Rigidity Expander Graphs

Alan Lew
Carnegie Mellon University

Joint w/ Eran Nevo, Yuval Peled, Orit Raz

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- G=(V,E) a graph

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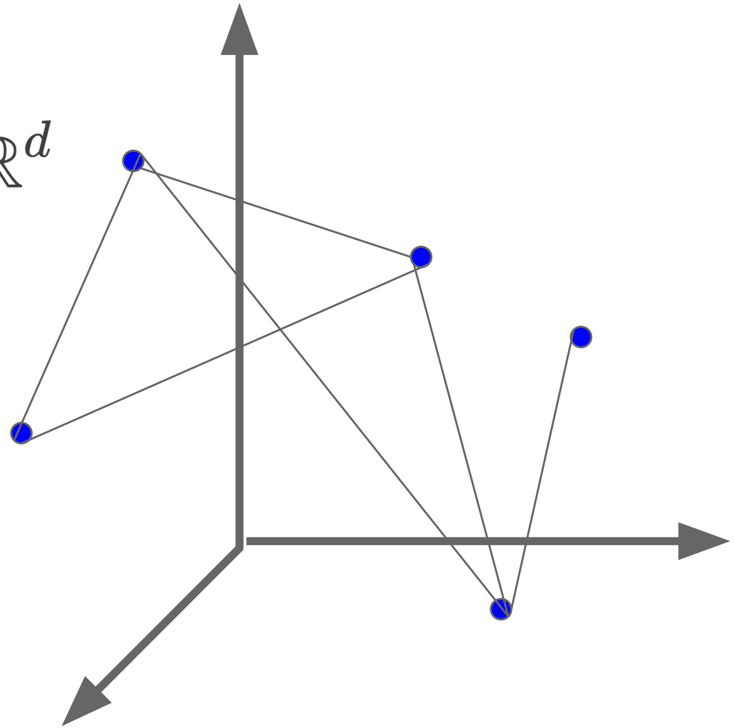

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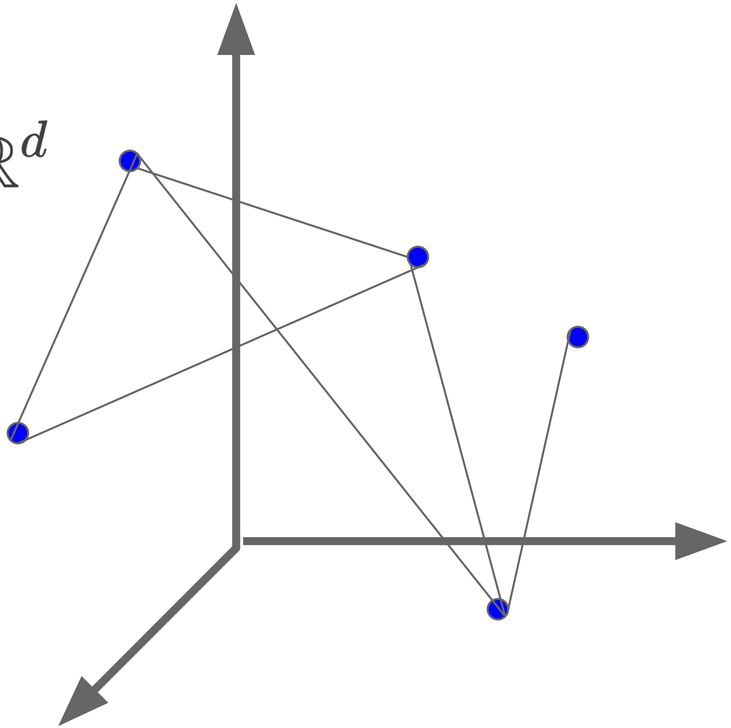

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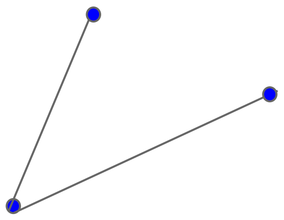

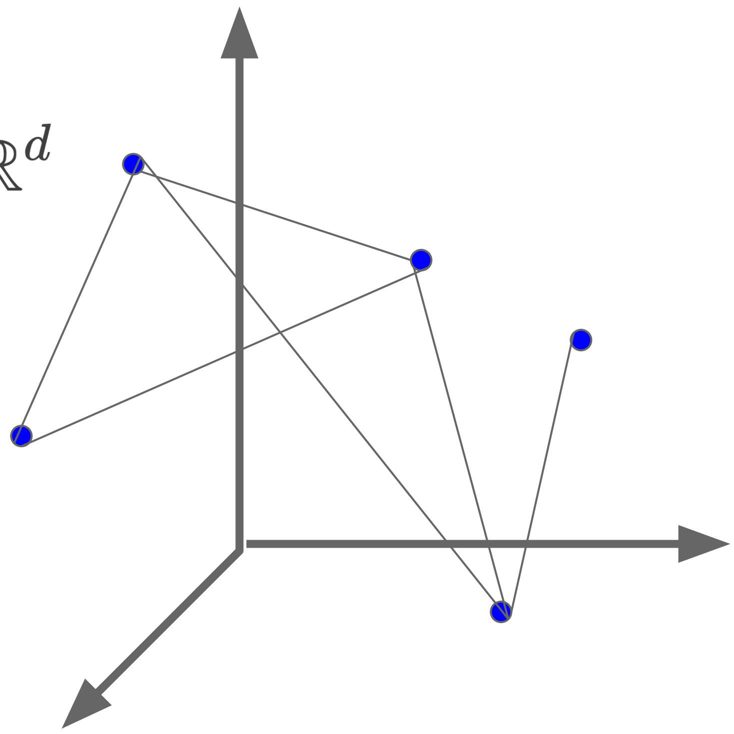

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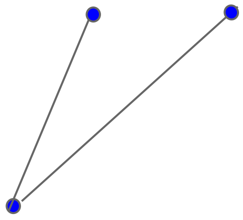

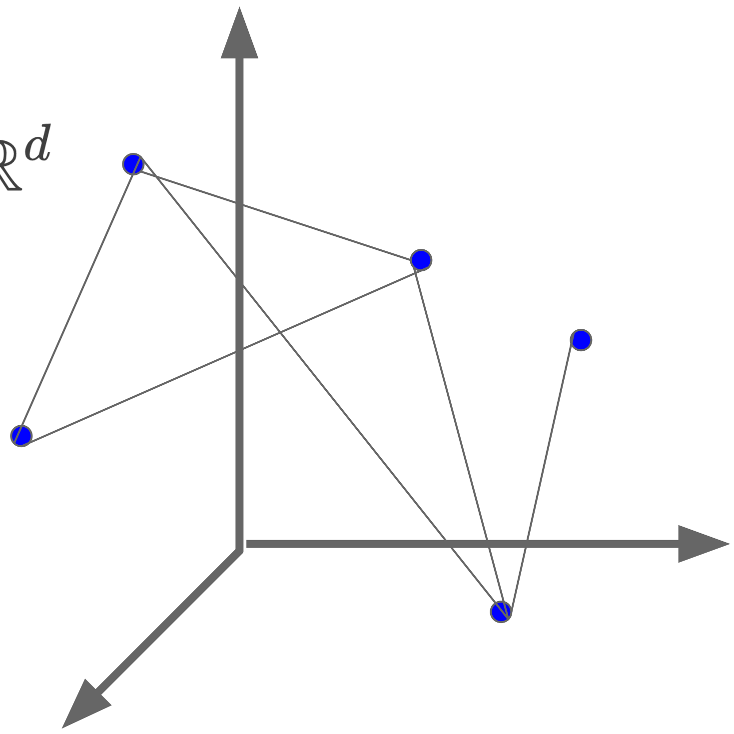

Rigidity

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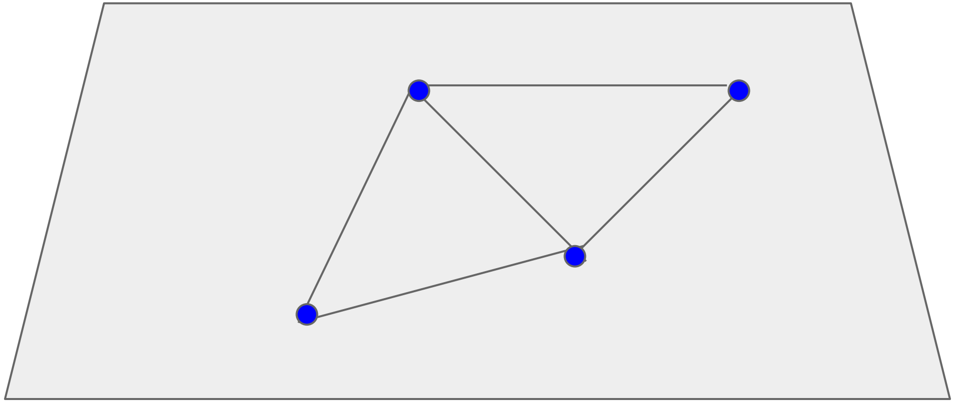
Or: Is there a continuous motion of the vertices that preserves the lengths of all edges, except translations and rotations?

Some Examples



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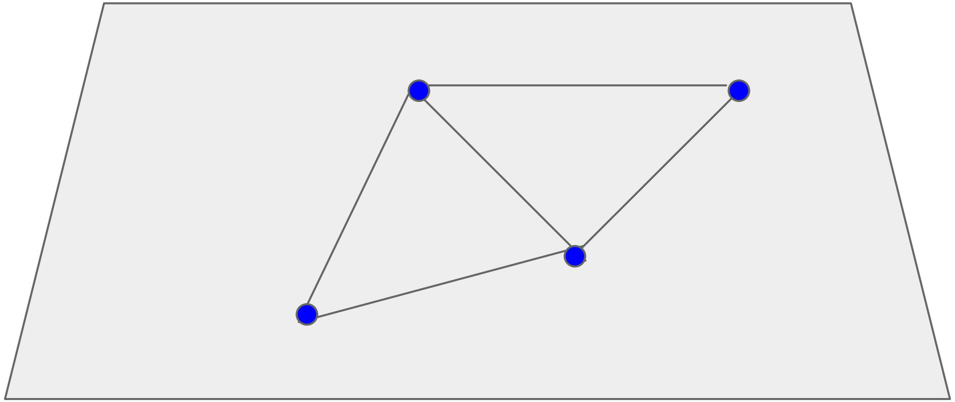
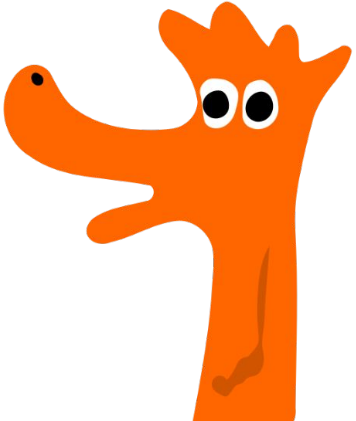


Rigid in $\mathbb{R}^2$

(or **2-rigid**)



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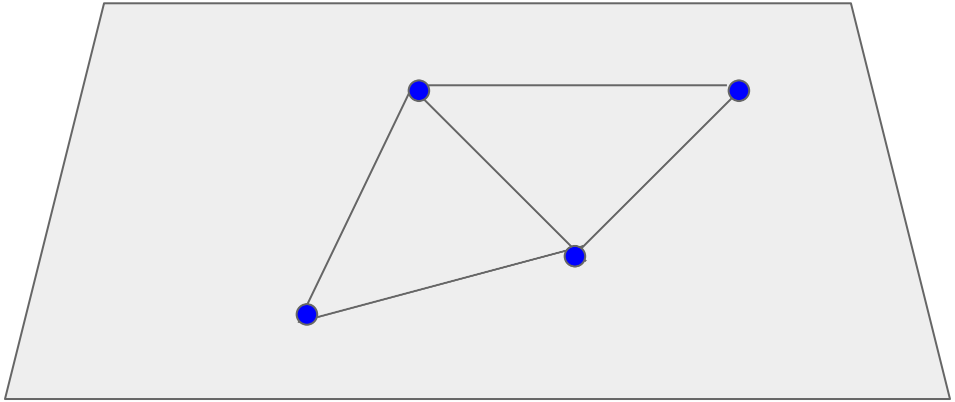

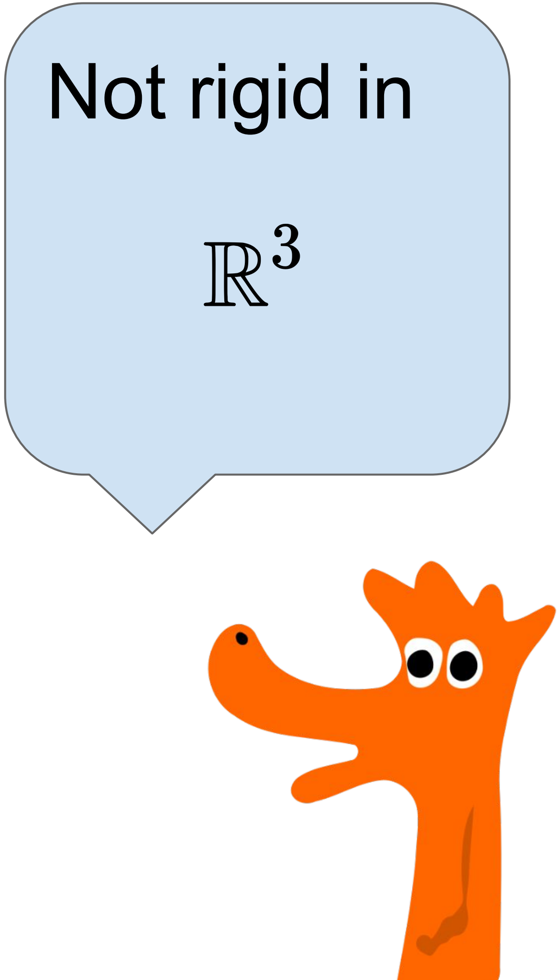

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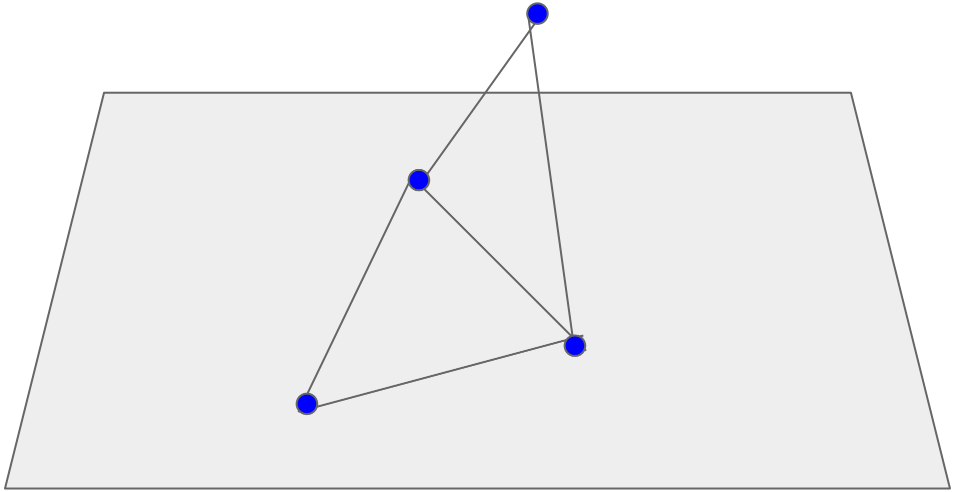
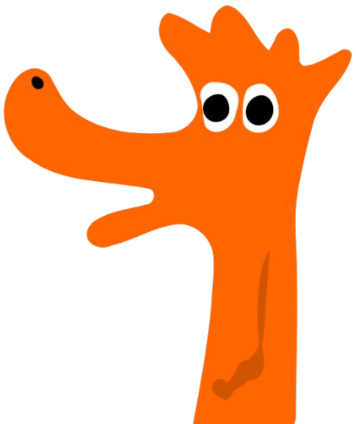


Not rigid in

$$\mathbb{R}^3$$



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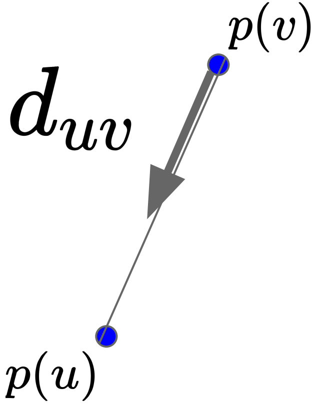

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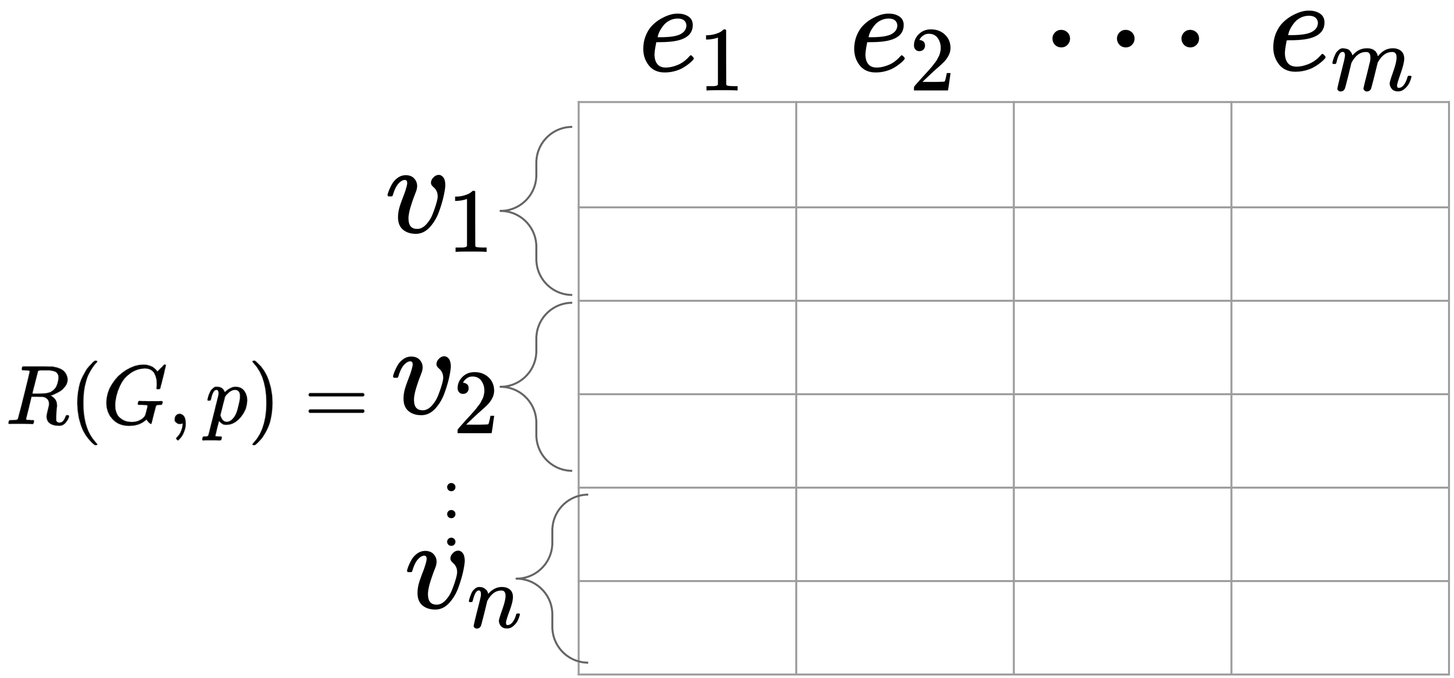

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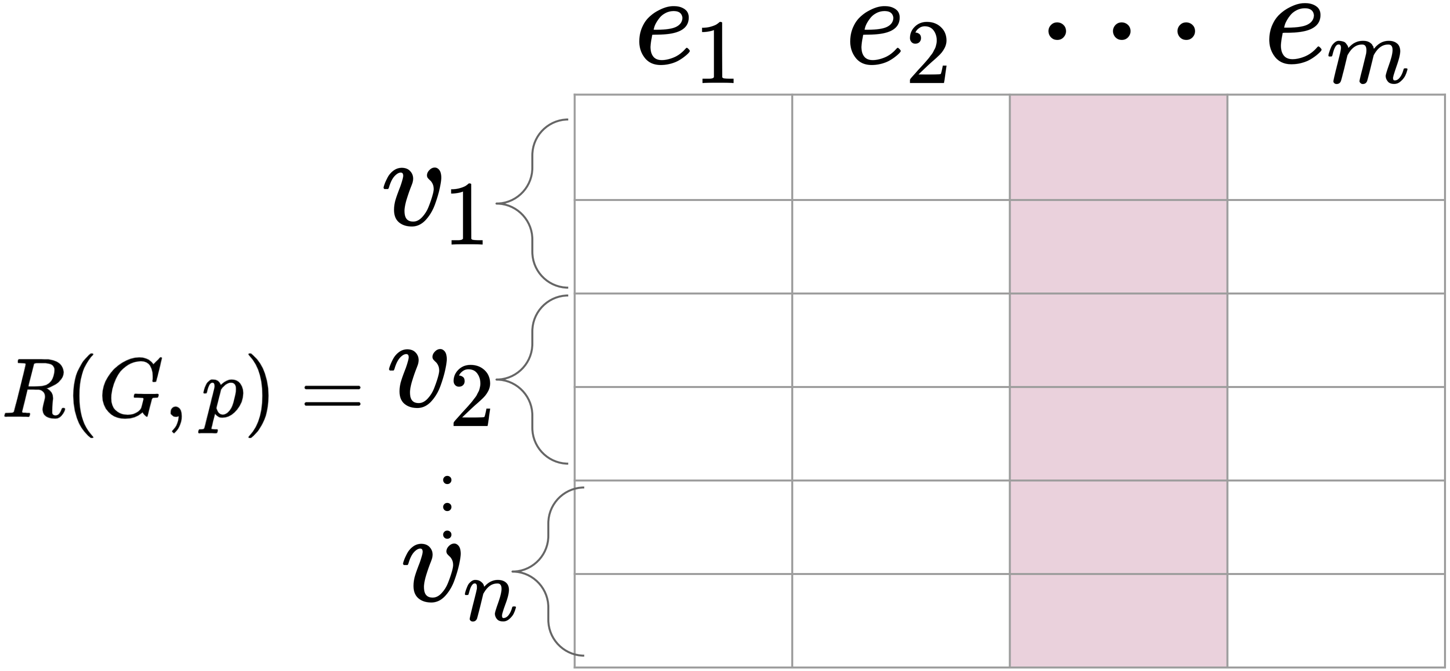

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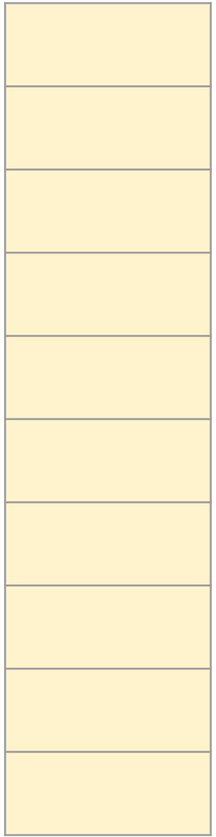
$$e = \{u, v\}$$



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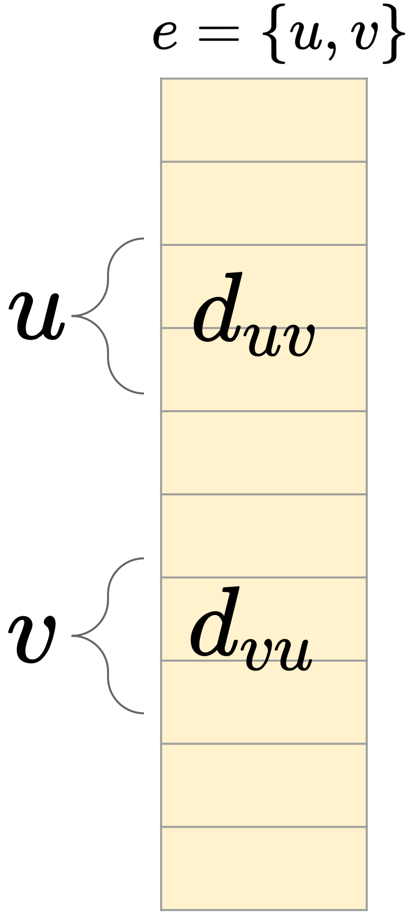

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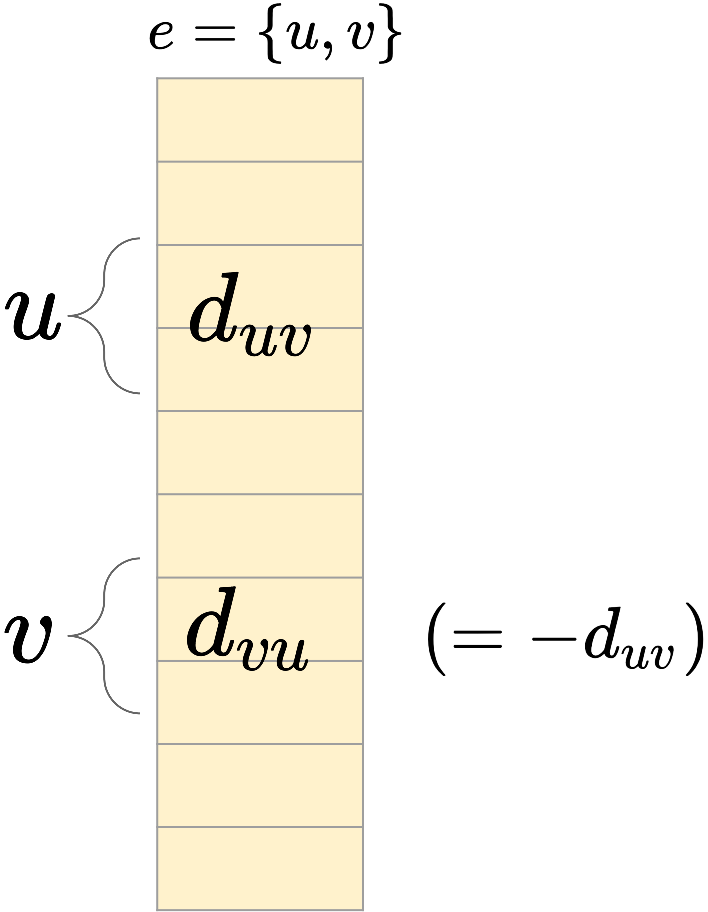

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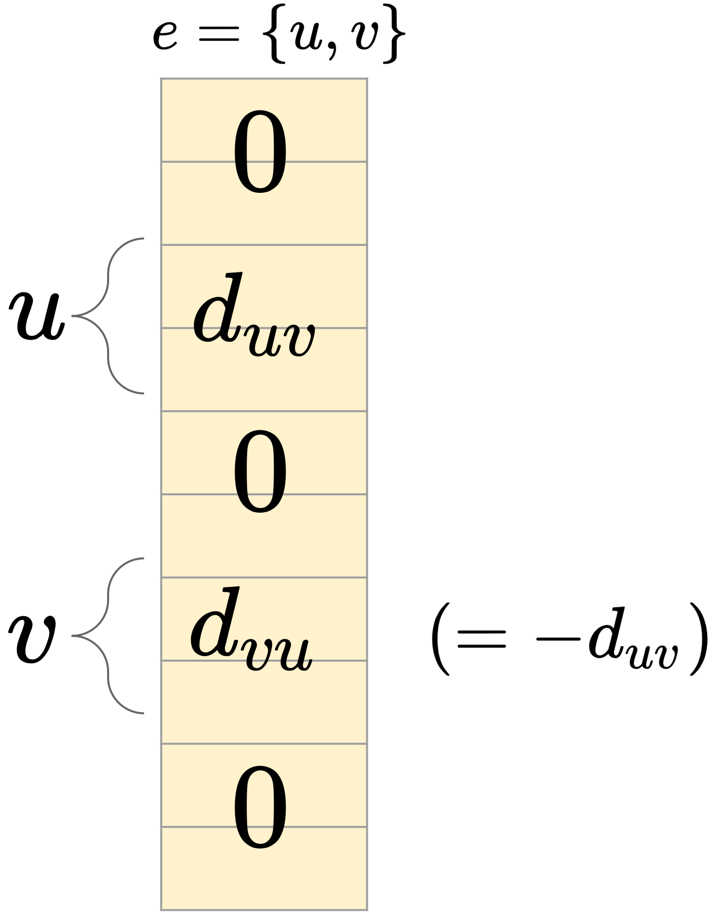

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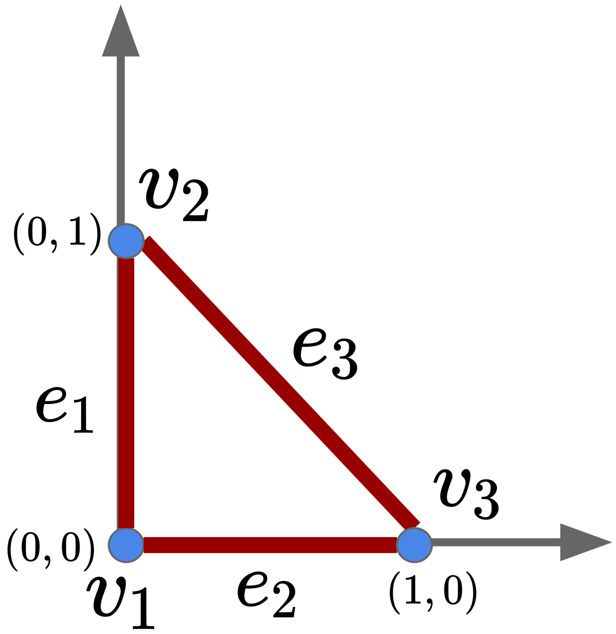
Example:



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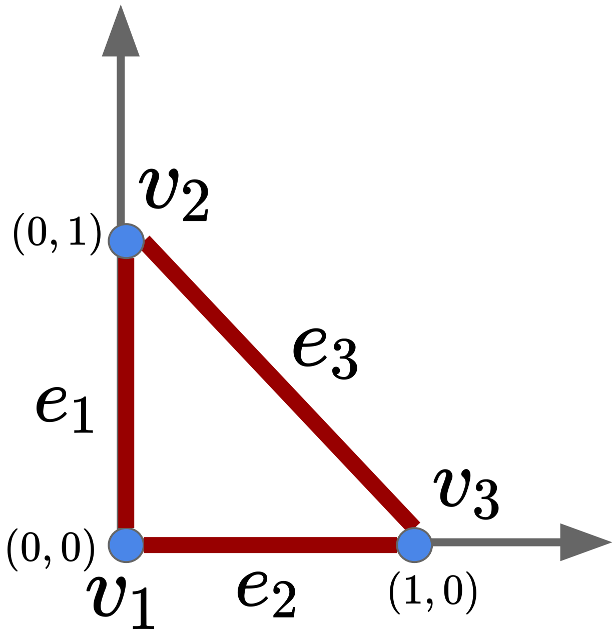
Example:

$R(G, p) =$

	e_1	e_2	e_3
v_1			
v_2			
v_3			



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Example:

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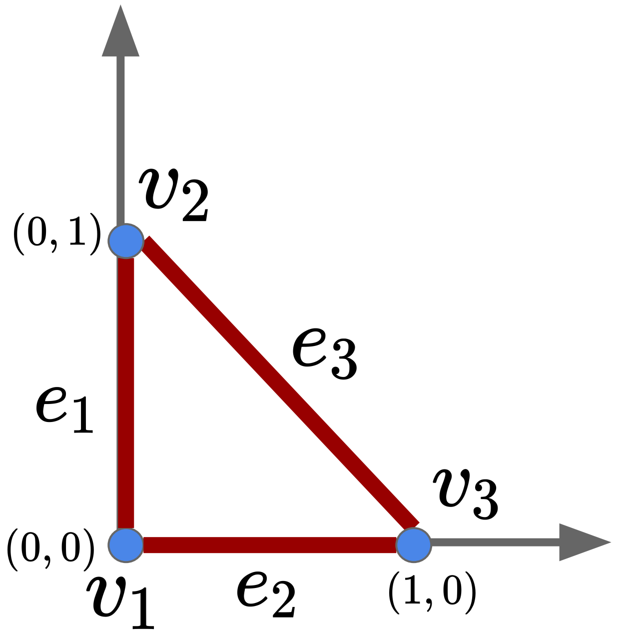
	e_1	e_2	e_3
v_1	0		
	-1		
v_2	0		
	1		
v_3	0		
	0		



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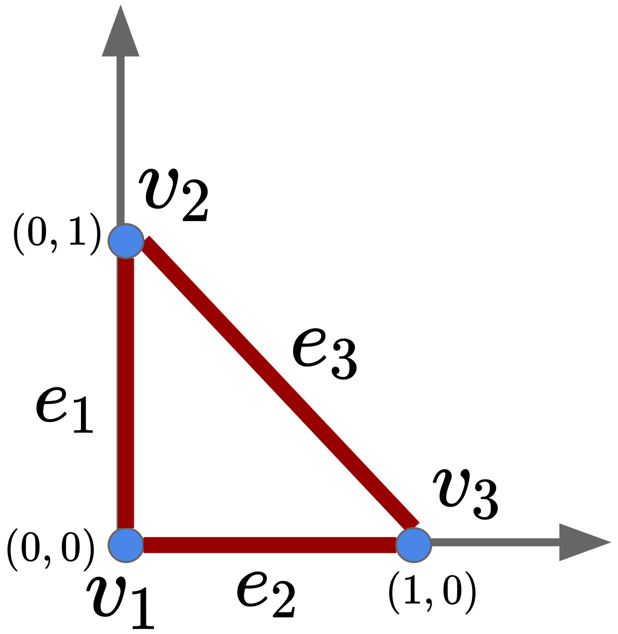
v_3		0	1	
		0	0	



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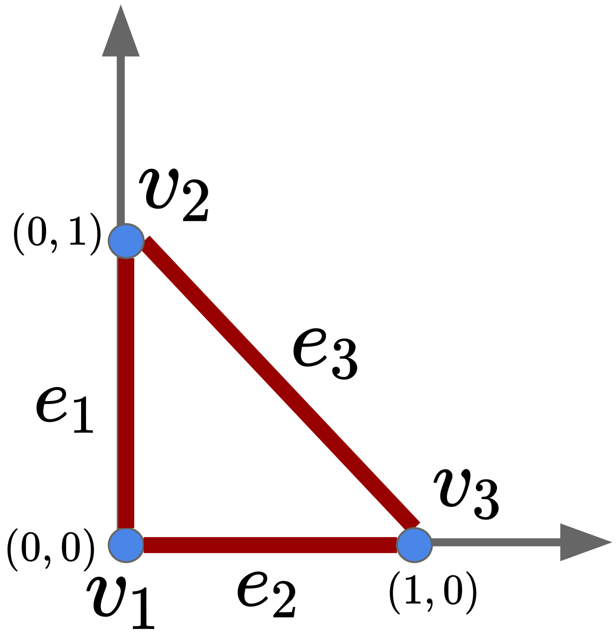
	e_1	e_2	e_3
v_1	0	-1	0
	-1	0	0
v_2	0	0	$-1/\sqrt{2}$
	1	0	$1/\sqrt{2}$
v_3	0	1	$1/\sqrt{2}$
	0	0	$-1/\sqrt{2}$



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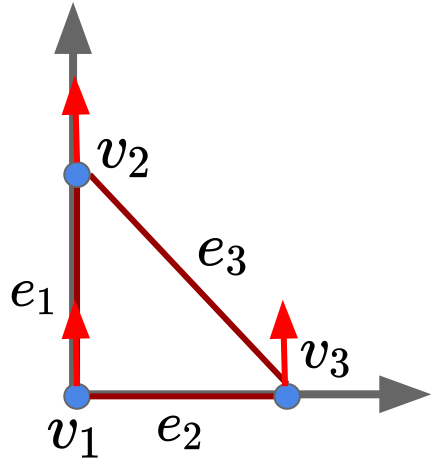
0	1	0	1	0	1

0	-1	0
-1	0	0
0	0	$-1/\sqrt{2}$
1	0	$1/\sqrt{2}$
0	1	$1/\sqrt{2}$
0	0	$-1/\sqrt{2}$

=

0
0
0
0
0
0



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1	0	1	0	1	0

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$=$

0
0
0
0
0
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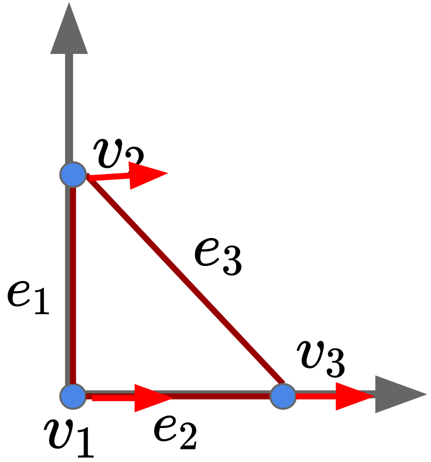

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0
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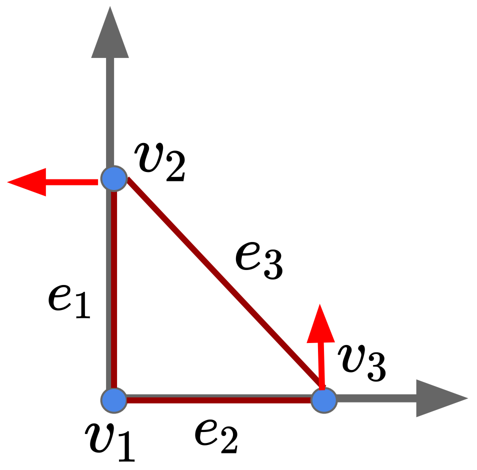

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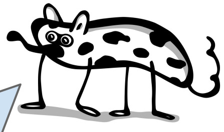

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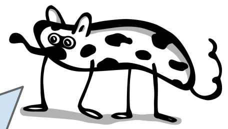

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$$a_1(G) = a(G)$$

the algebraic connectivity (a.k.a Laplacian spectral gap) of G.

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Jordán-Tanigawa ('20):

-If $a_d(G) > k$, then G remains d-rigid after removing any k vertices.

-Let G be a d-rigid graph, and let $c \geq 1$. For any t such that

$$\frac{2\log\left((dn - \binom{d+1}{2})c\right)}{a_d(G)} < t \leq 1,$$

a random subgraph of G obtained by keeping each edge of G with probability t is d-rigid with probability at least 1-1/c.

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Jordán-Tanigawa ('20), Zhu ('13):

$a_1(K_n) = n$

$a_2(K_n) = n/2$

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-What is $a_d(K_n)$?

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L-Nevo-Peled-Raz ('22+): For $d \geq 3$, $\frac{1}{2}\left\lfloor \frac{n}{d} \right\rfloor \leq a_d(K_n) \leq \frac{2n}{3(d-1)} + \frac{1}{3}$

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Conjecture (L-Nevo-Peled-Raz '22+):

$$a_d(K_n) = \begin{cases} 1 & \text{if} \quad d+1 \leq n \leq 2d, \\ \frac{n}{2d} & \text{if} \quad 2d \leq n. \end{cases}$$

Rigidity Expanders

A family of graphs $\{G_i = (V_i, E_i)\}_{i=1}^{\infty}$ with $\lim_{i \to \infty} |V_i| = \infty$ is a family of **d-rigidity expander graphs** if there is $\epsilon > 0$ such that $a_d(G_i) \geq \epsilon$ for all i.

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What happens for d>1?

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Conjecture (Jordán-Tanigawa '20, L-Nevo-Peled-Raz '22+):

For any $d \geq 1$, there **do not exist** families of **2d-regular** d-rigidity expander graphs.

A lower bound on d-dimensional algebraic connectivity

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Theorem (L-Nevo-Peled-Raz ‘22+):

$$a_d(G) \geq \min\left(\{a(G[A_i])\}_{i=1}^{d} \cup \left\{\tfrac{1}{2}a(G(A_i, A_j))\right\}_{i<j} \right).$$

A lower bound on d-dimensional algebraic connectivity

$d = 3$

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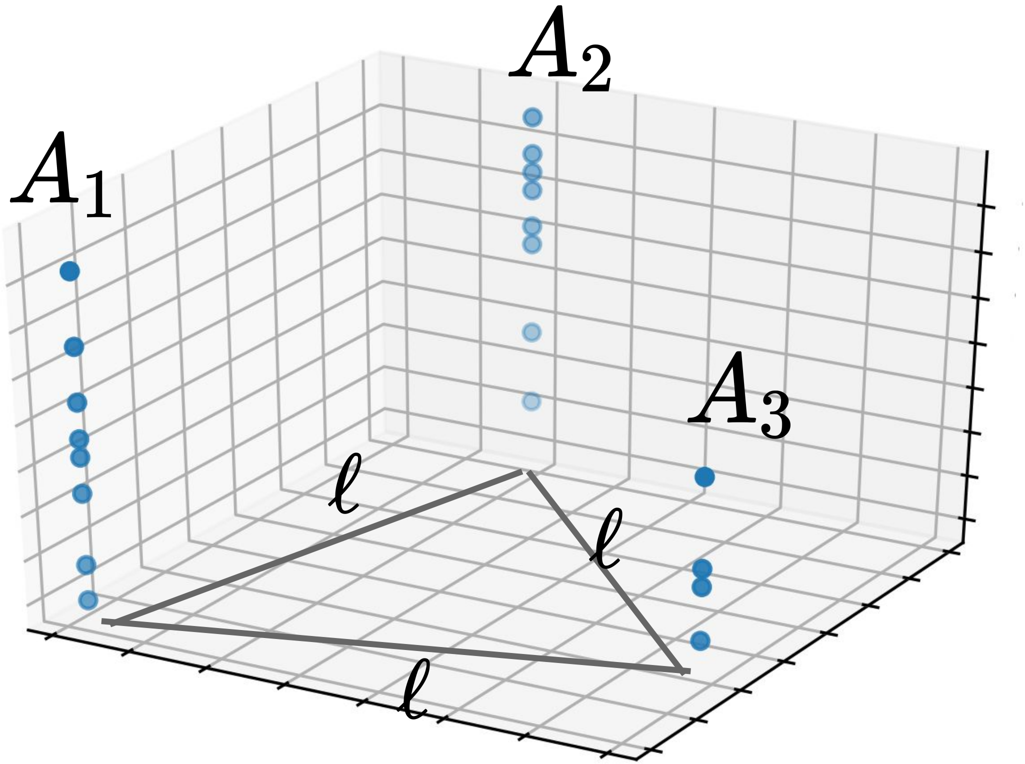

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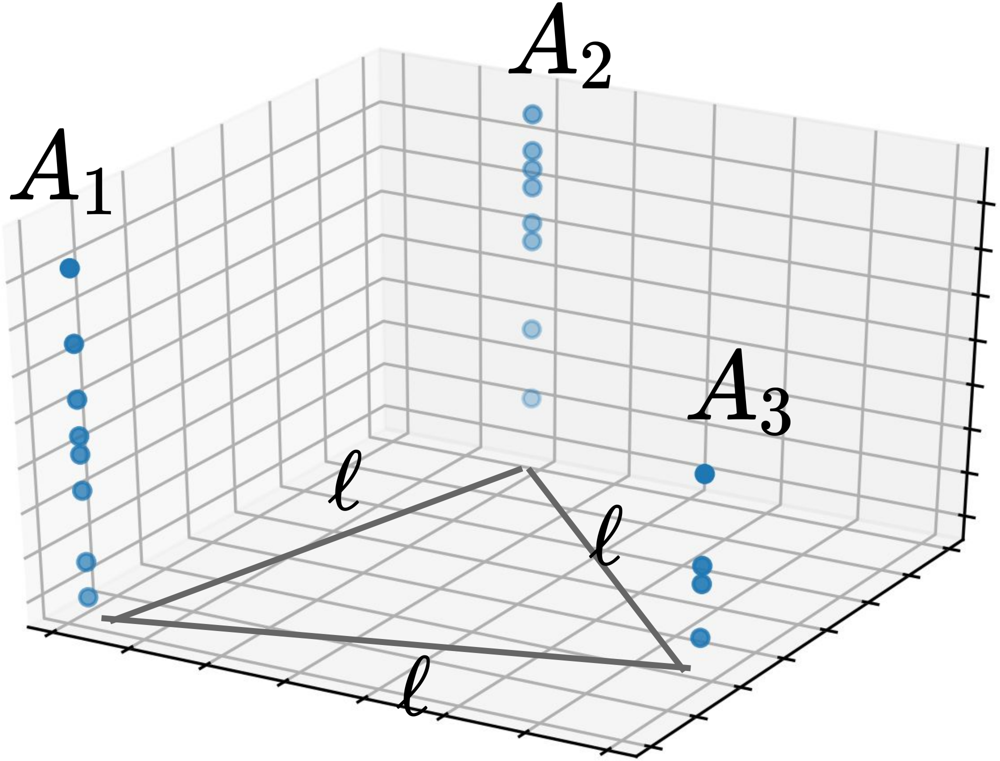

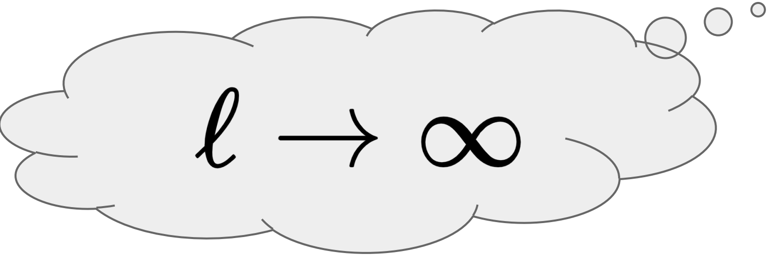

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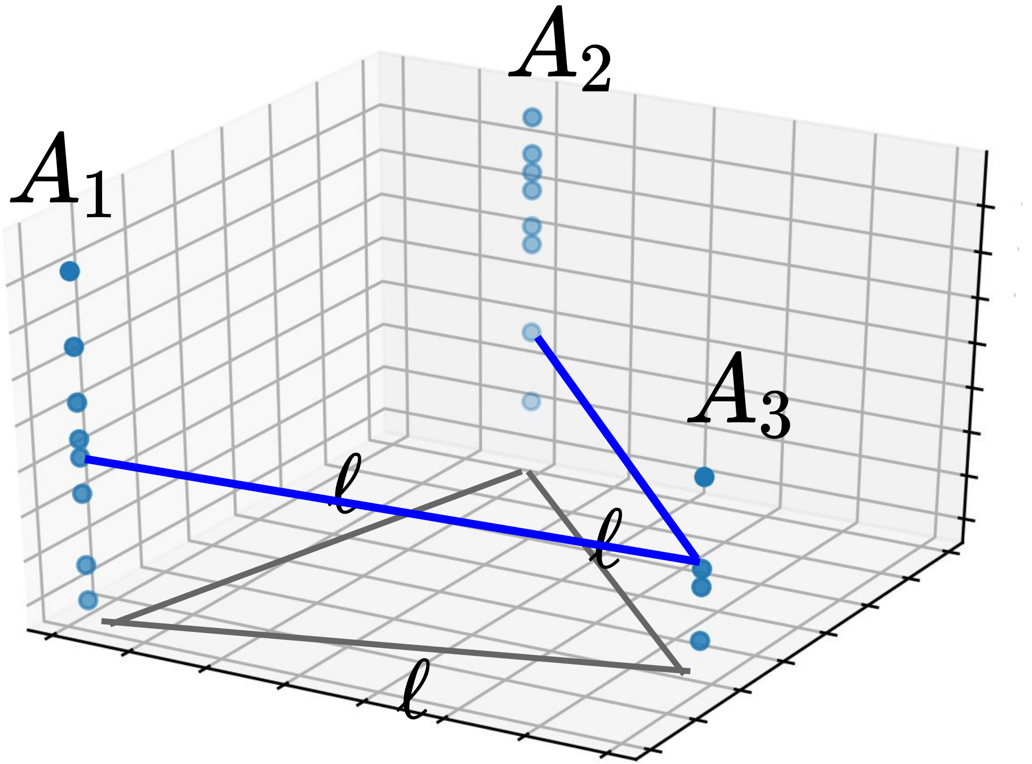

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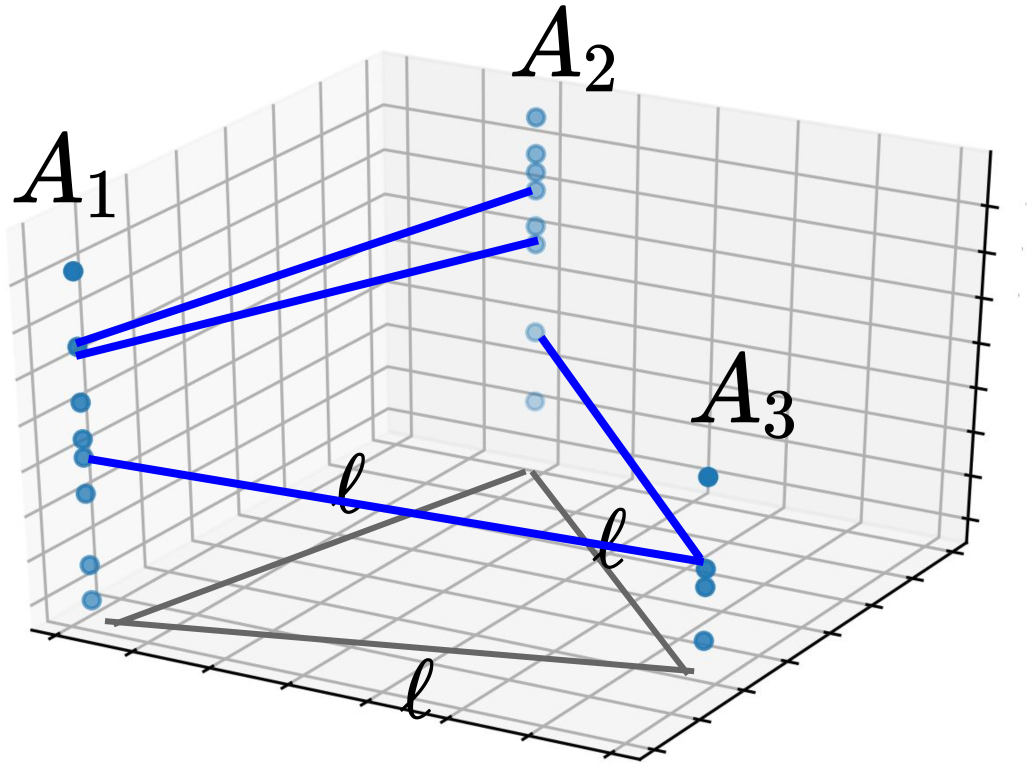

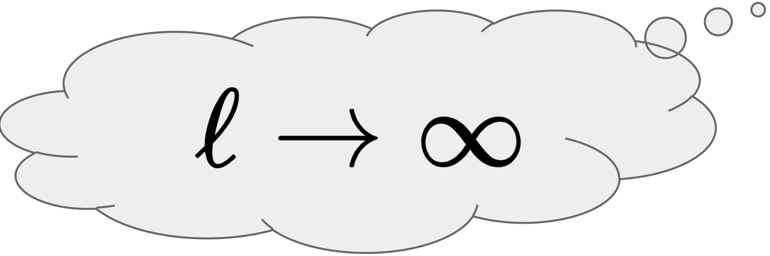

Lower bound for complete graphs:

Partition vertex set into d sets of size $\left\lfloor \frac{n}{d} \right\rfloor$ or $\left\lceil \frac{n}{d} \right\rceil$ each.

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By the theorem:

$$a_d(K_n) \geq \frac{1}{2} \left\lfloor \frac{n}{d} \right\rfloor$$

Construction of Rigidity Expanders

Lemma (L-Nevo-Peled-Raz '22+):

There is some $\epsilon = \epsilon(d) > 0$ such that for any n there exists a graph H_n on dn vertices, with exactly n vertices of degree 3 and (d-1)n vertices of degree 2, and $a(H_n) \geq \epsilon$.

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Proof idea: Take a 3-regular expander on n vertices, and subdivide some of its edges (each edge at most $\sim (d-1)/3$ times).

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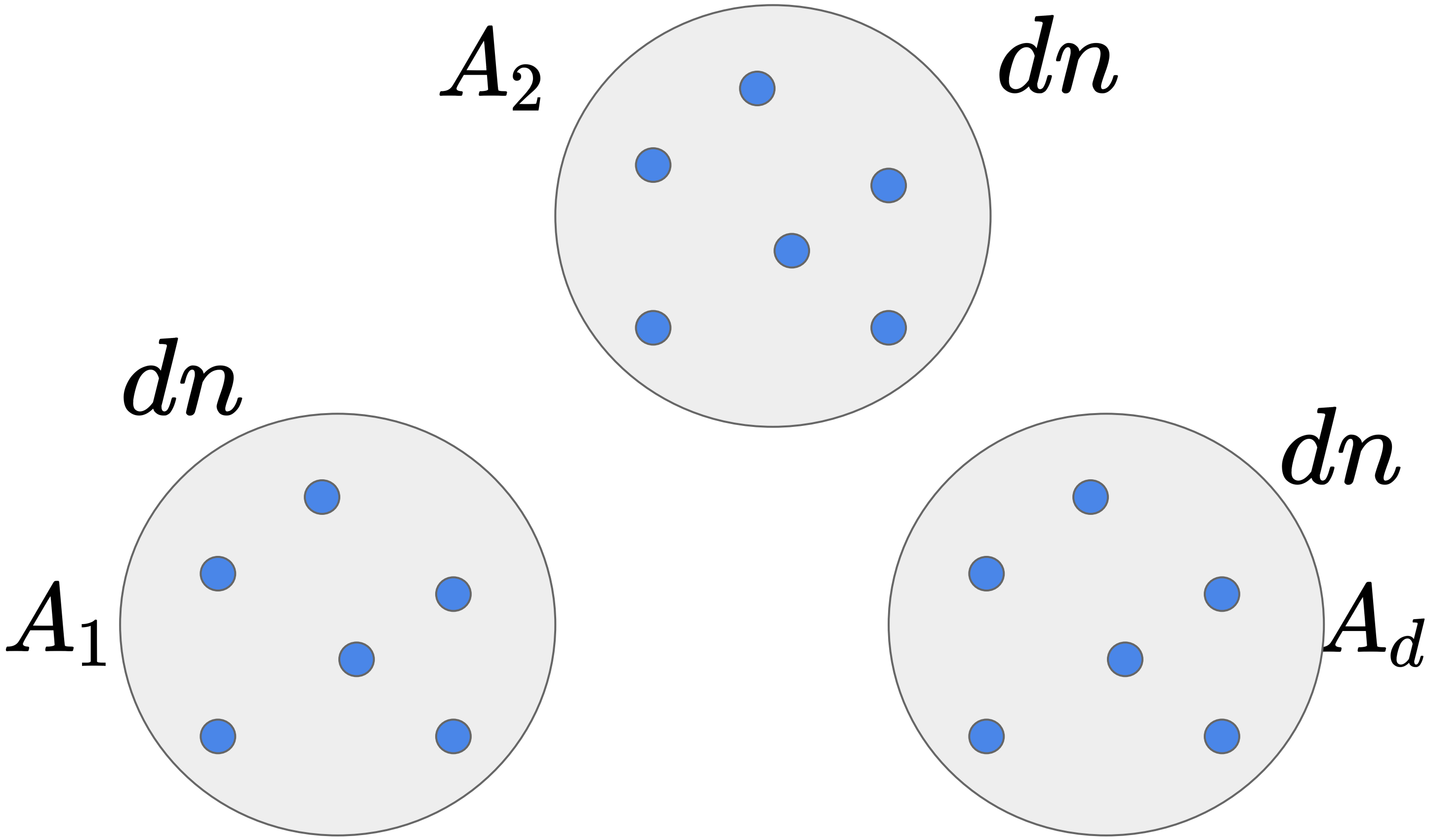

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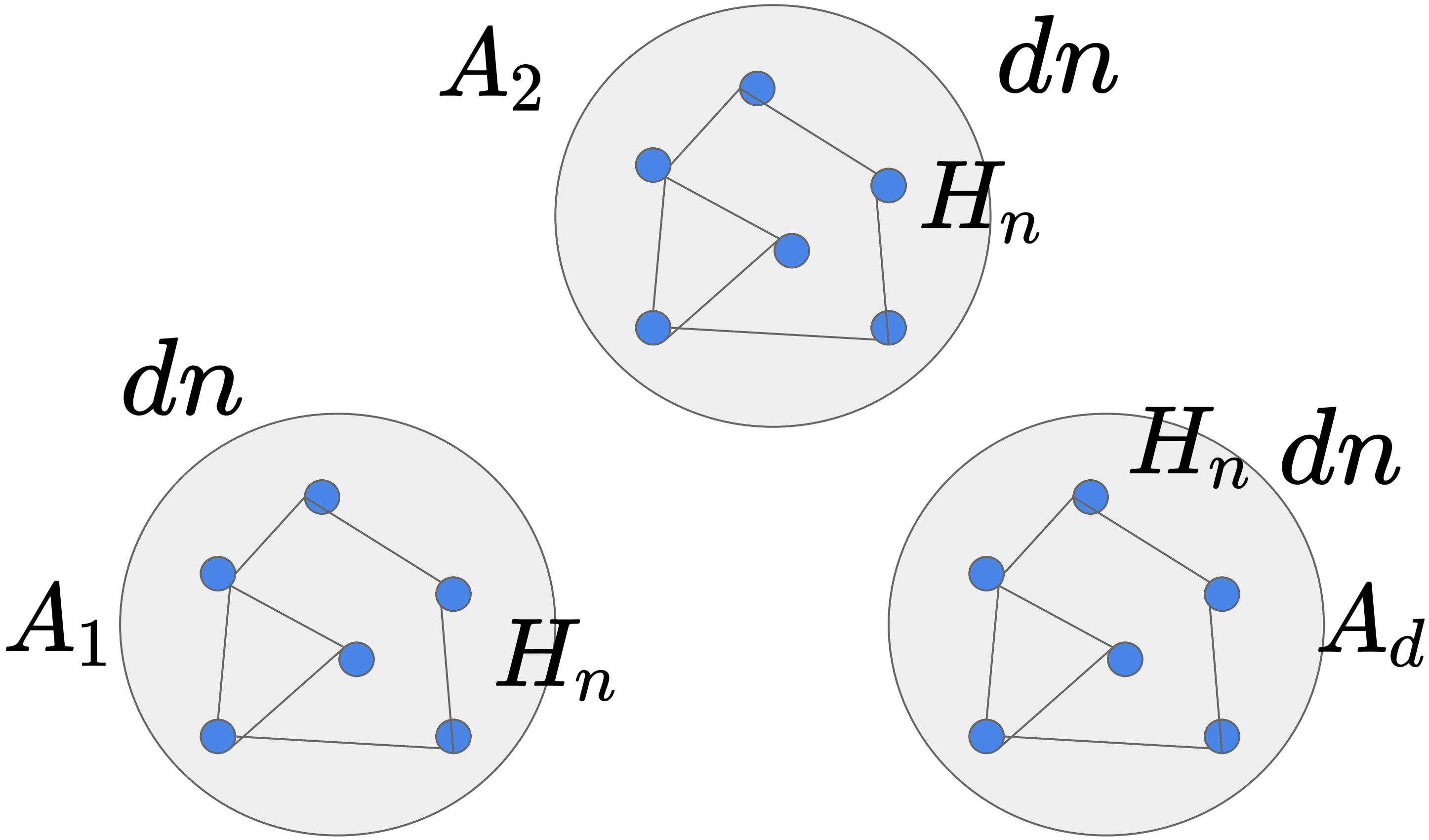

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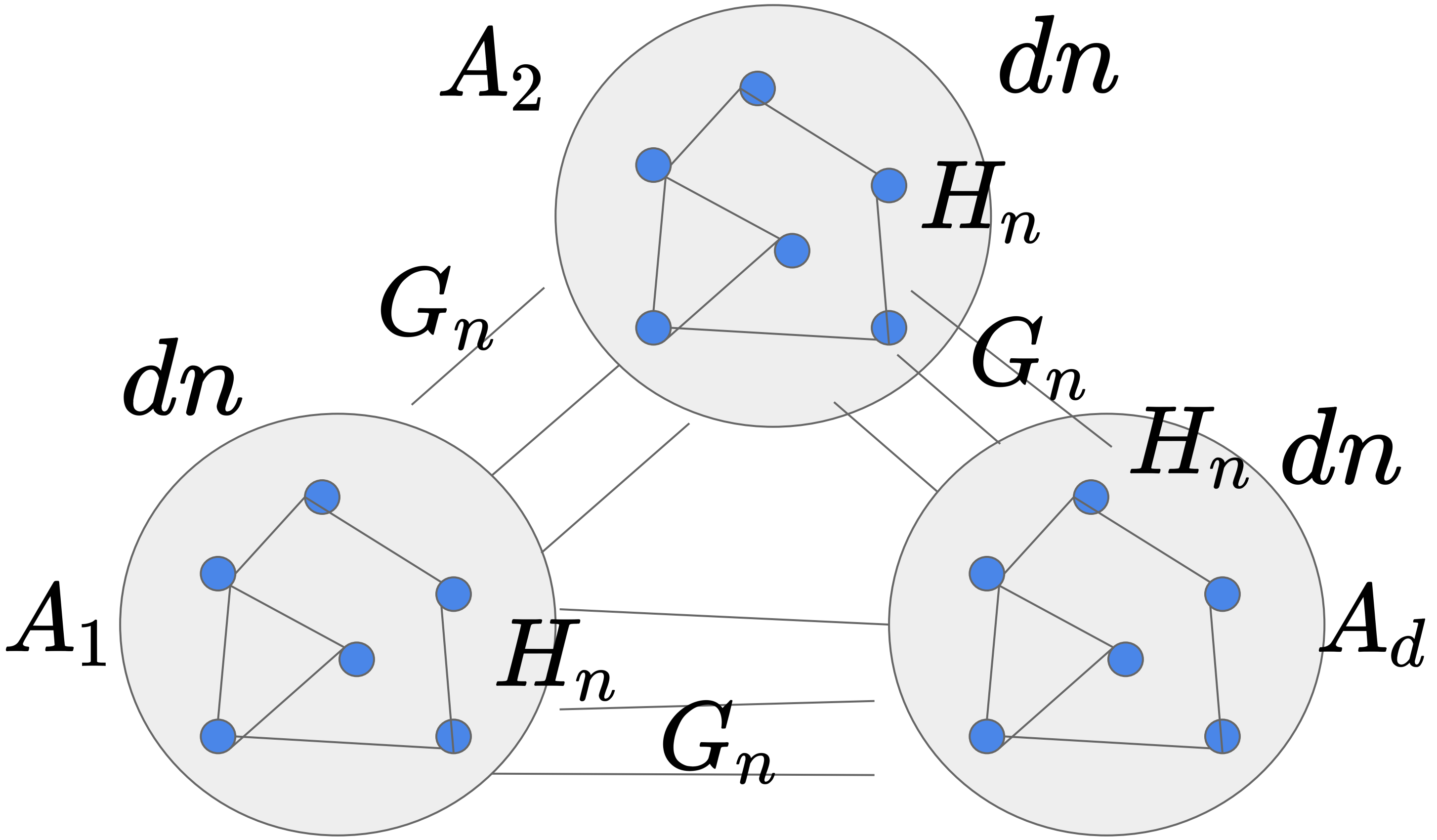

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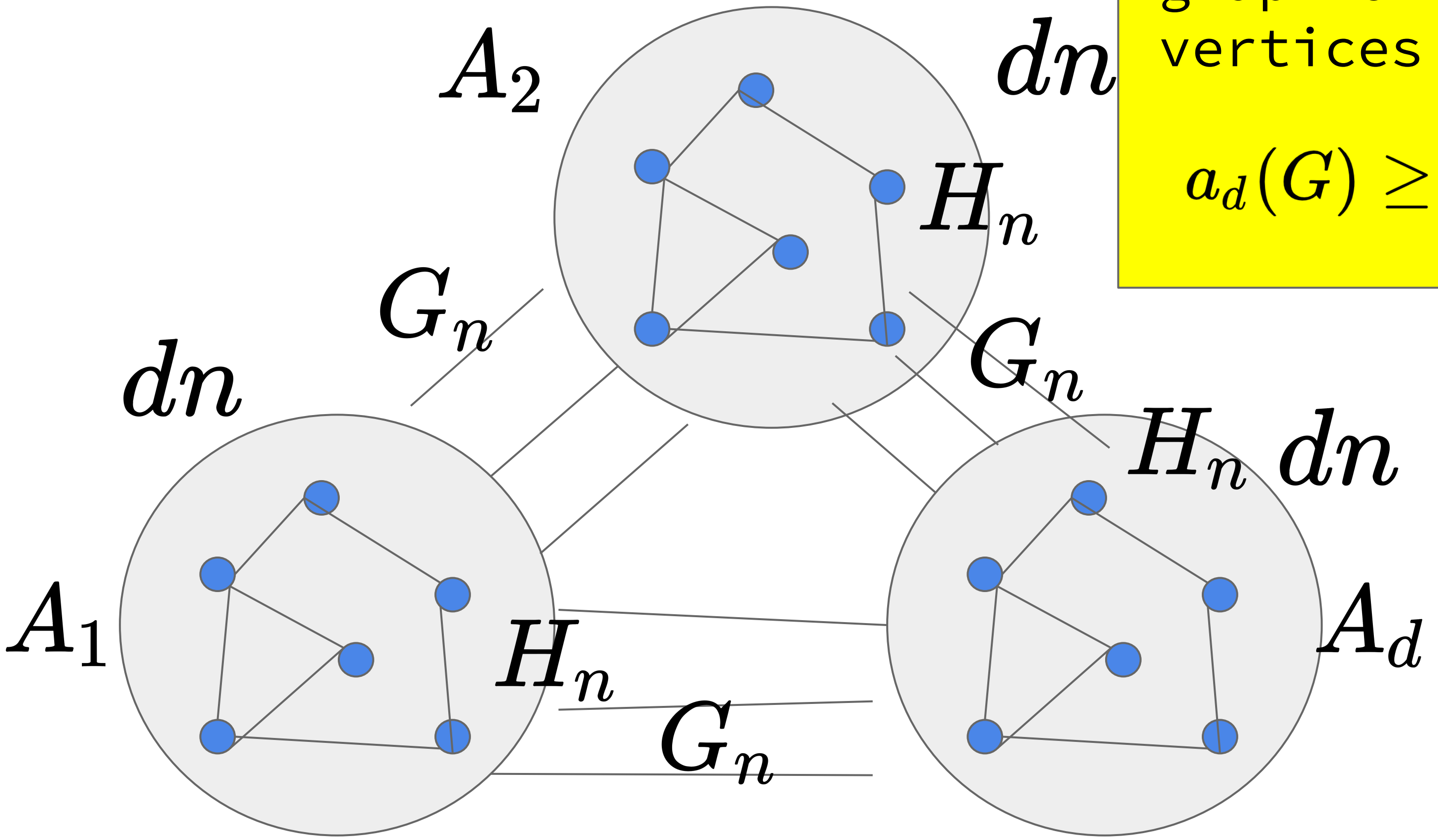

(2d+1)-regular graph on d^2n vertices and

$$a_d(G) \geq \frac{1}{2}\epsilon(d)$$

Some open problems

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- What is the best possible expansion constant for a family k-regular graphs? (Alon-Boppana-type bound)
- What can we say about the d-dimensional algebraic connectivity of minimally d-rigid graphs?
- Understand the behaviour of $a_d(G)$ when G is fixed and d varies.

Thank you for listening!