Representability and boxicity of simplicial complexes

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Boxicity

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Let $G$ be a graph. $\text{box}(G) = \text{minimal } k \text{ such that } G \text{ is the intersection of } k \text{ interval graphs.}$

$\text{box}(G) = 1 \iff G \text{ is an interval graph.}$
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Boxicity- Example

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$\text{box}(G) \leq k \iff G$ is the intersection graph of a family of axis-parallel boxes in $\mathbb{R}^k$. 
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Boxicity- Roberts’ Theorem

Theorem (Roberts ’69, Witsenhausen ’80)

Let $G$ be a graph on $n$ vertices. Then

$$\text{box}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$
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Representability

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Representability- Important properties

Helly's Theorem
Let $C_1, \ldots, C_m$ be a family of convex sets in $\mathbb{R}^d$. 
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\[ \text{Homology of a } d\text{-representable complex:} \]
Let $X$ be $d$-representable. Then, for any $k \geq d$, the $k$-th homology group of $X$ vanishes. This is a consequence of the Nerve Theorem: The homology of $N(F)$ is the same as that of the union of $\bigcup_{C \in F} C$. 

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\[
\mathcal{F} = \begin{align*}
\text{and} \quad N(\mathcal{F}) = \begin{cases}
\text{triangle} \\
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Boxicity in terms of representability

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Vertex set: $V$, 

\[ \text{box}(G) \leq k \text{ if and only if } X(G) \text{ can be written as the intersection of } k \text{ 1-representable complexes}. \]

(Follows from fact that $G = G_1 \cap \cdots \cap G_k$ iff $X(G) = X(G_1) \cap \cdots \cap X(G_k)$.)
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\[ \text{Boxicity } \text{of } G = \text{the minimum integer } k \text{ such that } G \text{ is } k-representable. \]
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**d-Boxicity**

$H_2(X) \neq 0 \Rightarrow X \text{ is not } 2\text{-representable}$

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Example ($d = 2$):

[Diagram of a simplicial complex $X$]
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$X = \begin{array}{c}
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$\text{box}_2(X) = 2$
Missing faces

Let $X$ be a simplicial complex on vertex set $V$. 

$h(X) = \text{maximum dimension of a missing face.}$

• $X$ is the clique complex of a graph $\iff h(X) = 1$ (missing faces are the edges of the complement graph of $G$).
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Let $X$ be a simplicial complex on vertex set $V$.
$\tau \subset V$ is a missing face of $X$ if $\tau \notin X$ but $\sigma \in X$ for all $\sigma \subsetneq \tau$.

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$h(X) = 2$

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Proof: Let $X = N(\mathcal{F})$.
Assume for contradiction that there is a missing face $\tau$ of dimension $d + 1$. 

Fact: If $X = X_1 \cap \cdots \cap X_k$, then $h(X) \leq \max\{h(X_i) : i = 1, \ldots, k\}$.

As a consequence:
If $\text{box}_d(X) < \infty$ then $h(X) \leq d$. 
Claim: If $X$ is $d$-representable, then $h(X) \leq d$.

Proof: Let $X = N(\mathcal{F})$.
Assume for contradiction that there is a missing face $\tau$ of dimension $d + 1$.
This corresponds to a family of sets $F_1, \ldots, F_{d+2} \in \mathcal{F}$, such that any $d + 1$ of them intersect, but $\cap_{i=1}^{d+2} F_i = \emptyset$. 
Claim: If $X$ is $d$-representable, then $h(X) \leq d$.

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Claim: If $X$ is $d$-representable, then $h(X) \leq d$.

Proof: Let $X = N(F)$.
Assume for contradiction that there is a missing face $\tau$ of dimension $d + 1$.
This corresponds to a family of sets $F_1, \ldots, F_{d+2} \in F$, such that any $d + 1$ of them intersect, but $\cap_{i=1}^{d+2} F_i = \emptyset$. This is a contradiction to Helly's Theorem.

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Steiner Systems

\[ \mathcal{F} = \text{family of subsets of size } k \text{ of a set } V \text{ of size } n. \]
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- If any subset of \( V \) of size \( t \) is contained in exactly one set of \( \mathcal{F} \), \( \mathcal{F} \) is a Steiner \((t, k, n)\)-system.

Example: Steiner triple systems \((2, 3, n)\)-systems

Any 2 vertices are contained in exactly one triple.

Example: Steiner \((1, 2, n)\)-system

Keevash (’14): For infinitely many values of \( n \), Steiner \((t, k, n)\)-systems exist.
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Steiner Systems

\( \mathcal{F} = \) family of subsets of size \( k \) of a set \( V \) of size \( n \).

- If any subset of \( V \) of size \( t \) is contained in exactly one set of \( \mathcal{F} \), \( \mathcal{F} \) is a Steiner \((t, k, n)\)-system.

Example: Steiner triple systems \(((2, 3, n)\)-systems)

Any 2 vertices are contained in exactly one triple.

Example: Steiner \((1, 2, n)\)-system

- Keevash ('14): For infinitely many values of \( n \), Steiner \((t, k, n)\)-systems exist.
Previously known results

Theorem (Witsenhausen '80):

Let $X$ be a simplicial complex with $n$ vertices satisfying $h(X) = d$. Then

$$\text{box}_d(X) \leq \left\lfloor \frac{1}{2} \binom{n}{d} \right\rfloor.$$
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Let $X$ be a simplicial complex with $n$ vertices satisfying $h(X) = d$. Then

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Remarks.

- For $d = 1$, we recover Roberts’ Theorem.
- For $d \geq 2$, this improves previous bounds due to Witsenhausen.
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Moreover, $\text{box}_d(X) = \frac{1}{d + 1} \binom{n}{d}$ if and only if the missing faces of $X$ form a Steiner $(d, d + 1, n)$-system.
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A bound on representability

A main ingredient in the proof of the bound $\text{box}_d(X) \leq \left\lfloor \frac{1}{d+1} \binom{n}{d} \right\rfloor$ is the following result:
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**Missing faces:**

\[
X = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
U
\end{array}
\]

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\[ X = \begin{matrix}
1 & 3 & 5 \\
2 & 4 & U
\end{matrix} \quad \begin{matrix}
1 & 3 & 4 & 5 \\
2 & 3 & 5 & 4
\end{matrix} \]

\[ \implies X \text{ is 2-representable.} \]
A bound on representability– Example

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1 \\
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5 \\
\end{array}$

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\[ X = \]

Missing faces:

\[ F_2 \quad F_4 \quad F_5 \]
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\[ \text{Missing faces:} \]

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\begin{align*}
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\end{align*}

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**Theorem (Wegner ’67):**

Let $X$ be a simplicial complex with $n$ vertices. Then $X$ is $(n - 1)$-representable.  

Moreover, if $X$ is not the boundary of an $(n - 1)$-dimensional simplex, then it is $(n - 2)$-representable.
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Let $X$ be a simplicial complex on vertex set $V$. For $U \subset V$, let $X[U] = \{ \sigma \in X : \sigma \subset U \}$ be the subcomplex of $X$ induced by $U$. 

- $H^k(X)$ be the $k$-th (reduced) homology group of $X$ with coefficients in $Q$.
- If $H^k(X[U]) = 0$ for all $U \subset V$ and all $k \geq d$, $X$ is called $d$-Leray.
- $X$ is $d$-representable $\Rightarrow$ $X$ is $d$-Leray. (Since any induced subcomplex of $X$ is also $d$-representable).

Theorem (L. ‘20): Let $X$ be a simplicial complex whose set of missing faces $M$ forms a Steiner $(d, d+1, n)$-system. Then, $X$ cannot be written as the intersection of less than $1^{d+1}(nd)$ $d$-Leray complexes. 

$\Rightarrow$ box $d(X) = 1^{d+1}(nd)$. 
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$\implies \text{box}_d(X) = \frac{1}{d+1}(\binom{n}{d})$. 
A tool for computing homology:
Let $K$ be a simplicial complex on vertex set $W$, and $\mathcal{N}$ its set of missing faces.
The extremal case - Sketch of proof

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Let $K$ be a simplicial complex on vertex set $W$, and $\mathcal{N}$ its set of missing faces. Define

$$\Gamma(K) = \left\{ \mathcal{N}' \subset \mathcal{N} : \bigcup_{\tau \in \mathcal{N}'} \tau \neq W \right\}.$$
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Let $X$ be a simplicial complex whose set of missing faces $M$ forms a Steiner $(d, d + 1, n)$-system. Then, $X$ cannot be written as the intersection of less than $\frac{1}{d+1} \binom{n}{d}$ $d$-Leray complexes.
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Proof:
Assume for contradiction that $X = X_1 \cap \cdots \cap X_k$, where the $X_i$’s are $d$-Leray and $k < \frac{1}{d+1} \binom{n}{d}$. 

\[ \text{Fact: } M_i = \bigcup_{i=1}^{k} M_i. \]
\[ \text{Since } |\mathcal{M}| = \frac{1}{d+1} \binom{n}{d} > k, \text{ there is some } i \text{ such that } |M_i| \geq 2. \]
\[ \text{Choose } i \text{ and } \tau_1, \tau_2 \in M_i \text{ such that } |\tau_1 \cap \tau_2| \text{ is maximal.} \]
\[ Y = X_i [\tau_1 \cup \tau_2]. \]
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Let $\mathcal{M}_i$ be the set of missing faces of $X_i$. 
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Let \(\mathcal{M}_i\) be the set of missing faces of \(X_i\). \textbf{Fact:} \(\mathcal{M} = \bigcup_{i=1}^{k} \mathcal{M}_i\).
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Choose \(i\) and \(\tau_1, \tau_2 \in \mathcal{M}_i\) such that \(|\tau_1 \cap \tau_2|\) is maximal.
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Let $M_i$ be the set of missing faces of $X_i$. **Fact:** $M = \bigcup_{i=1}^{k} M_i$.
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Let $Y = X_i[\tau_1 \cup \tau_2]$. Since $X_i$ is $d$-Leray, we must have $H_j(Y) = 0$ for all $j \geq d$. 

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**Claim:** $\Gamma(Y)$ is disconnected.
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Therefore,

\[
H_{|\tau_1 \cup \tau_2| - 3}(Y) = H_0(\Gamma(Y)) \neq 0.
\]
Claim: $\Gamma(Y)$ is disconnected. (We omit the proof)

Therefore,

$$H|_{\tau_1 \cup \tau_2}|^{-3}(Y) = H_0(\Gamma(Y)) \neq 0.$$ 

Since $\mathcal{M}$ is a Steiner $(d, d + 1, n)$-system, $|\tau_1 \cap \tau_2| < d.$
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**Claim:** $\Gamma(Y)$ is disconnected. (We omit the proof)

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Since $\mathcal{M}$ is a Steiner $(d, d + 1, n)$-system, $|\tau_1 \cap \tau_2| < d$. So,

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The extremal case - Sketch of proof

Claim: $\Gamma(Y)$ is disconnected. (We omit the proof)

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A contradiction to $H_j(Y) = 0$ for all $j \geq d$. 

Representability of complexes without large missing faces

Let \( X \) be a simplicial complex on vertex set \( V \).

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\text{rep}(X) = \text{minimal } d \text{ such that } X \text{ is } d\text{-representable.}
\]

Assume \( |V| = n \).

- Wegner ('67): \( \text{rep}(X) \leq n - 1 \).
  (Equality iff \( X \) is boundary of \( (n-1) \)-dimensional simplex).
- Roberts, Witsenhausen: If \( X \) is a clique complex (i.e. \( h(X) = 1 \)), then \( \text{rep}(X) \leq n^2 \).
  (Equality iff missing faces form a complete matching).

What is the correct bound if \( h(X) \leq d \) for some \( d \geq 2 \)?
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rep($X$) = minimal $d$ such that $X$ is $d$-representable.
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Conjecture:
Let $X$ be a simplicial complex on $n$ vertices, with $h(X) \leq d$. Then

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Moreover, $\text{rep}(X) = \frac{dn}{d + 1}$ if and only if the missing faces of $X$ consist of $\frac{n}{d+1}$ pairwise disjoint sets of size $d + 1$. 
A special case:

Let $X$ be a complex whose missing faces form a Steiner triple system. What is $\text{rep}(X)$?
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Let $X$ be a complex whose missing faces form a Steiner triple system. What is $\text{rep}(X)$?

$\text{rep}(X) = 4$. Indeed, using a different construction, we can show $\text{rep}(X) = 4$. Does $\text{rep}(X) \leq 5$ hold?
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A special case:

Let $X$ be a complex whose missing faces form a Steiner triple system. What is $\text{rep}(X)$?

\[ \implies \text{rep}(X) \leq 5 > \left\lfloor \frac{2.7}{3} \right\rfloor = 4. \]

Indeed, using a different construction, can show $\text{rep}(X) = 4$.

\[ \implies \text{rep}(X) \leq 7 > \frac{2.9}{3} - 1 = 5 \]

Does $\text{rep}(X) \leq 5$ hold?
Thank you!