Spectral Gaps of Generalized Flag Complexes

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Graph Laplacian

$G = (V, E)$ a graph, $|V| = n$.

The Laplacian of $G$ is the $V \times V$ matrix $L_G$:

$$L_G(u, v) = \begin{cases} 
\deg(u) & \text{if } u = v, \\
-1 & \text{if } uv \in E, \\
0 & \text{otherwise.}
\end{cases}$$
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Eigenvalues of \( L_G \)

\[
0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G).
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\( \lambda_2(G) = \text{Spectral Gap of } G \).
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\( \lambda_2 > 0 \iff G \text{ is connected.} \)
Simplicial Cohomology

$X$ a simplicial complex on vertex set $V$.
$X(k) = k$-dimensional simplices.
$C^k(X) = k$-cochains = skew-symmetric maps from the set of ordered $k$-simplices to $\mathbb{R}$.
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(E.g. \( \phi([v_1, v_2, v_3]) = \phi([v_3, v_1, v_2]) = -\phi([v_1, v_3, v_2]) \))
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For $\sigma \in X(k + 1)$, $\tau \in \sigma(k)$ ordered simplices:
Let $\{v\} = \sigma \setminus \tau$.

$(\sigma : \tau) =$ sign of permutation mapping $\sigma$ to $v\tau$. 

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(E.g. $([v_1, v_2, v_3] : [v_3, v_1]) = \text{sign}(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}) = 1$).

The **Coboundary Operator** $d_k : C^k(X) \to C^{k+1}(X)$ is given by

$$d_k \phi(\sigma) = \sum_{\tau \in \sigma(k)} (\sigma : \tau) \phi(\tau).$$
Simplicial Cohomology

\[ Z^k(X) = k\text{-cocycles} = \text{Ker}(d_k). \]
\[ B^k(X) = k\text{-coboundaries} = \text{Im}(d_{k-1}). \]

\( k \)-th reduced cohomology group of \( X \):

\[ \tilde{H}^k(X; \mathbb{R}) = Z^k(X)/B^k(X). \]
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Inner product on \( C^k(X) \):

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Adjoint of coboundary operator: \( d_k^* : C^{k+1}(X) \to C^k(X) \)
\[ \langle d_k \phi, \psi \rangle = \langle \phi, d_k^* \psi \rangle. \]
Higher Laplacians

\[
C^{k-1}(X) \xrightarrow{d_{k-1}} C^k(X) \xrightarrow{d_k} C^{k+1}(X)
\]

The reduced \textit{k-Laplacian} of \(X\) is the positive semidefinite operator

\[
L_k = d_{k-1}d^*_{k-1} + d^*kd_k : C^k(X) \to C^k(X).
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Matrix form of the \( k \)-Laplacian

\[ L_k(\sigma, \tau) = \begin{cases} \deg(\sigma) + k + 1 & \text{if } \sigma = \tau, \\ (\sigma : \sigma \cap \tau)(\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k, \sigma \cup \tau \not\in X, \\ 0 & \text{otherwise.} \end{cases} \]
Higher Laplacians

Example

$$X = \text{boundary of a tetrahedron.}$$
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\[
L_1(X) = \begin{pmatrix}
4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}
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\end{pmatrix} \]

\[ L_2(X) = \begin{pmatrix}
3 & 1 & -1 & 1 \\
1 & 3 & 1 & -1 \\
-1 & 1 & 3 & 1 \\
1 & -1 & 1 & 3 \\
\end{pmatrix} \]
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\[ \mu_k(X) = k\text{-th spectral gap of } X = \text{minimal eigenvalue of } L_k(X). \]
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Relation with the graph Laplacian
Let \( G \) = 1-skeleton of \( X \). Then

\[ L_0(X) = L_G + J, \]

\[ \mu_0(X) = \lambda_2(G). \]
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Simplicial Hodge Theorem

\[
\operatorname{Ker}(L_k) \cong \tilde{H}^k(X; \mathbb{R}).
\]

In particular:

\[
\mu_k > 0 \iff \tilde{H}^k(X; \mathbb{R}) = 0.
\]
Flag Complexes

The flag complex (or clique complex) $X(G)$ of graph $G = (V, E)$:
Vertex set: $V$, Simplices: all cliques of $G$. 
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Example

$$G = \begin{array}{c}
\begin{tikzpicture}
  \draw (0,0) -- (1,1) -- (2,0) -- (0,0);
  \draw (0,0) -- (0,1) -- (1,1) -- (0,0);
  \draw (1,1) -- (2,0) -- (1,1);
\end{tikzpicture}
\end{array}$$  $$X(G) = \begin{array}{c}
\begin{tikzpicture}
  \draw (0,0) -- (1,1) -- (2,0) -- (0,0);
  \draw (0,0) -- (0,1) -- (1,1) -- (0,0);
  \draw (1,1) -- (2,0) -- (1,1);
  \filldraw[blue] (0,0) circle (2pt);
  \filldraw[blue] (1,1) circle (2pt);
  \filldraw[blue] (2,0) circle (2pt);
\end{tikzpicture}
\end{array}$$
Spectral Gaps of Flag Complexes

\[ G = (V, E) \text{ a graph with } |V| = n. \text{ Let } X = X(G). \]

**Theorem [Aharoni-Berger-Meshulam]:**

For \( k \geq 1 \)

\[ k \mu_k(X) \geq (k + 1) \mu_{k-1}(X) - n. \]

In particular

\[ \mu_k(X) \geq (k + 1) \lambda_2(G) - kn. \]
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**Corollary:**

\[ \lambda_2(G) > \frac{kn}{k + 1} \implies \mu_k(X) > 0 \implies \tilde{H}^k(X; \mathbb{R}) = 0. \]
Extremal Example [Aharoni-Berger-Meshulam]:
Let $n = r\ell$ and let $G$ be the Turán graph $T(n, r)$, i.e. the complete $r$-partite graph.

$G = \begin{array}{c}
a_1 \\
b_1 \\
a_2 \\
b_2 \\
a_3 \\
b_3 \\
\end{array}$

$X(G) = \begin{array}{c}
a_1 \\
a_2 \\
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\[
\lambda_2(G) = \ell(r - 1) = \frac{r-1}{r} n, \text{ but } \tilde{H}^{r-1}(X(G); \mathbb{R}) \neq 0.
\]
Generalized Flag Complexes

Missing Faces

$X$ a simplicial complex on vertex set $V$.

$\tau \subset V$ is a missing face of $X$ if $\tau \not\in X$ but $\eta \in X$ for all $\eta \subset \tau$.

$h(X) = \max \{ \dim(\tau) : \tau \text{ is a missing face of } X \}$. 

Example

The missing faces: $\{v_1, v_2, v_3\}, \{v_1, v_4\}$

$X$ is a flag complex $\iff h(X) = 1$ (missing faces = edges of the complement of $G$).
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Spectral Gaps of Generalized Flag Complexes

$X$ a simplicial complex on vertex set $V$, $|V| = n$, with $h(X) = d$.

**Theorem:**
For $k \geq d$

$$(k - d + 1)\mu_k(X) \geq (k + 1)\mu_{k-1}(X) - dn.$$
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**Corollary:**
$\mu_{d-1}(X) > \left( 1 - \binom{k+1}{d}^{-1} \right) n \implies \mu_k(X) > 0 \implies \tilde{H}^k(X; \mathbb{R}) = 0.$
Extremal Examples

Let \( d = 2 \). Then

\[
\mu_1(X) > \left(1 - \left(\frac{k + 1}{2}\right)^{-1}\right)n \iff \tilde{H}^k(X; \mathbb{R}) = 0.
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Let $X$ be the complex whose missing faces are the lines of the affine plane over $\mathbb{F}_3$: 

![Diagram of a complex with missing faces labeled as lines in the affine plane over F3]
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Let $X$ be the complex whose missing faces are the lines of the affine plane over $\mathbb{F}_3$:

$$
\mu_1(X) = 6 = \left(1 - \left(\frac{2+1}{2}\right)^{-1}\right)n, \text{ but } \tilde{H}^2(X; \mathbb{R}) = \mathbb{R} \neq 0.
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Several more examples for $d = 2$ and $k \leq 4$, all arising from finite geometries.
Homological Connectivity

The homological connectivity of a complex $X$:

$$\eta(X) = \min\{i : \tilde{H}^i(X) \neq 0\} + 1.$$
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Examples

$\eta(X) = 2$

$\eta(X) = \infty$
Connectivity of Independence Complexes

Let $G = (V, E)$ be a graph. The Independence Complex: $I(G) = X(\bar{G})$. 

$\eta(I(G))$ can be bounded by different "domination parameters" of the graph $G$. For example: A subset $S \subseteq V$ is totally dominating if every vertex $v \in V$ has a neighbor in $S$. Let $\bar{\gamma}(G)$ be the minimal size of a totally dominating set. 

Theorem [Aharoni-Chudnovsky, Meshulam]: $\eta(I(G)) \geq \bar{\gamma}(G)/2$. 

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Connectivity of Independence Complexes

Example

\[ \tilde{\gamma}(G) = 4 \]

\[ \eta(I(G)) = 2 = \tilde{\gamma}(G) / 2 \]

\[ G = \]

\[ I(G) = \]

\[ 1 \quad 2 \quad 3 \]

\[ 4 \quad 5 \quad 6 \]
Connectivity of Independence Complexes

Example

\[ \gamma(G) = 4 \]
\[ \eta(I(G)) = 2 = \frac{\gamma(G)}{2} \]
Connectivity of Independence Complexes

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\[ \tilde{\gamma}(G) = 4 \]

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Vector Domination of a Graph

A vector representation of $G$:
$P : V \rightarrow \mathbb{R}^\ell$ such that for any $v, w \in V$

$$P(v) \cdot P(w) \geq \begin{cases} 1 & \text{if } \{v, w\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$
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\end{cases}$$

Identify $P$ with an $\mathbb{R}^{|V| \times \ell}$ matrix.

A vector $0 \leq \alpha \in \mathbb{R}^V$ is dominating for $P$ if $\alpha PP^T \geq 1$, i.e.

$$\sum_{v \in V} \alpha(v) P(v) \cdot P(u) \geq 1$$

for all $u \in V$. 
Vector Domination of a Graph

The value of $P$:

$$|P| = \min\{\alpha \cdot 1 : \alpha \text{ is dominating}\}$$

$$= \max\{\alpha \cdot 1 : \alpha \geq 0, \alpha PP^T \leq 1\}.$$ 

Define $\Gamma(G)$ to be the supremum of $|P|$ over all vector representations of $G$. 
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Theorem [Aharoni-Berger-Meshulam]:

$$\eta(I(G)) \geq \Gamma(G).$$
Vector Domination of a Simplicial Complex

Let $X$ be a simplicial complex on vertex set $V$.
$\mathcal{M}(k) =$ missing faces of dimension $k$ of $X$.
$J = \{ k \in \mathbb{N} : \mathcal{M}(k) \neq \emptyset \}$.
$S(X) = \bigcup_{k \in J} \binom{V}{k-1}$.
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If $X$ is a clique complex: $J = \{1\}$, $S(X) = \{\emptyset\}$. 
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For each $\sigma \in S(X)$ fix $\ell = \ell(\sigma)$.

A vector representation of $X$ with respect to $\sigma$: $P_\sigma : V \rightarrow \mathbb{R}^\ell$, such that

$$P_\sigma(v) \cdot P_\sigma(w) \geq \begin{cases} 1 & \text{if } vw\sigma \in M(|\sigma| + 1), \\ 0 & \text{otherwise}. \end{cases}$$
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$$P_\sigma(v) \cdot P_\sigma(w) \geq \begin{cases} 1 & \text{if } vw\sigma \in M(|\sigma| + 1), \\ 0 & \text{otherwise}. \end{cases}$$

$P = \{P_\sigma : \sigma \in S(X)\}$ is called a vector representation of $X$. 
The value of $P$:

$$|P| = \max\{\alpha \cdot \mathbf{1} : \alpha \geq 0, \alpha P_{\sigma} P_{\sigma}^T \leq \mathbf{1} \ \forall \sigma \in S(X)\}.$$ 

Define $\Gamma(X)$ to be the supremum of $|P|$ over all vector representations of $X$. 

Remark: For a graph $G$, we have $\Gamma(G) = \Gamma(I(G))$. 

Theorem: 

$$\sum_{k \in J} k(\eta(X)k) \geq \Gamma(X).$$
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**Colorful Simplices**

Let $V_1, \ldots, V_m$ be a partition of $V$. A simplex $\sigma \in X$ is **colorful** if $|\sigma \cap V_i| = 1$ for all $i = 1, \ldots, m$.

**Theorem [Aharoni-Haxell, Meshulam]:**

If for all $\emptyset \neq I \subset \{1, 2, \ldots, m\}$

$$\eta(X[\cup_{i \in I} V_i]) \geq |I|,$$

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**Theorem:**
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General Position in $\mathbb{R}^d$

A set $S \subset \mathbb{R}^d$ is in general position if any $k$-dimensional flat contains at most $k + 1$ points of $S$ (for $k \leq d - 1$).
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Example

In $\mathbb{R}^2$:
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Let $V \subset \mathbb{R}^d$ a finite set. $V_1, V_2, \ldots, V_m$ a partition of $V$. A set $S \subset V$ is colorful if $|S \cap V_i| = 1$ for $i = 1, \ldots, m$. 
Hall-type Theorem for General Position

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Theorem [Holmsen-Martínez Sandoval-Montejano]: If for every $\emptyset \neq I \subset \{1, 2, \ldots, m\}$

$$\varphi(\bigcup_{i \in I} V_i) > \begin{cases} |I| - 1 & \text{if } |I| \leq d + 1, \\ d\left(\frac{2|I| - 2}{d}\right) & \text{if } |I| \geq d + 2, \end{cases}$$

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\( S \subset \mathbb{R}^d \) a finite set. A weight function \( f : S \to \mathbb{R}_{\geq 0} \) is in fractional general position if for every \( 0 \leq k \leq d - 1 \), \( k \)-dimensional flat \( F \) and \( \sigma \subset F \cap S \) of size \( k \)

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$\varphi^*(S) \geq \varphi(S)$ (the characteristic function of any subset of $S$ in general position is in fractional general position).
Examples

In $\mathbb{R}^2$: 

\[ \phi = 2 \phi^* = 5 \cdot 1.2 = 2.12 \]

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Examples

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$\varphi = 2$

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If for every $\emptyset \neq I \subset \{1, 2, \ldots, m\}$

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Sketch of Proof

For $V$ be a finite set of points in $\mathbb{R}^d$, build a simplicial complex $X$:
Vertex set: $V$, Simplices: all subsets $S \subset V$ in general position.
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Missing faces of $X$: $S \subseteq V$, $|S| \leq d + 1$, such that $S$ is affinely dependent but any $|S| - 1$ points in $S$ are independent.
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**Vector Representation of $X$:**

Let $1 \leq k \leq d$. Let $\mathcal{F}_k$ be the set of $(k-1)$-dimensional flats spanned by points in $V$.

For $\sigma \subset V$, $|\sigma| = k - 1$, define $P_{\sigma} : V \rightarrow \mathbb{R}^{\mathcal{F}_k}$ by

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If $vw\sigma$ is a missing face of $X$, then it is contained in a $(k - 1)$-dimensional flat $F$, spanned by any $k$ points in $vw\sigma$. So $\text{span}(v\sigma) = \text{span}(w\sigma) = F$, therefore $P_\sigma(v) \cdot P_\sigma(w) = 1$. 
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If $f : V \to \mathbb{R}$ is in fractional general position, then $\alpha(v) = f(v)/d$ satisfies $\alpha P_\sigma P_\sigma^T \leq 1$ for all $\sigma \in S(X)$. So

$$\Gamma(X) \geq |P| \geq \varphi^*(V)/d.$$
$X$ a simplicial complex on vertex set $V$, $|V| = n$, with $h(X) = d$.
Let $k \geq 0$.
$\delta_k(X) = \text{minimal degree of a simplex in } X(k)$. 

Theorem: 

$\mu_k(X) \geq (d + 1)(\delta_k(X) + k + 1) - dn$.

As a consequence:

Theorem [Adamaszek]:
$\tilde{H}_k(X; \mathbb{R}) = 0$ for all $k > \frac{d^2 + 1}{dn - 1}$. 


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Extremal Examples

$X, Y$ simplicial complexes on disjoint vertex sets. The join $X \ast Y = \{\sigma \cup \tau : \sigma \in X, \tau \in Y\}$.
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Define

$$X = \Delta_d^{(d-1)} \ast \Delta_d^{(d-1)} \ast \cdots \ast \Delta_d^{(d-1)} \ast \Delta_{r-1}.\tag{t times}$$

Vertices: $n = (d + 1)t + r$.

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For all $k$ we have 

$$\mu_k(X) = (d + 1)(\delta_k(X) + k + 1) - dn.$$
Let $d = 1$, $t = 3$, $r = 3$: 

$$X = \begin{array}{cccc}
\ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\end{array}$$
Extremal Examples

Let $d = 1$, $t = 3$, $r = 3$:

For example for $k = 2$:

$\mu_2(X) = 3,$
$\delta_2(X) = 3.$

Indeed $\mu_2(X) = 2(\delta_2(X) + 2 + 1) - n = 12 - 9 = 3.$
For flag complexes \((h(X) = 1)\) these are the only extremal examples:

**Theorem:**
Let \(X\) be a flag complex on vertex set \(V, |V| = n\), such that \(\mu_k(X) = 2(k + 1) - n\) for some \(k \geq 0\). Then

\[
X \cong \Delta_1^{(0)} \ast \Delta_1^{(0)} \ast \cdots \ast \Delta_1^{(0)} \ast \Delta_2^{(k+1) - n - 1} \quad \text{(n – k – 1) times}
\]