \sigma \in K \) \( \forall \sigma \not\supseteq \tau \).
Missing faces

- $\mathcal{K}$ is simp. complex on vertex set $V$.
- A missing face is a set $\tau \subseteq V$ s.t.: 1. $\tau \notin \mathcal{K}$, 2. $\sigma \in \mathcal{K}$ for all $\sigma \nsubseteq \tau$

E.g.

$K = \triangle$
Missing faces

- $K = \text{simp. complex on vertex set } V.$
- A missing face is a set $\tau \subset V$ st: $1. \tau \notin K, 2. \sigma \in K \land \forall \sigma \notin \tau$

E.g.

$K =$ [diagram of a polyhedron with missing faces highlighted]
Missing faces

- \( K = \text{simp. complex on vertex set } V \).
- A missing face is a set \( \tau \subseteq V \) s.t.: 1. \( \tau \notin K \), 2. \( \sigma \in K \) \( \forall \sigma \in \tau \).

E.g.

\[ K = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \]
Missing faces

- $K$ = simp. complex on vertex set $V$.
- A missing face is a set $\tau \subseteq V$ s.t.: 1. $\tau \notin K$, 2. $\sigma \in K$ for all $\sigma \not\subseteq \tau$

E.g.

$K =$ [diagram of a complex with a missing face highlighted]
Missing faces

- $K = \text{ simp. complex on vertex set } V$.
- A missing face is a set $\tau \subseteq V$ s.t.:
  1. $\tau \notin K$,
  2. $\sigma \in K \quad \forall \sigma \notin \tau$

E.g.

$K =$ [Diagram of a 3D shape with missing faces]
Missing faces

- \( K = \) simp. complex on vertex set \( V \).
- A missing face is a set \( \tau \subseteq V \) st: 1. \( \tau \notin K \), 2. \( \sigma \in K \) \( \forall \sigma \notin \tau \).

E.g.

\[ K = \triangle \]
Missing faces

- $K$ = simp. complex on vertex set $V$.

- A **missing face** is a set $\tau \subseteq V$ st: 1) $\tau \notin K$, 2) $\sigma \in K$ \hspace{1cm} $\forall \sigma \nsubseteq \tau$

**E.g.**

\[ K = \quad \]
Missing faces

- \( K \) = simp. complex on vertex set \( V \).
- A missing face is a set \( \tau \subseteq V \) such that:
  1. \( \tau \notin K \),
  2. \( \sigma \in K \quad \forall \sigma \not\in \tau \).

E.g.

\[ K = \begin{align*}
\text{[Diagram of a complex with missing face highlighted.]}\end{align*} \]
Missing faces

- \( K = \text{simp. complex on vertex set } V. \)
- A missing face is a set \( \tau \subseteq V \) s.t.: ① \( \tau \notin K \), ② \( \sigma \in K \) \( \forall \sigma \in \tau \)

E.g.

\( K = \)
Missing faces

- $K$ is simp. complex on vertex set $V$.
- A missing face is a set $\tau \subseteq V$ st: 1. $\tau \notin K$, 2. $\sigma \in K \forall \sigma \notin \tau$

E.g.

$K =$ \[
\begin{array}{c}
\text{\circ} \\
\text{\circ} \\
\text{\triangle} \\
\end{array}
\]
Missing faces

- $K = \text{simp. complex on vertex set } V$.
- A missing face is a set $\tau \subseteq V$ st: 1. $\tau \notin K$, 2. $\sigma \in K \forall \sigma \subseteq \tau$

**E.g.**

$K = \Delta$

$h(K) =$ maximum dimension of a missing face in $K$
**Missing faces**

- $K = \text{simp. complex on vertex set } V$.
- A **missing face** is a set $\tau \subseteq V$ s.t.:  
  1. $\tau \notin K$,  
  2. $\sigma \in K \quad \forall \sigma \subsetneq \tau$

**E.g.**

$K = \Delta$

$h(K) = 2$

$h(K) =$ **maximum dimension of a missing face in $K$**
Helly in terms of missing faces

- Helly's Thm is equivalent to:

Thm: If $K$ is $d$-representable, then $h(K) \leq d$. 
Helly for d-Leray complexes

Thm: If $K$ is d-Leray then $h(K) \leq d$. 
Erdős–Gallai numbers

$H \leq 2^e$ family of sets.
Erdös–Gallai numbers

- A non-empty family of sets $H$.
- A set $C \subseteq V$ covers $H$ if $\bigcap_{A \in H} C \neq \emptyset$ for all $A \in H$. 
Erdős-Gallai numbers

- $H \subseteq \mathcal{V}$ family of sets.
- A set $C \subseteq \mathcal{V}$ covers $H$ if $A \cup C \neq \emptyset$ for all $A \in H$
- $\mathcal{I}(H) =$ minimum size of cover
Erdős-Gallai numbers

- \( H \leq 2 \) family of sets.
- A set \( C \subseteq V \) covers \( H \) if \( A \cap C \neq \emptyset \) for all \( A \in H \).

\[ I(H) = \text{minimum size of cover} \]
Erdős-Gallai numbers

- $\mathcal{H} \subseteq 2^V$ family of sets.
- A set $C \subseteq V$ covers $\mathcal{H}$ if $A \cap C \neq \emptyset$ for all $A \in \mathcal{H}$

- $\mathcal{I}(\mathcal{H}) = \text{minimum size of cover}$
Erdős–Gallai numbers

- A set $C \subseteq V$ covers $H$ if $A \cap C \neq \emptyset$ for all $A \in H$

- $\tau(H) = \min \text{ size of cover}$
Erdös-Gallai numbers

\( \zeta(r, t) = \text{minimum } m \text{ such that:} \)
Erdős-Gallai numbers

\( \psi(r, t) = \text{minimum } m \text{ such that:} \)

A family \( H \) of sets of size \( \leq t \) each, if \( \psi(H') \leq t - 1 \) for any \( H' \subseteq H \) with \( \left| \bigcup H' \right| \leq m \), then \( \psi(H) \leq t - 1 \).
Erdös-Gallai numbers

\(\eta(t,t) = \text{minimum } m \text{ such that:}\)

A family \(H\) of sets of size \(\leq t\) each, if \(\eta(H') \leq t-1\) for any \(H' \subseteq H\) with \(|UH'| \leq m\), then \(\eta(H) \leq t-1\).

Erdös-Gallai ('61)
Erdős-Gallai numbers

\[ \eta(r,t) = \text{minimum } m \text{ such that:} \]

A family \( H \) of sets of size \( \leq r \) each, if \( \tau(H') \leq t-1 \) for any \( H' \subseteq H \) with \( |UH'| \leq m \), then \( \tau(H) \leq t-1 \).

\textbf{Erdős-Gallai ('61)}

- \( \eta(2,t) = 2t \)
Erdős-Gallai numbers

\( \eta(r, t) = \text{minimum } m \text{ such that:} \)

A family \( H \) of sets of size \( \leq r \) each, if \( \tau(H') \leq t-1 \) for any \( H' \subseteq H \) with \( |UH'| \leq m \), then \( \tau(H) \leq t-1. \)

\textbf{Erdős-Gallai (’61)}

- \( \eta(2, t) = 2t \)
- \( \eta(r, 2) = \left\lfloor \left( \frac{r+2}{2} \right)^2 \right\rfloor \)
Erdős-Gallai numbers

\( \eta(r_t) = \min \{ m \mid \text{such that:} \}

A family \( \mathcal{H} \) of sets of size \( \leq r \) each, if \( \tau(\mathcal{H}') \leq t-1 \) for any \( \mathcal{H}' \subseteq \mathcal{H} \) with \( |\cup \mathcal{H}'| \leq m \), then \( \tau(\mathcal{H}) \leq t-1 \).

Erdős-Gallai (’61)

- \( \eta(2, t) = 2t \)
- \( \eta(r, 2) = \left\lfloor \left( \frac{r+2}{2} \right)^2 \right\rfloor \)

Tuza (’88):

- \( \eta(r, t) \leq \binom{r+t-1}{r-1} + \binom{r+t-2}{r-1} \)
Erdös-Gallai numbers

\( \eta(r,t) = \text{minimum } m \text{ such that:} \)

A family \( H \) of sets of size \( \leq r \) each, if \( \sum(H') \leq t-1 \) for any \( H' \subseteq H \) with \( \bigcup H' \leq m \), then \( \prod(H) \leq t-1 \).

Erdös-Gallai (’61)
- \( \eta(2,t) = 2t \)
- \( \eta(r,2) = \left\lceil \left( \frac{r+2}{2} \right)^2 \right\rceil \)

Tuza (’88):
- \( \eta(r,t) < \binom{r+t-1}{r-1} + \binom{r+t-2}{r-1} \)
- \( \eta(r,t) = O(t^2) \) for fixed \( t \)
Erdős–Gallai numbers

$\eta(r, t) =$ minimum $m$ such that:

A family $H$ of sets of size $\leq r$ each, if $\tau(H') \leq t-1$ for any $H' \subseteq H$ with $\bigcup H' \leq m$, then $\tau(H) \leq t-1$.

Erdős–Gallai (’61)

- $\eta(2, t) = 2t$
- $\eta(r, 2) = \lceil (\frac{r+2}{2})^2 \rceil$

Tuza (’88):

- $\eta(r, t) < \binom{r+t-1}{r-1} + \binom{r+t-2}{r-1}$
- $\eta(r, t) = O(t^2)$ for fixed $t$
- $\eta(r, t) = O(t^{r-1})$ for fixed $r$
Helly with tolerance

- $F = \text{family of sets.}$
Helly with tolerance

- $F =$ family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\cap F' \neq \emptyset$. 
Helly with tolerance

- $F$ = family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\cap F' \neq \emptyset$.

E.g.

$F =$
Helly with tolerance

- $F =$ family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t.
  $|F'| \geq |F| - t$ and $\bigcap F' \neq \emptyset$.

E.g.

$F = \{ \text{overlapping shapes} \}$

$\bigcap F = \emptyset$
Helly with tolerance

- $F$ = family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\bigcap F' \neq \emptyset$.

E.g.

$F = \{\text{shapes}\}$

$\bigcap F = \emptyset$
Helly with tolerance

- $F$ = family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\bigcap F' \neq \emptyset$.

E.g.: $|F'| = |F| - 2$
Helly with tolerance

- $F$ = family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\bigcap F' \neq \emptyset$.

E.g. $F' = \{\text{intersections}\}$ where $|F'| = |F| - 2$ and $\bigcap F' \neq \emptyset$. 
Helly with tolerance

- $F =$ family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\forall F' \neq \emptyset$.

E.g.

$F = \{ \text{shapes} \}$

So, $F$ has a point in common with tolerance 2.
Helly with tolerance

Thm (Montejano-Oliveros '10):
Helly with tolerance

Thm (Montejano-Oliveros '10):
Let $F$ be a finite family of convex sets in $\mathbb{R}^d$. 
Helly with tolerance

Thm (Montejano–Oliveros '10):

Let $F$ be a finite family of convex sets in $\mathbb{R}^d$.

If any $F' \subseteq F$ of size $\eta(d+1,t+1)$ has a point in common with tolerance $t$, ...
Helly with tolerance

Thm (Montejano-Oliveros '10):

Let $F$ be a finite family of convex sets in $\mathbb{R}^d$.
If any $F' \subseteq F$ of size $\chi(d+1,t+1)$ has a point in common with tolerance $t$,
then $F$ has a point in common with tolerance $t$. 
Tolerance Complexes

* $K = \text{simp. complex on vertex set } V.$
Tolerance Complexes

- $K = \text{simp. complex on vertex set } V.$
- $t \geq 0 \text{ an integer.}$
Tolerance Complexes

- $K = \text{simp. complex on vertex set } V.$
- $t \geq 0$ an integer.
- The $T$-tolerance complex of $K$.
Tolerance Complexes

- $K$ = simp. complex on vertex set $V$.
- $t \geq 0$ an integer.
- The $t$-tolerance complex of $K$:

$$
T_t(K) = \left\{ \sigma \subseteq V \mid \exists \sigma' \subseteq \sigma \\
|10^{-1} \geq |10^{-t} - t|
\sigma' \in K \right\}
$$
Tolerance Complexes

* $K = \text{simp. complex on vertex set } V$.
* $t \geq 0$ an integer.
* The $T$-tolerance complex of $K$:

$$\mathcal{T}_t(K) = \left\{ \sigma \subseteq V \mid \exists \sigma' \subseteq \sigma \text{ s.t. } \sigma' \in K, \left| 10^{-1} - t \right| \leq \left| \sigma' \sigma \right| \right\}$$

E.g.

$$K = \bullet \quad \bullet \quad \bullet \quad \bullet$$
Tolerance Complexes

- $K$ = simp. complex on vertex set $V$.
- $t \geq 0$ an integer.
- The $T$-tolerance complex of $K$:

$$\mathcal{T}_t(K) = \left\{ \sigma \in \mathcal{V} \left| \exists \sigma' \leq \sigma \quad \left( ^{\sigma'} \mathcal{O} \right)^{-1} \mathcal{O} \geq ^{\sigma'} \mathcal{O} - t ^{\sigma'} \mathcal{O} \mathcal{K} \right. \right\}$$

E.g.

$$K = \bullet \quad \mathcal{T}_t(K) = \text{complex with vertices and edges}$$
Tolerance Complexes

\[ K = \text{family of sets } V, \]

\[ F = \text{family of sets} \]
Tolerance Complexes

- $K = \text{the finite point set } V$.
- $\mathcal{F} = \text{family of sets}$.

$\tau_t(N(\mathcal{F})) = \left\{ F' \leq F \mid \text{in common with } F' \text{ has pt. tolerance } t \right\}$
Helly's property for tolerance complexes

Thm (Montejano-Oliveros '10):

If $K$ is $d$-representable, then

$$h(T_e(K)) \leq h(d+1, t+1) - 1$$
Helly's property for tolerance complexes

Thm (Montejano-Oliveros '10):

If $h(K) \leq d$, then

$$h(T_t(K)) \leq h(d+1, t+1) - 1$$
Helly's property for tolerance complexes

Thm (Montejano-Oliveros '10):

If $h(K) \leq d$, then
$$h(T_e(K)) \leq h(d+1,t+1) - 1$$

• If we assume $K$ is $d$-collapsible/d-Leray, can we obtain a stronger conclusion?
Collapsibility and Leray numbers of $T_t(K)$

Conjecture: If $K$ is $d$-Leray, then $T_t(K)$ is $(h(d+1, t+1) - 1)$-Leray.
Collapsibility and Leray numbers of $J_t(K)$

**Conjecture:** If $K$ is d-Leray, then $J_t(K)$ is $(\eta(d+1, t+1) - 1)$-Leray.

**Conjecture:** If $K$ is d-collapsible, then $J_t(K)$ is $(\eta(d+1, t+1) - 1)$-collapsible.
Extremal examples

$C =$ facets of $d$-dim simplex,
$t+1$ copies of each.
Extremal examples

\[ C = \text{facets of } d\text{-dim simplex,} \]
\[ t+1 \text{ copies of each.} \]

\[ C = \begin{array}{c}
  t+1 \\
  t+1 \\
  t+1 
\end{array} \]
Extremal examples

\( C = \text{facets of } d \)-dim simplex,
\( t+1 \) copies of each.

\[
C = \begin{pmatrix}
\text{t + 1} \\
\text{t + 1} \\
\text{t + 1}
\end{pmatrix}
\]

\( N(C) = d \)-representable
Extremal examples

$\mathcal{C} = \text{ facets of } d\text{-dim simplex, }$

$t+1 \text{ copies of each.}$

$\mathcal{C} = \begin{array}{c}
\begin{array}{c}
\bigtriangleup \\
\bigtriangleup \\
\bigtriangleup \\
\end{array}
\end{array}
$

$N(\mathcal{C}) = d\text{-representable (d-collapsible)}$

$\text{d-collapsible}$
Extremal examples

$C = \text{facets of } d\text{-dim simplex, } t+1 \text{ copies of each. } T_t(N(C)) = ?$

$C = \begin{array}{c}
\text{t+1} \\
\text{t+1} \\
\text{t+1}
\end{array}$

$N(C) = d\text{-representable (d-collapsible)}$
Extremal examples

\( C = \text{facets of } d\text{-dim simplex}, \text{ } \forall t+1 \text{ copies of each.} \)

\( T_t(N(C)) = ? \)

\( C \setminus \{a\} = \begin{tikzpicture}
\begin{scope}[scale=0.5]
\node (a) at (0,0) {}; \
\node (b) at (1,0) {}; \\
\node (c) at (0.5,1) {}; \\
\draw (a) -- (b) -- (c) --cycle; \\
\end{scope}
\end{tikzpicture} \)

\( t \)

\( t+1 \)

\( t+1 \)

\( N(C) = \text{d-representable (d-collapsible)} \)

(d-Leray)
Extremal examples

$C = \text{ facets of } d\text{-dim simplex, } t + 1 \text{ copies of each.}$

$\text{N}(C) = d\text{-representable}$

$N = \{x \in \mathbb{R}^d : \langle x, \alpha \rangle = 1, \alpha \in C \}$

$T(\text{N}(C)) = ?$
Extremal examples

\( C = \text{facets of d-dim simplex, } t+1 \text{ copies of each.} \quad T_t(\mathcal{N}(e)) = ? \)

\[ C \setminus \{a\} = \begin{array}{c}
\text{t} \\
\text{t+1} \\
\text{t+1}
\end{array} \]

\( N(C) = \text{d-representable (d-collapsible)} \quad \text{(d-Leray)} \)
Extremal examples

\[ C = \text{facets of } d\text{-dim simplex, } t+1 \text{ copies of each.} \]

\[ C \setminus \{A\} = t \quad t+1 \]

\[ T_t(N(C)) = \text{boundary of } (d+1)(t+1)-1\text{-dim. Simplex} \]

\[ N(C) = \text{d-representable (d-collapsible) } \]

\[ d\text{-acyclic} \]
Extremal examples

\[ C = \text{facets of } d\text{-dim simplex, } \]
\[ t+1 \text{ copies of each. } \]

\[ T_t(N(C)) = (\Gamma(t+1)(t+1)-2)\text{-dim. Sphere} \]

\[ N(C) = d\text{-representable (}d\text{-collapsible)} \]

\[ d\text{-kerdyn} \]
Extremal examples

$C =$ facets of $d$-dim simplex, $t+1$ copies of each.

$C \setminus \{a\} =$

$\begin{array}{c}
\text{N}(C) = d\text{-representable} \\
(d\text{-collapsible})
\end{array}$

$T_t(N(C))$

$= (d+1)(t+1)-2)$-dim. Sphere

$\Downarrow$

$T_t(N(C))$ is NOT

$(d+1)(t+1)-2)$-Leray
Extremal examples

$C$ = facets of $d$-dim simplex, $t+1$ copies of each.

$C \setminus \{a\} = \begin{array}{c}
\text{t+1} \\
\text{t+1} \\
\text{t+1}
\end{array}$

$N(C) = d$-representable ($d$-collapsible)

$T_t(N(C))$ is NOT $(d+1)(t+1)-2$-Leray

$T_t(N(C)) = (d+1)(t+1)-2$-dim. Sphere

For $d=1$, $(d+1)(t+1)-2 = 2t$
Extremal examples

\[ C = \text{facets of } d \text{-dim simplex, } t+1 \text{ copies of each.} \]

\[ T_t(N(C)) = (d+1)(t+1)-2 \text{-dim. sphere} \]

\[ T_t(N(C)) \text{ is NOT } (d+1)(t+1)-2 \text{-Leray} \]

\[ N(C) = \text{d-representable (d-collapsible) } \]

For \( d = 1 \), \( (d+1)(t+1)-2 = 2T = \chi(2, t+1)-2 \)
Extremal examples

Montejano-Oliveros ('10):
\[ \exists C \subset \mathbb{R}^d \text{ family of convex sets in } \mathbb{R}^d\]
\[ \text{st. } \mathcal{T}_h(N(C)) \text{ is not } \left( \left( \frac{d+3}{2} \right)^2 - 2 \right) - \text{Leray} \]

- For \( d=1 \), \( \tau_{d+1}(r+1)^{-2} = 2 \tau = \zeta(2, t+1) - 2 \)
Extremal examples

Montejano-Oliveros ('10):

\exists C \subseteq \text{family of convex sets in } \mathbb{R}^d

\text{st. } \mathcal{I}_d(\mathcal{N}(C)) \text{ is not } \left(\left\lfloor \frac{(d+3)^2}{2} \right\rfloor -2\right)\text{-Leray}

• For \( d=1 \), \( \Gamma_{d+1}(t+1)-2 = 2t = \zeta(2, t+1) - 2 \)
• \( t=1 \): \( \left\lfloor \frac{(d+3)^2}{2} \right\rfloor -2 = \zeta(d+1, 2) - 2 \)
Main results

\[ h(t,d) = \begin{cases} d & ; t = 0 \\
\left\lceil \sum_{s=\min(t,d)}^{d} (s)(h(t-s,d) + 1) \right\rceil + d & ; t > 0 \end{cases} \]
Main results

\[ h(t,d) = \begin{cases} 
  d & \text{if } t = 0 \\
  \left[ \sum_{s=0}^{\min(t,d)} (d \cdot s \cdot (h(t-s,d) + 1)) \right] + d & \text{if } t > 0 
\end{cases} \]

Thm (Kim-L. ’21):

Let \( K \) be a \( d \)-collapsible complex. Then, \( T_t(K) \) is \( h(t,d) \)-Leray.
Main results

\[ h(t,d) = \begin{cases} 
  d, & t = 0 \\
  \min_{t\geq d} \left( \sum_{s=4}^{d} (d)(h(t-s,d)+1) \right) + d, & t > 0 
\end{cases} \]

Thm (Kim-L. '21):
Let \( K \) be a \( d \)-collapsible complex. Then, \( T_t(K) \) is \( h(t,d) \)-Leray.

\[ h(t,1) = 2t + 1 \]
Main results

$$h(t,d) = \begin{cases} d, & t = 0 \\ \left\lceil \sum_{s=0}^{\min\{t,d\}} (s)(h(t-s,d)+1) \right\rceil + d, & t > 0 \end{cases}$$

Thm (Kim-L. '21): Let $K$ be a $d$-collapsible complex. Then, $T_t(K)$ is $h(t,d)$-Leray.
Main results

\[ h(t,d) = \begin{cases} 
  0 & \text{, } t = 0 \\
  \left( \sum_{s=0}^{\min(t,d)} p_s (h(t-s,d) + 1) \right) + d & \text{, } t > 0 
\end{cases} \]

**Thm (Kim-L. '21):**

Let \( K \) be a \( d \)-collapsible complex. Then, \( T_t(K) \) is \( h(t,d) \)-Leray.

\[
\begin{align*}
  h(t,d) &= 2t + 1 \\
  h(1,d) &= d^2 + 2d \\
  \text{For fixed } t, \quad h(t,d) &= O(d^{t+1})
\end{align*}
\]
Main results

- For $t=1$, we obtained:
Main results

For $t=1$, we obtained:

$K$ $d$-collapsible

$\downarrow$

$T_1(K)$ $(d^2 + 2d)$-Leray
Main results

For $t=1$, we obtained:

$K$ $d$-collapsible

$\Rightarrow$ $\left\lfloor \frac{(d+3)^2}{2} \right\rfloor - 1$

$\tilde{T}_1(K)$ $(d^2 + 2d)$-Leray
Main results

- For $t=1$, we obtained:
  
  $K$ $d$-collapsible
  
  $? \downarrow \left\lfloor \frac{(d+3)^3}{2} \right\rfloor - 1$

  $T_1(K) (d^2 + 2d)$-Leray

- True for $d=2$: 
Main results

• For $t=1$, we obtained:

\[ K \text{ d-collapsible} \]

\[ \Downarrow \]

\[ (d+3)^3 - 1 \]

\[ T_1(K) \text{ (d}^2 + 2d)\text{-Leray} \]

• True for $d=2$:

Thm (Kim-L. '21): Let $K$ be 2-collapsible. Then, $T_1(K)$ is 5-Leray.
Some ideas from the proof

Some definitions:
Some ideas from the proof

Some definitions:

\[ k = \text{simp. complex on } V. \quad \sigma \in k \]
Some ideas from the proof

Some definitions:

\( k = \text{simp. complex on } V, \sigma \in k \)

\( \mathcal{L}_k(K, \sigma) = \{ t \in K : \sigma \cup t = \emptyset \} \)
Some ideas from the proof

Some definitions:

\( k = \text{simp. complex on } V, \sigma \in k \)

\( \text{lk}(k, \sigma) = \{ t \in k : \sigma \cup t = \emptyset \} \cap \text{out} \in k \)

\( \text{cost}(k, \sigma) = \{ t \in k : \sigma \not\subseteq t \} \)
Some ideas from the proof

Thm (Tancer '10): $k$ is $d$-collapsible iff either dim$(k) < d$ or
Some ideas from the proof

**Thm (Tancer '10):** $K$ is $d$-collapsible

iff either $\dim(K) < d$ or

$\exists \sigma \in K$, $|\sigma| = d$, contained in unique
max face $T \not= \sigma$ and

$\text{cost}(K, \sigma)$ is $d$-collapsible
Some ideas from the proof

Let $K$ be d-col.
Some ideas from the proof

Let $K$ be $d$-col.

If $\dim(K) < d$ then

$$\dim(T_t^c(K)) < d + t < h(t, d)$$
Some ideas from the proof

Let $K$ be $d$-col.

If $\dim(K) < d$ then

$$\dim(T_t(t)) < d + t < h(t, d)$$

\[ \checkmark \]
Some ideas from the proof

Let $K$ be $d$-col.

If $\dim(K) < d$ then

$$\dim(T_t(K)) < d + t < h(t,d)$$

Otherwise: $\exists \sigma \in K$, $|\sigma| = d$, s.t. $\sigma$ is contained in unique max. face

$$t = \sigma U U (U \neq \emptyset), \text{ cost}(K, \sigma) \text{ d-col.}$$
Some ideas from the proof

We want to show: \( \tilde{H}_k(T_t(k)) = 0 \) for \( k \geq h(r, d) \).
Some ideas from the proof

We want to show: $\tilde{H}_k(\mathbb{T}_t(k)) = 0$ for $k \geq h(t, d)$.

Look out long exact seq. of the pair $\mathbb{T}_t(\text{cost}(k, 0)) \leq \mathbb{T}_t(k)$.
Some ideas from the proof

We want to show: \( \widetilde{H}_k(T_t(k)) = 0 \)

for \( k \geq h(t,d) \).

Look at long exact seq. of
the pair \( T_t(cost(k,\sigma)) \subseteq T_t(k) \):

\[ \cdots \rightarrow \widetilde{H}_k(T_t(cost(k,\sigma))) \rightarrow \widetilde{H}_k(T_t(k)) \rightarrow \]

\[ \rightarrow H_k(T_t(k), T_t(cost(k,\sigma))) \rightarrow \cdots \]
Some ideas from the proof

We want to show: \( \tilde{H}_k (T_t(k)) = 0 \)

for \( k \geq hr(t,d) \).

Look at long exact seq. of the pair \( (T_t, \text{cost}(k,\sigma)) \leq T_t(k) \):

\[
\cdots \to \tilde{H}_k (T_t(\text{cost}(k,\sigma))) \to \tilde{H}_k (T_t(k)) \to \]

by induct. \( \cong 0 \)

hypothesis

\[
\to \tilde{H}_k (T_t(k), T_t(\text{cost}(k,\sigma))) \to \cdots
\]
Some ideas from the proof

Prop:

\[ H_k(T_t(K), T_t(\text{cost}(k, \sigma))) \cong \]

\[ \bigoplus \tilde{H}_{k-d-1}(U T_{t-1}(\ll k(k, \sigma, \sigma') [\text{Uuw}] )) \]

\[ W \in V_1(\sigmaUU) \]

\[ \begin{array}{c}
\sigma' \cong \\
15\sigma' \leq t \\
1W1 = t
\end{array} \]
Some ideas from the proof

Prop:

\[ H_k(T_t(K), T_t(\text{cost}(K, \sigma))) \cong \]

\[ \bigoplus \widetilde{H}_{k-d-1}(U \cup T_{t-1} \cup \{k(k, \sigma, \sigma') [uvw]\}) \]

\[ w \in V1(\sigma uv) \]

\[ \|w\| = t \]

\[ d \text{-collapsible} \]

(Khmelevitsky '18)
Some ideas from the proof

Prop:

\[ H_k(T_t(K), T_t(\text{cost}(K, \sigma))) \cong \]

\[ \bigoplus H_{k-d-1}(U T_{t-\delta}(\text{lK}(K, \sigma, \sigma'), [UuW])) \]

\[ W \in V_1(\sigma U V) \]

\[ \|W\| = t \]

\[ h(t-\delta, \sigma, \sigma') \text{-Leray} \]

(by induction on \( t \))
Some ideas from the proof

Prop:

\[ H_k \left( \Gamma_t(K), \Gamma_t(\text{cost}(K, \sigma)) \right) \cong \]

\[ \bigoplus \widetilde{H}_{k-d-1} \left( \bigcup_{\sigma' \leq \sigma} \Gamma_{t-10^{-1}}(L(K(K, \sigma, \sigma'))[UwW]) \right) \]

\[ w \in V \setminus (\sigma \cup V) \]

\[ |w| = t \]

\[ \sum_{0 \leq \sigma' \leq \sigma} \bigoplus_{15 \leq t' \leq t} \left[ \langle \chi_{h(t-10^{-1}, d) + 1} \rangle - 1 \right] \] - Leray

\[ (Kawaribi-Meshulam '06) \]
Some ideas from the proof

**Prop:**

\[ H_k \left( \tau_t(K), \tau_t \left( \text{cost}(K, \sigma) \right) \right) \cong \bigoplus \tilde{H}_{k-d-1} \left( \bigcup_{\sigma' \leq \sigma} \bigcup_{101 \leq t} \left[ h(t,d) - d - 1 \right] \right) \]

\[ \text{W} \in \nu \setminus \nu \nu \]
\[ |W| = t \]

(Kovari-Meshulam '06)
Some ideas from the proof

Prop:

\[ H_k(T_t(K), T_t(\text{cost}(K, \sigma))) \cong \]

\[ \bigoplus H_{k-d-1}(U T_{t-1} \tau_{1,0} [k(K, \sigma, \sigma') [UuW]]) \]

\[ W \in V_1((\sigma \cup U)) \]

\[ \forall w \ni t \]

\[ \text{for } k \geq h(t, d) \]
Some ideas from the proof

Prop:

\[ H_k(T_t(K), T_t(\text{cost}(k, \sigma))) \equiv \]

\[ \bigoplus H_{k-d-1} \left( U T_{t-1}(lK(k, \sigma \cdot \sigma') [UuW]) \right) \]

\[ w \in V_1(\sigma uu) \quad \sigma', \sigma \leq \sigma \]

\[ \|w\|_t = t \]

\[ = 0 \quad \checkmark \]

for \( k \geq h(t, d) \)
A geometric application

**Thm (Kim, L. 21')**: Let $C_1, C_2, C_3, C_4, C_5, C_6$ be families of convex sets in the plane.
A geometric application

**Thm (Kim, L. 21'):** Let $C_1, C_2, C_3, C_4, C_5, C_6$ be families of convex sets in the plane. If every \{A_1, A_2, A_3, A_4, A_5, A_6\} has a pt. in common with Tolerance 1.
A geometric application

Thm (Kim, L. 21'): Let

$C_1, C_2, C_3, C_4, C_5, C_6$ be families of convex sets in the plane.

If every $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ has a pt. in common with Tolerance 1, then one of the $C_i$ has pt. in common with tol. 1
A geometric application

Thm (Kim, L. 21'): Let $C_i, C_j, C_k$...

Follows from application of Karlvi and Meshulam's topological colorful Helly theorem ('05).

To pt. in common with tolerance 1, then one of the $C_i$ has pt. in common with tol. 1.
THANKS FOR LISTENING!
Leray numbers of
Tolerance complexes

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Simplicial complexes

- $V$ = finite set
Simplicial complexes

- $V$ is a finite set
- $K \subseteq 2^V$ is called a simplicial complex if:
Simplicial complexes

- $V$ is a finite set
- $K \subseteq \mathcal{P}(V)$ is called a simplicial complex if:

$$
A \in K \implies \forall B \subseteq A, \quad B \in K
$$
Simplicial complexes

- $V$ = finite set
- $K \subseteq 2^V$ is called a simplicial complex if:
  \[ A \in K \implies \forall B \subseteq A, B \in K \]

- A set $A \in K$ is called a simplex of $K$
Simplicial complexes

- $V$ finite set
- $K \subseteq 2^V$ is called a simplicial complex if:

\[ A \in K \implies \forall B \in A, B \in K \]

- A set $A \in K$ is called a (face) simplex of $K$
Simplicial complexes

- $V$ is finite set
- $K \subseteq 2^V$ is called a simplicial complex if:
  
  $$A \in K \Rightarrow \forall B \subseteq A, B \in K$$

- A set $A \in K$ is called a \textit{(face) simplex of} $K$.
- The dimension of a simplex $A$ is $|A|-1$. 
Simplicial complexes

- \( V \) is a finite set
- \( K \subseteq 2^V \) is called a simplicial complex if:
  \[
  A \in K \implies \forall B \subseteq A, \quad B \in K
  \]

- A set \( A \in K \) is called a \((\text{face})\) simplex of \( K \).
- The dimension of a simplex \( A \) is \( |A| - 1 \).
- The dimension of \( K \) is the maximum dimension of a simplex.
Simplicial complexes

- We can view simp. complexes as geometric objects:
Simplicial complexes

- We can view simp. complexes as geometric objects:

\[ K = \]

\[ \begin{array}{ccc}
1 & 2 & 3 \\
4 & & \\
\end{array} \]
Simplicial complexes

- We can view simp. complexes as geometric objects:

\[ K = \begin{array}{ccc}
1 & 2 & 3 \\
& 4 \\
\end{array} \iff K = \begin{Bmatrix}
\{1,2,3\}, \{1,4\}, \\
\{1,2\}, \{13\}, \{2,3\}, \\
\{1\}, \{2\}, \{3\}, \\
14, \emptyset
\end{Bmatrix} \]
Simplicial complexes

- We can view simp. complexes as geometric objects:

\[ K = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array} \iff K = \begin{cases}
\{1,2,3\}, \{1,4\} \\
\{1,2,3\}, \{1,3\}, \{2,3\} \\
\{1,2\}, \{2,3\}, \{3\} \\
\{4\}, \emptyset
\end{cases} \]

- We can study the topology of a complex.
Homology

\[ \tilde{H}_i(K) = \text{\textit{i}-dimensional reduced homology group of } K \text{ (with real coefficients)} \]
Homology

1. $\tilde{H}_i(K) = \text{i-dimensional reduced homology group of } K \text{ (with real coefficients)}$

2. Informally: $\tilde{H}_i(K)$ counts "i-dimensional holes" in $K$. 
**Homology**

- $\tilde{H}_i(K) = \text{i-dimensional reduced homology group of } K \text{ (with real coefficients)}$

- Informally: $\tilde{H}_i(K)$ counts "i-dimensional holes" in $K$.

**E.g.**

$K = \begin{array}{c}
\text{pentagon}
\end{array}$
Homology

- $\tilde{H}_i(K)$ = $i$-dimensional reduced homology group of $K$ (with real coefficients)

- Informally: $\tilde{H}_i(K)$ counts "$i$-dimensional holes" in $K$.

E.g.

$K =$ \begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,0);
\end{tikzpicture}

$\tilde{H}_i(K) = \begin{cases} \mathbb{R} & i=1 \\ 0 & \text{otherwise} \end{cases}$
Homology

- \( \tilde{H}_i(K) \) is the \( i \)-dimensional reduced homology group of \( K \) (with real coefficients).

- Informally, \( \tilde{H}_i(K) \) counts "\( i \)-dimensional holes" in \( K \).

Example:

\[ K = \text{[diagram of a complex shape]} \]
Homology

- $\tilde{H}_i(K)$: $i$-dimensional reduced homology group of $K$ (with real coefficients)

- Informally: $\tilde{H}_i(K)$ counts "$i$-dimensional holes" in $K$.

Example:

$K = \begin{array}{c}
\begin{array}{c}
\text{Diagram of a 2-dimensional manifold}
\end{array}
\end{array}$

$\tilde{H}_i(K) = \begin{cases}
\mathbb{R} & \text{if } i=2 \\
0 & \text{otherwise}
\end{cases}$
Homology

- $\tilde{H}_i(K) =$ i-dimensional reduced homology group of $K$ (with real coefficients)

- Informally: $\tilde{H}_i(K)$ counts "i-dimensional holes" in $K$.

E.g.

$d$-dimensional $K = sphere$
Homology

- $\tilde{H}_i(K)$ = $i$-dimensional reduced homology group of $K$ (with real coefficients)

- Informally: $\tilde{H}_i(K)$ counts "$i$-dimensional holes" in $K$.

E.g. $d$-dimensional sphere $K = \{ x \in \mathbb{R}^d : \|x\| = r \}$

$\tilde{H}_i(K) = \begin{cases} \mathbb{R}^d & i = d \\ 0 & \text{otherwise} \end{cases}$
Leray numbers

- \( K = \text{simp. complex on vertex set } V \)
Leray numbers

\[ K = \text{simp. complex on vertex set } V \]

For \( U \subseteq V \),

\[ K[U] = \{ \sigma \subseteq U : \sigma \in K \} \]
Leray numbers

- $K_\ast$ simp. complex on vertex set $V$
- For $U \subseteq V$,

$$K[U] = \{ \sigma \subseteq U : \sigma \in K \}$$

(subcomplex induced by $U$)
Leray numbers

- $K = \text{simp. complex on vertex set } V$
- For $U \subseteq V$, $K[U] = \{ \sigma \subseteq U : \sigma \in K \}$ (subcomplex induced by $U$)
Leray numbers

\[ K = \text{simp. complex on vertex set } V \]

- For \( U \subseteq V \),
  \[ K[U] = \{ \sigma \subseteq U : \sigma \in K \} \]
  (subcomplex induced by \( U \))
Levy numbers

- $K =$ simp. complex on vertex set $V$
- For $U \subseteq V$, $K[U] = \{ \sigma \subseteq U : \sigma \in K \}$ (subcomplex induced by $U$)
Leray numbers

- $K$ = simp. complex on vertex set $V$

- For $U \subseteq V$, $\tilde{K}[U] = \{ \sigma \subseteq U : \sigma \in K \}$ (subcomplex induced by $U$)

- $K$ is $d$-Leray if $\tilde{H}_i(\tilde{K}[U]) = 0$ for all $i \geq d$ and $U \subseteq V$. 
Leray numbers

- $K = \text{simp. complex on vertex set } V$
- For $U \subseteq V$,
  $K[U] = \{ \sigma \subseteq U : \sigma \in K \}$
  (subcomplex induced by $U$)
- $K$ is $d$-Leray if
  $\tilde{H}_i(K[U]) = 0$ for all $i \geq d$ and $U \subseteq V$. 
Leray numbers

- $K = \text{simp. complex on } V$
- For $U \subseteq V,$
  \[ K[U] = \{ \sigma \subseteq U : \sigma \in K \} \]  
  (subcomplex induced by $U$)
- $K$ is $d$-Leray if
  \[ H_i(K[U]) = 0 \text{ for all } i \geq d \text{ and } U \subseteq V. \]
Collapsibility

Let $\sigma \in K$ s.t. $10 \leq d$
Collapsibility

Let $o \in K$ s.t. $1 \leq d$ and $o$ is contained in unique maximal face $t \in K$
Collapsibility

Let $o \in K$ st. $1 \leq d$ and $o$ is contained in unique maximal face $T \subseteq K$
Collapsibility

Let \( \sigma \in K \) st. \( 1 \leq d \) and \( \sigma \) is contained in unique maximal face \( t \in K \)

Elementary \( d \)-collapse:
Collapsibility

Let $o \in K$ st. $10 \leq d$ and $o$ is contained in unique maximal face $t \in K$

Elementary $d$-collapse:
Collapsibility

Let $o \in K$ s.t. $10 \leq d$

and $o$ is contained in unique maximal face $t \in K$

Elementary $d$-collapse:
Collapsibility

Let $o \in K$ st. $1 \leq d$ and $o$ is contained in unique maximal face $t \in K$

Elementary $d$-collapse:

- If $E$ sequence of elem. $d$-coll.
  from $K$ to $\emptyset$: $K$ is $d$-collapsible
Collapsibility

E.g.

\[ K = \]
Collapsibility

E.g.

$k = \quad \begin{array}{c}
\begin{array}{c}
\text{Not} \\
\text{1-collapsible}
\end{array}
\end{array}$
Collapsibility

E.g.

\[ K = \]

we will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = \sigma_i \]

we will show that

K is 2-collapsible.
Collapsibility

E.g.

\[ K = \sigma_i, \tau_i \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K_1 = \]

we will show that

\( K \) is 2-collapsible.
Colapsibility

E.g.

\[ K_1 = \]

We will show that \( K \) is 2-collapsible.
Collapsibility

\[ K_1 = \]

\[ \tau_2 \]

\[ \sigma_2 \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K_2 \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = k_2 \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = \tau_3 \]

we will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = K_3 \]

we will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K_3 = \begin{array}{c}
\delta_4
\end{array} \]

we will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = \begin{array}{c}
\sigma_4 = \tau_4
\end{array} \]

we will show that

\[ K \text{ is 2-collapsible.} \]
Collapsibility

E.g.

\[ K = \]

we will show that 

\[ K \] is 2-collapsible.
Collapsibility

E.g.

\[ K = \sigma_5^u \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = \]

we will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

$k = 5$

we will show that $K$ is 2-collapsible.
Collapsibility

E.g.

\[ K = 5 \]

we will show that

\[ K \] is 2-collapsible.
Collapsibility

E.g.

\[ K_5 \]

\[ \sigma'_6 = \emptyset \]

\[ \tau_6 \]

We will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = K_6 \]

we will show that \( K \) is 2-collapsible.
Collapsibility

E.g.

\( K = \)

\( K \) is 2-collapsible.
Collapsibility

E.g.

\[ K = \quad K \text{ is } 2\text{-collapsible}. \]

- Collapsibility of \( K = \) minimal \( d \) s.t. \( K \) is \( d \)-collapsible.
Collapsibility

E.g.

\[ K = \quad K \text{ is } 2\text{-collapsible}. \]

- \[ C(K) \]

\text{Collapsibility of } K = \text{minimal } d \text{ s.t. } K \text{ is } d\text{-collapsible.}
Collapsibility

E.g.

\[ K = \]

\[ K \text{ is } 2\text{-collapsible.} \]

\[ C(K) = 2 \]

\[ C(K) \]

\[ \text{Collapsibility of } K = \]

minimal \( d \) \( \text{s.t. } K \text{ is } d\text{-collapsible.} \)
Representability

\[ F = \{F_1, F_2, \ldots, F_n\} \] family of sets.
Representability

- $F = \{F_1, F_2, \ldots, F_n\}$ family of sets.
- $N(F) = \text{nerve of } F$.
Representability

- $F = \{ F_1, F_2, \ldots, F_n \}$ family of sets.
- $N(F) =$ nerve of $F$:
  - A vertex is assigned to each set in the family.
Representability

$ F = \{ F_1, F_2, \ldots, F_n \} $ family of sets.

$ N(F) = \text{nerve of } F $:

- A vertex is assigned to each set in the family.
- Simplices correspond to subfamilies with non-empty intersection.
Representability

E.g.

\[ F = \]
Representability

E.g.

\[ F = \quad N(F) = \]
Representability

E.g.

\[ F = \quad \quad N(F) = \]
Representability

E.g.,

\[ F = \quad N(F) = \]

\begin{align*}
\text{\includegraphics{example1.png}}
\end{align*}
Representability

E.g.

\[ F = \]  

\[ N(F) = \]
A d-nerve representable complex = nerve of a family of convex sets in Rd.

\[ N(F) = \text{?} \]

Example:

Representability
Representability

E.g.

\[ F = \]

2-representable

\[ N(F) = \]

- \( d \)-representable complex = nerve of a family of convex sets in \( \mathbb{R}^d \)
$K$ is $d$-representable
$K$ is $d$-collapsible

$K$ is $d$-representable
Wegner's Theorem (75) \[ K \text{ is } d\text{-collapsible} \]
\[ K \text{ is } d\text{-representable} \]
\[ K \text{ is } d\text{-Leray} \]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets.
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common,
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i=1, \ldots, m$).
Helly's Theorem

Let $C_1, ..., C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i = 1, ..., m$).

E.g.

\[
\begin{align*}
&d=1 \\
&\mathbb{R}
\end{align*}
\]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ $(i=1, \ldots, m)$.

E.g.

\[ d=1 \]

\[ \text{IR} \]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ (i=1, ..., m).

E.g.

\[ d=1 \]
Helly's Theorem

Let $C_1,\ldots,C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i=1,\ldots,m$).

E.g.

\[ d=1 \quad \text{\begin{tikzpicture}
\draw[->] (-1,0) -- (1,0) node[right] {$\mathbb{R}$};
\foreach \i in {1,2,3,4,5,6}
\draw[red, line width=1pt] (0,0.1) -- (0,-0.1);
\end{tikzpicture}} \]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ $(i=1, \ldots, m)$.

E.g.

d=1

\[ \mathbb{R} \]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i=1, \ldots, m$).

E.g.

\begin{align*}
d=1 & \quad \begin{array}{c}
\text{1D} \\
\text{Line segments}
\end{array} \\
\text{d=2} & \quad \begin{array}{c}
\text{2D} \\
\text{Intersecting regions}
\end{array}
\end{align*}
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ $(i=1, \ldots, m)$.

E.g.

\[ d=1 \quad d=2 \]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i = 1, \ldots, m$).

E.g.

\[ d=1 \begin{array}{c}
\text{line} \\
\text{1D} \\
\text{1 point}
\end{array} \quad \begin{array}{c}
\text{plane} \\
\text{2D} \\
\text{2 point}
\end{array} \]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i = 1, \ldots, m$).

E.g.

$d=1$  \hspace{2cm} d=2
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i = 1, \ldots, m$).

E.g.

\[
\begin{align*}
&d = 1 \\
&d = 2
\end{align*}
\]
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i=1, \ldots, m$).

E.g.

$$d=1 \quad \text{and} \quad d=2$$
Helly's Theorem

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be convex sets. If every $d+1$ of these sets have a point in common, then there is a point in common to all $C_i$ ($i = 1, \ldots, m$).

E.g.


d = 1
d = 2

not convex!
Missing faces

- $K =$ simp. complex on vertex set $V$. 
Missing faces

- K = simp. complex on vertex set V.
- A missing face is a set \( \mathcal{T} \subseteq V \)

s.t.: 
Missing faces

- $K = \text{simp. complex on vertex set } V$.
- A missing face is a set $\tau \subseteq V$ s.t.: $\circ \ 1 \ \tau \notin K$
Missing faces

- $K = \text{simp. complex on vertex set } V$.
- A missing face is a set $\tau \subseteq V$ s.t: 1. $\tau \notin K$, 2. $\sigma \in K \forall \sigma \notin \tau$
Missing faces

- $K = \text{simp. complex on vertex set } V.$
- A missing face is a set $\tau \subseteq V$ st: 1) $\tau \notin K$, 2) $\sigma \in K \forall \sigma \subseteq \tau$

E.g.: $K = \triangle$
Missing faces

- $K = \text{ simp. complex on vertex set } V.$
- A missing face is a set $\tau \subseteq V$ s.t.: ① $\tau \not\subseteq K$, ② $\sigma \in K \setminus \forall \sigma \not\in \tau$

E.g.

$K = \begin{array}{c}
\begin{array}{c}
\text{Red triangle}
\end{array}
\end{array}$
**Missing faces**

- $K = \text{simp. complex on vertex set } V.$
- A **missing face** is a set $T \subseteq V$ s.t.:  
  1. $T \notin K,$  
  2. $\sigma \in K \forall \sigma \notin T$

**E.g.**

$K = \begin{array}{c}
\text{triangle} \\
\text{star}
\end{array}$
Missing faces

- \( K = \) simp. complex on vertex set \( V \).
- A **missing face** is a set \( \tau \subseteq V \)
  such that:
  1. \( \tau \notin K \),
  2. \( \sigma \in K \) for all \( \sigma \subseteq \tau \).

**E.g.**

\( K = \)
Missing faces

- $K$ = simp. complex on vertex set $V$.
- A missing face is a set $T \subseteq V$ s.t.: 
  1. $T \notin K$, 
  2. $S \in K \ \forall S \subseteq T$

Ex.

$K = \begin{array}{c}
\text{triangle} \\ \text{triangle} \\
\text{triangle} \ \ \text{triangle}
\end{array}$
Missing faces

- $K =$ simp. complex on vertex set $V$
- A missing face is a set $\tau \subseteq V$
  s.t.: 1. $\tau \notin K$, 2. $\sigma \in K \quad \forall \sigma \in \tau$

E.g.

$K =$ 

\[ \begin{array}{c}
\text{E.g.}\n\end{array} \]

\[ K = \]
Missing faces

- \( K \) = simp. complex on vertex set \( V \).
- A missing face is a set \( \tau \subseteq V \) s.t.: ① \( \tau \notin K \), ② \( \sigma \in K \) \( \forall \sigma \notin \tau \)

E.g.

\( K = \)
Missing faces

- $K = \text{simp. complex on vertex set } V$
- A missing face is a set $T \subseteq V$ s.t.:  
  1. $T \notin K$,
  2. $\sigma \in K \forall \sigma \notin T$

E.g.

$K = \begin{array}{c}
\text{vertex set } V = \{1, 2, 3, 4\}, \\
\text{edge set } E = \{(1, 2), (2, 3), (3, 4), (4, 1), (1, 3), (2, 4)\}
\end{array}$
Missing faces

- $K$ = simp. complex on vertex set $V$.
- A missing face is a set $\tau \subset V$ s.t.:
  1. $\tau \notin K$,
  2. $\sigma \in K \implies \forall \sigma \not\in \tau$

E.g.

$K = \begin{array}{c}
\text{ } \text{ } \\
\text{ } \text{ } \\
\text{ } \text{ } \\
\text{ } \text{ } \\
\end{array}$
Missing faces

- \( K = \text{ simp. complex on vertex set } V \).
- A missing face is a set \( \tau \subseteq V \)
  s.t.:
  1. \( \tau \notin K \),
  2. \( \sigma \in K \) \( \forall \sigma \in \tau \)

E.g.

\[ K = \triangle \]
Missing faces

- $K = \text{simp. complex on vertex set } V$.
- A missing face is a set $\mathcal{T} \subseteq V$ s.t.: 
  1. $\mathcal{T} \notin K$,
  2. $\sigma \in K$ \quad $\forall \sigma \nsubseteq \mathcal{T}$

E.g.

$K = \begin{array}{c}
\begin{array}{c}
\text{triangle}
\end{array}
\end{array}$

$h(K) =$ maximum dimension of a missing face in $K$
Missing faces

- $K = \text{simp. complex on vertex set } V$.

- A missing face is a set $\mathcal{T} \subseteq V$ s.t.: (1) $\mathcal{T} \notin K$, (2) $\sigma \in K \forall \sigma \subseteq \mathcal{T}$

**E.g.**

$K = \begin{array}{c}
\begin{array}{c}
\bullet \\
\text{h(K) = 2}
\end{array}
\end{array}$

$h(K) =$ maximum dimension of a missing face in $K$
Helly in terms of missing faces

- Helly's Thm is equivalent to:
Helly in terms of missing faces

- Helly's Thm is equivalent to:

  **Thm:** If $K$ is $d$-representable, then $h(K) \leq d$. 
Helly for d-Leray complexes

Thm: If $K$ is $d$-Leray, then $h(K) \leq d$. 
Erdős-Gallai numbers

- \( H \leq 2 \) family of sets.
Erdős-Gallai numbers

- \( H \leq 2 \) family of sets.
- A set \( C \subseteq V \) covers \( H \) if \( A \cap C \neq \emptyset \) for all \( A \in H \).
Erdős-Gallai numbers

- $H \leq 2$ family of sets.
- A set $C \subseteq V$ covers $H$ if $\forall A \in H, A \cap C \neq \emptyset$ for all $A \in H$.
- $\Gamma(H) =$ minimum size of cover.
Erdős-Gallai numbers

- $H \subseteq \mathcal{P}$ family of sets.
- A set $C \subseteq V$ covers $H$ if $\bigcup_{A \in H} A \cap C \neq \emptyset$ for all $A \in H$.

$\mathcal{I}(H) =$ minimum size of cover
Erdős-Gallai numbers

- $H \subseteq \mathcal{V}$ family of sets.
- A set $C \subseteq \mathcal{V}$ covers $H$ if $A \cup C \neq \emptyset$ for all $A \in H$
- $\Gamma(H) =$ minimum size of cover
Erdös–Gallai numbers

- $\mathcal{H} \leq 2^\mathcal{V}$ family of sets.
- A set $C \subseteq \mathcal{V}$ covers $\mathcal{H}$ if $\bigcap_{A \in \mathcal{H}} C \neq \emptyset$ for all $A \in \mathcal{H}$

- $\tau(\mathcal{H}) =$ minimum size of cover
Erdős–Gallai numbers

$\varepsilon(r, t) = \text{minimum } m \text{ such that:}$
Erdős-Gallai numbers

\[ \ell(r,t) = \text{minimum } m \text{ such that:} \]

A family \( H \) of sets of size \( \leq r \) each, if \( \ell(H') \leq t-1 \) for any \( H' \subseteq H \) with \( \bigcup H' \leq m \), then \( \ell(H) \leq t-1 \).
Erdős-Gallai numbers

\( \zeta(r, t) = \text{minimum } m \text{ such that:} \)

A family \( \mathcal{H} \) of sets of size \( \leq r \) each, if \( \zeta(\mathcal{H}) \leq t-1 \) for any \( \mathcal{H}' \subseteq \mathcal{H} \) with \( |\bigcup \mathcal{H}'| \leq m \), then \( \zeta(\mathcal{H}) \leq t-1 \).

Erdős-Gallai ('61)
Erdős–Gallai numbers

\( \eta(r, t) = \min \{ m \mid \text{there exists a family } H \text{ of sets of size } \leq t \text{ each, if } \sum H' \leq t-1 \text{ for any } H' \subseteq H \text{ with } \sum H' \leq m, \text{ then } \tau(H) \leq t-1 \}. \)

Erdős–Gallai (’61)

\( \eta(2, t) = 2t \)
Erdős–Gallai numbers

\( \eta(r,t) \) = minimum \( m \) such that:

A family \( H \) of sets of size \( \leq r \) each, is \( \tau(H') \leq t-1 \) for any \( H' \subseteq H \) with \( \#H' \leq m \), then \( \tau(H) \leq t-1 \).

Erdős–Gallai (’61)

- \( \eta(2,t) = 2t \)
- \( \eta(1,t) = \left\lceil \left( \frac{t+2}{2} \right)^2 \right\rceil \)
Erdős-Gallai numbers

\[ \eta(r, t) = \text{minimum } m \text{ such that:} \]

A family \( H \) of sets of size \( \leq r \) each, if \( \tau(H') \leq t-1 \) for any \( H' \subseteq H \) with \( \bigcup H' \leq m \), then \( \tau(H) \leq t-1 \).

Erdős-Gallai (’61)

- \( \eta(2, t) = 2t \)
- \( \eta(r, 2) = \left\lceil \left( \frac{r+2}{2} \right)^2 \right\rceil \)

Tuza (’88):

- \( \eta(r, t) < \left( \frac{r+t-1}{r-1} \right) \left( \frac{r+t-2}{r-1} \right) \)
Erdős-Gallai numbers

$\eta(r, t) = \text{minimum } m \text{ such that:}$

A family $H$ of sets of size $\leq t$ each, if $\tau(H') \leq t - 1$ for any $H' \subseteq H$ with $|\cup H'| \leq m$, then $\tau(H) \leq t - 1$.

Erdős-Gallai ('61)
- $\eta(2, 2) = 2t$
- $\eta(r, 2) = \left\lfloor \left( \frac{r+2}{2} \right)^2 \right\rfloor$

Tuza ('88):
- $\eta(r, t) < \binom{r+t-1}{t-1} + \binom{r+t-2}{t-1}$
- $\eta(r, t) = O(r^t)$ for fixed $t$
Erdős-Gallai numbers

$$\gamma(r, t) = \text{minimum } m \text{ such that: }$$

A family $H$ of sets of size $\leq t$ each, if $\tau(H') \leq t-1$ for any $H' \subseteq H$ with $|UH'| \leq m$, then $\tau(H) \leq t-1$.

**Erdős-Gallai ('61)**
- $\gamma(2, t) = 2t$
- $\gamma(r, 2) = \left\lceil \frac{(r+2)^2}{2} \right\rceil$

**Tuza ('88):**
- $\gamma(r, t) < \binom{r+t-1}{r-1} + \binom{r+t-2}{r-1}$

$\gamma(r, t) = O(t^2)$ for fixed $t$

$\gamma(r, t) = O(t^{\log_{r+2} r})$ for fixed $r$
Helly with tolerance

- $F =$ family of sets.
Helly with tolerance

- $F =$ family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\forall F' \neq \emptyset$. 
Helly with tolerance

- $F$ = family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\cap F' \neq \emptyset$.

E.g. $F =$
Helly with tolerance

- $F$ = family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\cap F' \neq \emptyset$.

E.g. $F =$ \[
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{example1.png}}
\end{array}
\]
$\cap F = \emptyset$
Helly with tolerance

- $F =$ family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\bigcap F' \neq \emptyset$.

E.g. $F =$

$\bigcap F = \emptyset$
Helly with tolerance

- $F$ = family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t.
  $|F'| \geq |F| - t$ and $\forall F' \neq \emptyset$.

E.g.

$F' = \{\text{intersecting shapes}\}$

$|F'| = |F| - 2$
Helly with tolerance

- $F = \text{family of sets}$.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\bigcap F' \neq \emptyset$.

E.g.

$F' = \{\text{red, green, blue shapes}\}$

$|F'| = |F| - 2$

$\bigcap F' \neq \emptyset$
Helly with tolerance

- $F$ = family of sets.
- $F$ has a point in common with tolerance $t$ if $\exists F' \subseteq F$ s.t. $|F'| \geq |F| - t$ and $\bigcap F' \neq \emptyset$.

E.g.: $F' = \{ \text{red}, \text{green}, \text{blue} \}$.

So, $F$ has a point in common with tolerance 2.
Helly with tolerance

Thm (Montejano-Oliveros '10):
Helly with tolerance

Thm (Montejano-Oliveros '10):

Let $F$ be a finite family of convex sets in $\mathbb{R}^d$. 
Helly with tolerance

Thm (Montejano-Oliveros '10):

Let $F$ be a finite family of convex sets in $\mathbb{R}^d$.

If any $F' \subseteq F$ of size $\aleph_0 (d+1, t+1)$ has a point in common with tolerance $t$, 

Helly with tolerance

Thm (Montejano-Oliveros '10):

Let $F$ be a finite family of convex sets in $\mathbb{R}^d$.

If any $F' \subseteq F$ of size $\gamma(d+1,t+1)$ has a point in common with tolerance $t$, then $F$ has a point in common with tolerance $t$. 
Tolerance Complexes

K = simp. complex on vertex set V.
Tolerance Complexes

* $K = \text{simp. complex on vertex set } V.$
* $t \geq 0 \text{ an integer.}$
Tolerance Complexes

- $K = \text{ simp. complex on vertex set } V$.
- $t \geq 0$ an integer.
- The $T$-tolerance complex of $K$: 
Tolerance Complexes

- $K =$ simp. complex on vertex set $V$.
- $t \geq 0$ an integer.

The $T$-tolerance complex of $K$:

$$T_t(K) = \left\{ \sigma \subseteq V \mid \exists \sigma' \leq \sigma \quad \left| 10^{-1} \sigma' \right| \geq 10^{-1} - t \right\}$$
Tolerance Complexes

- $K =$ simp. complex on vertex set $V$.
- $t \geq 0$ an integer.
- The $t$-tolerance complex of $K$:

$$\mathcal{T}_t(K) = \left\{ \sigma \subseteq V \mid \exists \sigma' \leq \sigma \quad \forall \sigma' \in K \quad \left( |\sigma'| \geq |\sigma| - t \right) \right\}$$

E.g.

$$K = \begin{array}{c}
\bullet \\
\bullet \\
\end{array}$$
Tolerance Complexes

- \( K = \text{simp. complex on vertex set } V. \)
- \( t \geq 0 \text{ an integer.} \)
- The \( T \)-tolerance complex of \( K \):

\[
\mathcal{T}_t(K) = \left\{ \sigma \subseteq V \mid \exists \sigma' \subseteq \sigma \exists \sigma' \in K \exists \sigma : 10^{-1} \geq |\sigma' - t| \sigma' \subseteq \sigma \right\}
\]

E.g.

\[ K = \cdot \quad \cdot \quad \mathcal{T}_t(K) = \]

Diagram of \( K \) and \( \mathcal{T}_t(K) \).
Tolerance Complexes

- \( K = \{ A : A \subseteq V, \exists x \in V \text{ s.t. } x \not\in A \} \)

- \( F = \) family of sets
Tolerance Complexes

$K = \{ U \subseteq V \mid U \text{ is a finite set} \}$

$F = \text{family of sets}$

$\tau_t(N(F)) = \{ F' \subseteq F \mid F' \text{ has pt. in common with } \text{tolerance } t \}$
Helly's property for tolerance complexes

Thm (Montejano-Oliveros '10):

If \( K \) is \( d \)-representable, then

\[
h(T_e(K)) \leq h(d+1, t+1) - 1
\]
Helly's property for tolerance complexes

Thm (Montejano-Oliveros '10):
If $h(K) \leq d$, then
$h(T_t(K)) \leq h(d+1, t+1) - 1$
Helly's property for tolerance complexes

Thm (Montejano-Oliveros '10):

If \( h(K) \leq d \), then

\[
h(T(K)) \leq h(d+1,t+1) - 1
\]

- If we assume \( K \) is \( d \)-collapsible/d-Leray, can we obtain a stronger conclusion?
Collapsibility and Leray numbers of $\mathcal{J}_t(K)$

Conjecture: If $K$ is $d$-Leray, then $\mathcal{J}_t(K)$ is $(h(d+1, t+1) - 1)$-Leray.
Collapsibility and Leray numbers of $\mathcal{I}_t(K)$

Conjecture: If $K$ is $d$-Leray, then
$\mathcal{I}_t(K)$ is $(h(d+1, t+1) - 1)$-Leray.

Conjecture: If $K$ is $d$-collapsible, then
$\mathcal{I}_t(K)$ is $(h(d+1, t+1) - 1)$-collapsible.
Extremal examples

$C =$ facets of $d$-dim simplex,
$t+1$ copies of each.
Extremal examples

$C = \text{facets of } d\text{-dim simplex, }$

$t+1 \text{ copies of each.}$

$C = \begin{array}{c}
\text{.}
\end{array}$

\begin{array}{c}
\text{.}
\end{array}$
Extremal examples

\[ C = \text{facets of } d\text{-dim simplex,} \]
\[ t+1 \text{ copies of each.} \]

\[ C = \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\end{array} \]

\[ t+1 \]

\[ t+1 \]

\[ N(C) = d\text{-representable} \]
Extremal examples

$C =$ facets of $d$-dim simplex,
    $t+1$ copies of each.

$C = \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} $

$N(C) = d$-representable ($d$-collapsible)
Extremal examples

\[ C = \text{facets of } d\text{-dim simplex, } t+1 \text{ copies of each.} \]

\[ T_t(N(C)) = ? \]

\[ C = \begin{array}{c}
\text{t+1} \\
\text{t+1} \\
\text{t+1}
\end{array} \]

\[ N(C) = d\text{-representable (d-collapsible)} \]

\[ d\text{-Leray} \]
Extremal examples

$C = \text{ facets of } d\text{-dim simplex, }$ $t+1 \text{ copies of each.}$ $T_t(N(C)) = ?$

$C \setminus \{a\} = \begin{array}{c}
  t \\
  t+1 \\
  t+1 
\end{array}$

$N(C) = d\text{-representable (d-collapsible/d-Leray)}$
Extremal examples

$C = \text{ facets of } d\text{-dim simplex, } t+1 \text{ copies of each. } T_t(N(C)) = ?$

$C \setminus \{a\} = \begin{array}{c}
\text{ } \\
\text{ } \\
t \times \\
t+1 \\
t+1 \\
\end{array}$

$N(C) = d\text{-representable (d-collapsible)}$

$N(C) = d\text{-acyclic}$
Extremal examples

\( C = \text{facets of } d\text{-dim simplex, } t+1 \text{ copies of each. } \)

\( \mathcal{T}_t(N(C)) = ? \)

\( C \setminus \{A\} = \begin{array}{c}
\times \\
(0,0,0) \quad (1,1,1) \\
(1,1,0) \\
\end{array} \)

\( N(C) = \text{d-representable (d-collapsible)} \)

\( \text{d-Leray} \)
Extremal examples

\( C = \) facets of \( d \)-dim simplex, \( t+1 \) copies of each.

\[ C \setminus \{a\} = \begin{array}{c}
\text{ } \\
\text{ } \\
t+1 \\
t+1
\end{array} \]

\( \mathcal{T}_t(N(C)) \) = boundary of \((d+1)(t+1)-1\)-dim. Simplex

\( N(C) = d \)-representable (\( d \)-collapsible, \( d \)-Leray)
Extremal examples

\( C = \text{facets of } d\text{-dim simplex,}\)
\( t+1 \text{ copies of each.} \)

\[ C \setminus \{A\} = \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
t+1
\end{array} \]

\[ T_t(N(C)) = (d+1)(t+1)-2\text{-dim. sphere} \]

\( N(C) = \text{d-representable (d-collapsible)} \)
Extremal Examples

$C = \text{facets of d-dim simplex, t+1 copies of each.}$

$C \setminus \{a\}$

$\implies$ 

$N(C) = d$-representable

$\implies$ $T_t(N(C))$

$= (d+1)(t+1)-\text{dim. sphere}$

$s_{2(d+1)(t+1)-2} \text{-Leray}$

$\implies$ $T_t(N(C))$ is \textbf{NOT} $s_{(d+1)(t+1)-2} \text{-Leray}$
Extremal examples

$C =$ facets of $d$-dim simplex, $t+1$ copies of each.

$N(C) =$ $d$-representable

$T_t(N(C)) = (\binom{d+1}{t+1}(t+1)-2)$-dim. sphere

$\downarrow$

$T_t(N(C))$ is $\textbf{NOT}$ $(\binom{d+1}{t+1}(t+1)-2)$-Leray

- For $d=1$, $\binom{d+1}{t+1}(t+1)-2 = 2t$
Extremal examples

\[ C = \text{facets of } d\text{-dim simplex, } \]
\[ t+1 \text{ copies of each.} \]

\[ C \setminus A = \begin{array}{c}
\text{t} \\
\text{t+1} \\
\text{t+1}
\end{array} \]

\[ N(C) = d\text{-representable (d-collapsible)} \]
\[ N(C) \text{ is NOT (d-}\text{Leray}) \]

\[ T_t(N(C)) = (d+1)(t+1)-2 \text{-dim. sphere} \]

\[ T_t(N(C)) \text{ is NOT (d+1)(t+1)-Leray} \]

- For \( d=1 \), \( (d+1)(C+1)-2 = 2t = \gamma(2, t+1)-2 \)
Extremal examples

Montejano-Oliveros ('10):

∃ C ∈ family of convex sets in \(R^d\)

st. \(\mathcal{L}(N(C))\) is not \(\left(\frac{d+3}{2}\right)^2\)-Leray

• For \(d=1\), \(\Delta(d+1)(e+1)^{-2} = 2 e = \zeta(2, e+1) - 2\)
Extremal Examples

Montejano-Oliveros (’10):

∃ C family of convex sets in \( \mathbb{R}^d \)

st. \( \mathcal{V}(N(C)) \) is not \( \left[ \left( \frac{d+3}{2} \right)^2 \right] - 2 \)-Leray

- For \( d=1 \), \( \left( d+1 \right) \left( c+1 \right) - 2 = 2 t = \zeta \left( 2, t+1 \right) - 2 
- t = 1: \left( \frac{d+3}{2} \right)^2 - 2 = \zeta \left( d+1, 2 \right) - 2 \)
Main results

\[ h(t,d) = \begin{cases} 
   d & ; t=0 \\
   \left\lfloor \sum_{s=0}^{\min(t,d)} (d) (h(t-s,d) + 1) \right\rfloor + d & ; t>0 
\end{cases} \]
Main results

\[ h(t,d) = \begin{cases} 
  d & ; t = 0 \\
  \left[ \sum_{s=0}^{\min\{t,d\}} (d)(h(t-s,d)+1) \right] + d & ; t > 0
\end{cases} \]

Thm (Kim-L. ’21):

Let \( K \) be a \( d \)-collapsible complex. Then, \( T_t(K) \) is \( h(t,d) \)-Leray.
Main results

\[ h(t,d) = \begin{cases} 
  d & ; t = 0 \\
  \left[ \sum_{s=1}^{\min\{t,d\}} (d)(h(t-s,d)+1) \right] + d & ; t > 0 
\end{cases} \]

Thm (Kim-L. '21):

Let \( K \) be a \( d \)-collapsible complex. Then, \( T_t(K) \) is \( h(t,d) \)-Leray.
Main results

\[ h(t,d) = \begin{cases} 
  d 
  & ; t = 0 
  
  \left\lfloor \frac{\min \{t, d\}^2}{\sum_{s=1}^{\min \{t, d\}} (h(t-s,d)+1) \right\rfloor + d ; t > 0 
\end{cases} \]

Thm (Kim-L. '21):
Let \( K \) be a \( d \)-collapsible complex. Then, \( T^t(K) \) is \( h(t,d) \)-Leray.

\[
\begin{align*}
  h(t,1) &= 2t + 1 \\
  h(1,d) &= d^2 + 2d
\end{align*}
\]
Main results

\[ h(t,d) = \begin{cases} d \left\lfloor \sum_{s=0}^{\min\{t,d\}} (s)(h(t-s,d)+1) \right\rfloor + d & ; t > 0 \\ 0 & ; t = 0 \end{cases} \]

Thm (Kim-L. '21):

Let \( K \) be a \( d \)-collapsible complex. Then, \( T_t(K) \) is \( h(t,d) \)-Leray.

For fixed \( t \),

\[ h(t,d) = 2t + 1 \]
\[ h(1,d) = d^2 + 2d \]
\[ h(t,d) = O(d^{t+1}) \]
Main results

- For $t=1$, we obtained:
Main results

- For $t=1$, we obtained:

$$K \Rightarrow \text{d-collapsible}$$

$$\Downarrow$$

$$J_t(K) \text{ (d}^2 + 2d\text{)-Leray}$$
Main results

* For \( t = 1 \), we obtained:

\[ K \text{ d-collapsible} \]

\[ \Downarrow \]

\[ \left\lfloor \left( \frac{d+3}{2} \right) \right\rfloor - 1 \]

\[ T_1(K) \text{ (d}^2 + 2d)\text{-Leray} \]
Main results

* For $t=1$, we obtained:
  \[ K \text{ d-collapsible} \]
  \[ ? \downarrow \left\lfloor \frac{(d+3)^3}{2} \right\rfloor - 1 \]
  \[ \mathcal{J}_t(K) (d^2 + 2d) \text{-Leray} \]

* True for $d=2$: 
Main results

- For $t = 1$, we obtained:

  $K$ $d$-collapsible

  $\Downarrow$ 

  $\left\lfloor \frac{d+3}{2} \right\rfloor - 1$

  $T_1(K)$ $(d^2 + 2d)$-Leray

- True for $d = 2$:

  Thm (Kim-L. '21): Let $K$ be 2-collapsible. Then, $T_1(K)$ is 5-Leray.
Some ideas from the proof

Some definitions:
Some ideas from the proof

Some definitions:

\( k = \text{simp. complex on } V, \sigma \in K \)
Some ideas from the proof

Some definitions:

\( k = \text{simp. complex on } V \), \( \sigma \in k \)

\( \mathcal{H}(K, \sigma) = \{ t \in K : \sigma \cap t = \emptyset \} \)
Some ideas from the proof

Some definitions:

- $k$ = simp. complex on $V$. $\sigma \in k$

- $\mathcal{L}k(k, \sigma) = \{ t \in k : \sigma \cup \tau = \emptyset \}$

- $\text{cost}(k, \sigma) = \{ t \in k : \sigma \not\subset \tau \}$
Some ideas from the proof

**Thm (Tancer '10):** $K$ is $d$-collapsible if and only if either $\dim(K) < d$ or...
Some ideas from the proof

**Thm (Tancer '10):** $K$ is $d$-collapsible iff either $\dim(K) < d$ or

$\exists \sigma \in K$, $|\sigma| = d$, contained in unique max face $t \not\in \sigma$ and

$\text{cost}(K, \sigma)$ is $d$-collapsible
Some ideas from the proof

Let $K$ be d-col.
Some ideas from the proof

Let $K$ be $d$-col.

If $\dim(K) < d$ then

$$\dim(T_t(k)) < d + t < h(d, t)$$
Some ideas from the proof

Let $K$ be $d$-col.

If $\dim(K) < d$ then

$$\dim(T_t(k)) < d + t < h(d, t)$$
Some ideas from the proof

Let $K$ be $d$-col.

If $\dim(K) < d$ then

$$\dim(T_t(k)) < d + t < h(d, t)$$

Now, $\exists \sigma \in K$, $|\sigma| = d$, $\sigma$ is contained in unique max. face $t = \sigma \vee U (U \neq \emptyset)$, $\text{cost}(k, \sigma)$ $d$-col.
Some ideas from the proof

We want to show: \( \bar{H}_k(\mathbb{C}^d(k)) = 0 \)
for \( k \geq h(t,d) \).
Some ideas from the proof

We want to show: \( \tilde{H}_k(\mathcal{I}_t(k)) = 0 \)

for \( k \geq 6 \epsilon t, d \).

Look out long exact seq. of the pair \( \mathcal{I}_t(\text{cost}(k, 0)) \leq \mathcal{I}_t(k) \):
Some ideas from the proof

We want to show: \( \widetilde{H}_k(T_t(k)) = 0 \) for \( k \geq h(t, d) \).

Look at long exact seq. of the pair \( (T_t, \text{cost}(k, \sigma)) \leq T_t(k) \):

\[
\cdots \rightarrow \widetilde{H}_k(T_t(\text{cost}(k, \sigma))) \rightarrow \widetilde{H}_k(T_t(k)) \rightarrow \ \rightarrow \widetilde{H}_k(T_t(k), T_t(\text{cost}(k, \sigma))) \rightarrow \cdots
\]
Some ideas from the proof

We want to show: \( \tilde{H}_k(T_t(k)) = 0 \) for \( k \geq h(t,d) \).

Look at long exact seq. of the pair \( T_t \leq T_t(k) \):

\[
\ldots \to \tilde{H}_k(T_t(\text{cost}(k,\sigma))) \to \tilde{H}_k(T_t(k)) \to \]

\[ \text{by indunct.} \quad \overset{11}{=} \quad \to \tilde{H}_k(T_t(k), T_t(\text{cost}(k,\sigma))) \to \ldots \]
Some ideas from the proof

Prop:

\[ H_k(T_t(K), T_t(\text{cost}(k, \sigma))) \cong \]

\[ \bigoplus \tilde{H}_{k-d-1} \left( \bigcup_{\sigma'} \left( U T_{t-10^{-1}k(k, \sigma' \sigma \sigma') \langle \text{UUW} \rangle} \right) \right) \]

\[ W \in V_1(\sigma \sigma' \sigma) \]

\[ 1W1 = t \]

\[ 1510 \leq t \]
Some ideas from the proof

Prop:

\[ H_k(T_t(K), T_t(\text{cost}(K, \sigma))) \cong \]

\[ \bigoplus \tilde{H}_{k-d-1} \left( U T_{t-1} \left( \frac{\text{lk}(K, \sigma \cup \sigma')}{\sigma_t, \sigma'_t \leq t, 1 \leq t} \right) \right) \]

\[ W \in V_1(\sigma \cup \sigma') \]

\[ \left\| W \right\| = t \]

\[ d \text{-collapsible} \]

(\text{Khmelevsky '18})
Some ideas from the proof

Prop:

\[ H_k(T_t(K), T_t(\text{cost}(k, \sigma))) \cong \]

\[ \bigoplus \widetilde{H}_{k-d-1} \left( \bigcup_{\sigma' \leq \sigma, 1 \leq i \leq t} T_{t-1}(lK(k, \sigma \cup \sigma'))[UuW] \right) \]

\[ w \in V_1(\sigma \cup \sigma) \]

\[ |w| = t \]

\[ h(t-1, d) \text{-Leray} \]

(by induction on \( t \))
Some ideas from the proof

Prop:

\[ H_k(\tau_t(k), \tau_t(\cos^2(k, \sigma))) \cong \]

\[ \bigoplus \tilde{H}_{k-d-1}(U \tau_{t-\lambda_1}(l_{k(k, \sigma', \sigma')}[\nu \nu \nu])) \]

\[ W \in V_1(\sigma \sigma \sigma) \]

\[ |W| = t \]

\[ \left[ \sum_{\sigma' \leq \sigma \leq \sigma} \lambda \left( t - \lambda_1(1, d) + 1 \right) - 1 \right] \text{-Leray} \]

\[ (Kahle-Meshulam '06) \]
Some ideas from the proof

Prop:

\[ H_k(\mathcal{T}_t(K), \mathcal{T}_t(\text{cost}(K, \sigma))) \cong \]

\[ \bigoplus \widetilde{H}_{k-d-1}( \bigcup_{0 \leq \sigma' \leq \sigma, \sigma' \leq t} \mathcal{T}_{t-\text{cost}}(K(K, \sigma' \sigma'))[UuW]) \]

\[ W \in V \setminus (UuV) \]

\[ |W| = t \]

\[ \left[ h(t, d) - d - 1 \right] \text{-Leray} \]

(Kalai-Meshulam '06)
Some ideas from the proof

Prop:

\[ H_k \left( T_t(K), T_t(\text{cost}(k, \sigma)) \right) \cong \bigoplus \tilde{H}_{k-d-1} \left( U T_{t-\text{log}_{10} \sigma} (lk(k, \sigma, \sigma')[WuW]) \right) \]

W \leq V \cup (\sigma Wu)

1W1 = t

= 0

for \( k \geq h(t, d) \)
Some ideas from the proof

Prop:

\[ H_k(T_t(K), T_t(\text{cost}(k, \sigma))) \cong \]

\[ \bigoplus H_{k-d-1}(U T_{t-1} \langle k(k, \sigma', \sigma')[UW] \rangle) \]

\[ W \subseteq V_1(\sigma UV) \]

\[ |W| = t \]

\[ 1 \leq t \]

\[ \sigma \preceq \sigma' \leq t \]

\[ \Rightarrow 0 \]

\[ = 0 \]

for \( k \geq h(t, d) \)
A geometric application

**Thm (Kim, L. 21)**: Let $C_1, C_2, C_3, C_4, C_5, C_6$ be families of convex sets in the plane.
A geometric application

Thm (Kim, L. 21'): Let

\( C_1, C_2, C_3, C_4, C_5, C_6 \) be families of convex sets in the plane. If every \( \{ A_1, A_2, A_3, A_4, A_5, A_6 \} \) has a pt. in common with Tolerance 1.
A geometric application

**Thm** (Kim, L. 21): Let $C_1, C_2, C_3, C_4, C_5, C_6$ be families of convex sets in the plane. If every \{A_1, A_2, A_3, A_4, A_5, A_6\} has a pt. in common with Tolerance 1, then one of the $C_i$ has pt. in common with tol. 1
A geometric application

Thm (Kim, L. 21'): Let $C_1, C_2, C_3$ be sets of points in a space $X$.

Follows from application of Kalai and Meshulam's topological colorful Helly theorem ('05) to the case where any pair of the $C_i$ has a point in common with tolerance 1, then one of the $C_i$ has a point in common with tol. 1.
THANKS FOR LISTENING!