Spectral gaps of generalized flag complexes

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Introduction

Let X be a simplicial complex without missing faces of dimension larger than d. Following the work of Aharoni, Berger and Meshulam, we prove lower bounds on the spectral gap of the k-Laplacian of X in terms of the spectral gap of the $(d-1)$-Laplacian. As an application, we prove a Hall-type theorem for sets of points in $\mathbb{R}^d$.

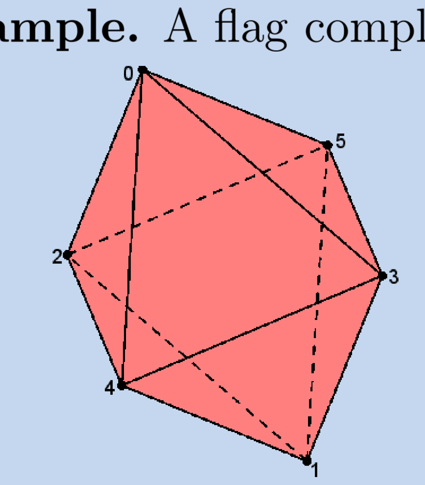
Flag complexes:

•$\sigma \subset V$ is a **missing face** of a complex X on vertex set V if $\sigma \notin X$ but $\sigma' \in X$ for all subsets $\sigma' \subsetneq \sigma$.

•Flag complex= complex without missing faces of dimension larger than 1.

•We look at complexes without missing faces of dimension larger than d as generalized flag complexes.

Example. A flag complex:



missing faces: {0,1}, {2,3}, {4,5} - of dimension 1.

Inequalities for spectral gaps of complexes without large missing faces

Theorem (Aharoni–Berger–Meshulam). Let X be a flag complex on n vertices, and $k \geq 1$. Then

$$k\mu_k(X) \geq (k+1)\mu_{k-1}(X) - n.$$

Corollary (Aharoni–Berger–Meshulam). If $\mu_0(X) > \frac{k}{k+1}n$, then $\tilde{H}^k(X) = 0$.

Extends to complexes without large missing faces:

Theorem. Let X be a complex on n vertices without missing faces of dimension larger than d, and $k \geq d$. Then

$$(k-d+1)\mu_k(X) \geq (k+1)\mu_{k-1}(X) - d \cdot n.$$

Corollary. If $\mu_{d-1}(X) > \frac{\binom{k+1}{d}-1}{\binom{k+1}{d}}n$, then $\tilde{H}^k(X) = 0$.

Some ideas from the proof

•For $\phi \in C^k(X)$ and a vertex $u \in V$ define $\phi_u \in C^{k-1}(X)$ by

$$\phi_u(\tau) = \begin{cases} \phi(u\tau) & \text{if } u\tau \in X(k) \\ 0 & \text{otherwise} \end{cases}$$

•$\sum_{u \in V} \|\phi_u\|^2 = (k+1)\|\phi\|^2$.

•For $\theta \subset V$, define: $m(\theta)$ = size of intersection of all missing faces of X contained in θ.

Key identity: for $\phi \in C^k(X)$,

$$(k-d+1)(L_k\phi, \phi) = \sum_{u \in V}(L_{k-1}\phi_u, \phi_u) - (B_k\phi, \phi)$$

where $B_k : C^k(X) \to C^k(X)$ has matrix representation in the standard basis:

$$(B_k)_{\sigma,\tau} = \begin{cases} \sum_{\eta \in \sigma(k-1)} \deg(\eta) - (k-d+1)\deg(\sigma) + (d-1)(k+1) & \text{if } \sigma = \tau, \\ (d+1-m(\sigma \cup \tau))\,[\sigma : \sigma \cap \tau] \cdot [\tau : \sigma \cap \tau] & \text{if } {|\sigma\cap\tau|=k, \atop \sigma\cup\tau \notin X(k+1)}, \\ 0 & \text{otherwise.} \end{cases}$$

For flag complexes, B_k is diagonal and its elements bounded by n.
For $d > 1$, we can bound the maximal eigenvalue of B_k by:

$$\max_{\sigma \in X(k)} \left(\sum_{\eta \in \sigma(k-1)} \deg(\eta) - (k-d+1)\deg(\sigma) + (d-1)(k+1) + \sum_{i=1}^{d}(d+1-i)\,|\{v \in V : m(v\sigma) = i\}| \right)$$

But for each $\sigma \in X(k)$, the expression just counts d times each vertex in V, except those with $m(v\sigma) = 1$. So $\lambda_{\max}(B_k) \leq d \cdot n$.

Simplicial cohomology and the Laplacian

- X - a finite simplicial complex.
- $X(k)$ the simplices of dimension k.
- $C^k(X)$= Real valued skew symmetric functions on the ordered k-simplices.
- The coboundary operator: $d_k : C^k(X) \to C^{k+1}(X)$.
- The k-th (reduced) real cohomology group:

$$\tilde{H}^k(X) = Ker(d_k)/Im(d_{k-1}).$$

- Inner product $(\phi, \psi) = \sum_{\sigma \in X(k)} \phi(\sigma)\psi(\sigma)$, norm $\|\phi\|^2 = (\phi, \phi)$.
- $d^* : C^{k+1}(X) \to C^k(X)$ - the adjoint of d_k.

The Laplacian operator:

$$L_k(X) = d_{k-1}d_{k-1}^* + d_k^* d_k.$$

- $\mu_k(X)$ - **The k-th spectral gap** = the minimal eigenvalue of $L_k(X)$.

Simplicial Hodge Theorem. $Ker(L_k(X)) \cong \tilde{H}^k(X)$ for $k \geq 0$.

Hence $\tilde{H}^k(X) = 0$ if and only if $\mu_k(X) > 0$.

An application: Hall type theorem for points in general position

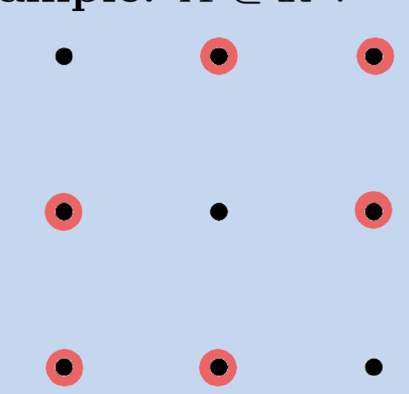
- $\{A_1, \ldots, A_m\}$ a family of finite sets in $\mathbb{R}^d$.
- $A \subseteq \cup_{i=1}^m A_i$ is **colorful** if $|A \cap A_i| = 1 \quad \forall i \in [m]$.
- $\varphi(A)$ = size of largest subset in general position.

Theorem (Holmsen–Martínez-Sandoval–Montejano). Let $\{A_1, \ldots, A_m\}$ be a family of finite sets in $\mathbb{R}^d$. If for all $\emptyset \neq I \subseteq [m]$

$$\varphi(\cup_{i \in I} A_i) > d\binom{2|I|-2}{d},$$

then $\cup_{i=1}^m A_i$ has a colorful subset in general position.

Example. $A \subset \mathbb{R}^2$:



•General position= no two points overlap, no three points lie in the same line.

$\varphi(A) = 6$, $\varphi^*(A) = 9$

A fractional extension:

- $\varphi^*(A)$= the maximum of $\sum_{p \in A} f(p)$ over all functions $f : A \to [0,1]$ such that $\sum_{p \in F \setminus S} f(p) \leq dim(F) + 1$ for every flat F spanned by points of A and set $S \subset F$ of size $dim(F)$.
- $\varphi^*(A) \geq \varphi(A)$ (take the characteristic function of the largest set in general position).

Theorem. Let $\{A_1, \ldots, A_m\}$ be a family of finite sets in $\mathbb{R}^d$. If for all $\emptyset \neq I \subseteq [m]$

$$\varphi^*(\cup_{i \in I} A_i) > d\sum_{k=1}^{d} k\binom{|I|-1}{k},$$

then $\cup_{i=1}^m A_i$ has a colorful subset in general position.

Vector representation and domination

- A **vector representation** P of X: For each $\sigma \subset V$ and $v \in V$ assign $P_\sigma(v) \in \mathbb{R}^l$ (for some l) such that for all $w \in V$

$$P_\sigma(v) \cdot P_\sigma(w) \geq \begin{cases} 1 & \text{if } vw\sigma \text{ is a missing face of dimension } |\sigma|+1, \\ 0 & \text{otherwise.} \end{cases}$$

- P_σ = matrix whose rows are $P_\sigma(v)$.
- For each σ assign $\alpha_\sigma \in \mathbb{R}_+^V$. $\{\alpha_\sigma\}_{\sigma \subset V}$ is **dominating** if

$$\sum_{\sigma \subset V} \alpha_\sigma P_\sigma P_\sigma^T \geq 1.$$

- $|P| = \min\left\{\sum_{\sigma \subset V}\sum_{v \in V} \alpha_\sigma(v) : \{\alpha_\sigma\}_{\sigma \subset V} \text{ is dominating}\right\}$.
- $\Gamma(X)$= supremum of $|P|$ over all representations P of X.
- **Homological connectivity** $\eta(X) = \min\{i : \tilde{H}^i(X) \neq 0\} + 1$.

Theorem. Let X be a complex without missing faces of dimension larger than d, then

$$\Gamma(X) \leq \sum_{k=1}^{d} k\binom{\eta(X)}{k}.$$

References

[1] Aharoni, R., Berger, E., & Meshulam, R. (2005). Eigenvalues and homology of flag complexes and vector representations of graphs. Geometric & Functional Analysis GAFA, 15(3), 555-566.

[2] Holmsen, A., Martinez-Sandoval, L., & Montejano, L. (2016). A geometric Hall-type theorem. Proceedings of the American Mathematical Society, 144(2), 503-511.