# Chapter 4

# Computing the volume of a convex body

The mathematical study of areas and volumes is as old as civilization itself, and has been conducted for both intellectual and practical reasons. As far back as 2000 B.C., the Egyptians[1] had methods for approximating the areas of fields (for taxation purposes) and the volumes of granaries. The exact study of areas and volumes began with Euclid[2] and was carried to a high art form by Archimedes[3]. The modern study of this subject began with the great astronomer Johann Kepler's treatise[4] *Nova stereometria doliorum vinariorum*, which was written to help wine merchants measure the capacity of their barrels.

We consider here the problem of computing the volume of a convex body in $\mathbb{R}^n$, where $n$ is assumed to be relatively large.

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[1]The Rhind Papyrus (copied ca. 1650 BC by a scribe who claimed it derives from the "middle kingdom" about 2000 - 1800 BC) consists of a list of problems and solutions, 20 of which relate to areas of fields and volumes of granaries.

[2]The exact study of volumes of pyramids, cones, spheres and regular solids may be found in Euclid's Elements (ca. 300 BC).

[3]Archimedes (ca. 240 BC) developed the method of exhaustion (found in Euclid) into a powerful technique for comparing volumes and areas of solids and surfaces. Manuscripts:

1. Measurement of the Circle. (Proves $3\frac{10}{71} < \pi < 3\frac{1}{7}$ ).
2. Quadrature of the Parabola
3. On the Sphere and Cylinder
4. On Spirals
5. On Conoids and Spheroids

[4]The application of modern infinitesimal ideas begins with Kepler's *Nova stereometria doliorum vinariorum* (New solid geometry of wine barrels), 1615.

## 4.1 The oracle model

A convex body $K \subseteq \mathbb{R}^n$ could be be given in a number of ways. For example $K$ could be a polyhedron and we are given a list of its faces, as we would be in the domain of Linear Programming. We could also be given a set of points in $\mathbb{R}^n$ and told that $K$ is its convex hull.

In general, however, $K$ may not be a polyhedron, and it might be difficult (or even impossible) to give a compact description of it. For example, if $K = \{(y, z) \in \mathbb{R}^{m+1} : v(y) \geq z\}$, where $v(y) = \max\{cx : Ax = y, x \geq 0\}$ is the value function of a linear program ($A$ is an $m \times n$ matrix.)

We want a way of defining convex sets which can handle all these cases. This can be achieved by taking an "operational" approach to defining $K$ i.e. we assume that information about $K$ can be found by asking an oracle. We assume that we have access to a *strong membership* oracle. Given $x \in \mathbb{R}^n$ we can "ask" the oracle whether or not $x \in K$. The oracle is assumed to answer immediately. Thus the work that the oracle does is hidden from us, but in most cases of interest it would be a polynomial time computation. For example, if $K$ is a polyhedron given by its facets, all the oracle needs to do is check whether or not $x$ is on the right side of each defining hyperplane.

With such an oracle, we will need to be given a litle more information. For $x \in \mathbb{R}^n$ and $r > 0$ we let $B(x, r)$ denote a ball of radius $r$ with centre $x$ and let $B = B(0, 1)$. We assume that there exists $d \in \mathbb{R}$ such that

$$B \subseteq K \subseteq dB. \tag{4.1}$$

In this case we say that the oracle is *well-guaranteed.*

Without a similar such guarantee, one could not be certain of finding even a single point of $K$ in finite time.

## 4.2 Sampling from a convex body

We discuss generating random points in a convex body $K$ by the use of random walks. Here $\delta > 0$ is some parameter to be defined later.

**Ball Walk:** $\mathcal{BW}$
Let $v_0 = 0$ and generate $v_1, v_2, \ldots, v_k, \ldots$ as follows: With probability $1/2$ put $v_{k+1} = v_k$. Otherwise, choose $y$ randomly from $B(v_k, \delta)$. If $y \in K$ put $v_{k+1} = y$, otherwise $v_{k+1} = v_k$.

A step of $\mathcal{BW}$ where $y \notin K$ is called an *improper* step and the other steps are called *proper* steps.

One immediate problem with this Markov chain is that the state space $K$ is far from finite and we have only been discussing finite Markov chains. Also, in practise we can only compute the coordinates of points $x \in K$ to finite precision. We therefore let $\eta = 2^{-\lceil 10n\log_2(\epsilon^{-1}dn)\rceil} \leq (\epsilon^{-1}dn)^{-10n}$ for some $0 < \epsilon < 1$ and let $\mathcal{L}$ be the lattice $\eta\mathbb{Z}^n$. Our random walk will therefore be on $K_{\mathcal{L}}$ where for convex set $S$ we let $S_{\mathcal{L}} = \cap\mathcal{L}$ and we re-define $\mathcal{BW}$. For $x \in \mathcal{L}$ and $r > 0$ we let $B_{\mathcal{L}}(x,r) = \{y \in \mathcal{L} : \ |y-x| \leq r\}$:

**Ball Walk:** $\mathcal{BW}$
Let $v_0$ be chosen with distribution $P$ and generate $v_1, v_2, \ldots, v_k, \ldots$ as follows: With probability 1/2 put $v_{k+1} = v_k$. Otherwise, choose $y$ randomly from $B_{\mathcal{L}}(v_k, \delta)$. If $y \in K$ put $v_{k+1} = y$, otherwise $v_{k+1} = v_k$.

The steady state distribution $Q$ of $\mathcal{BW}$ is uniform over $K_{\mathcal{L}}$: if $x, y \in K_{\mathcal{L}}$ and $|x-y| \leq \delta$ then

$$\mathbf{Pr}(v_{k+1} = y \mid v_k = x) = \frac{1}{2|B_{\mathcal{L}}(0,\delta)|}.$$

Thus the uniform distribution satisfies the detailed balance equations (1.13).

**Speedy Walk:** $\mathcal{SW}$
If we ignore the improper steps of $\mathcal{BW}$ then we have a sequence $v_0' = v_0, v_1', v_2', \ldots$ which define a new Markov chain called the *speedy walk*.

The steady state distribution of $\mathcal{SW}$ is not uniform. For $x \in K_{\mathcal{L}}$ let its *local conductance* $\ell_{\mathcal{L}}(x)$ be defined by

$$\ell_{\mathcal{L}}(x) = \frac{|K_{\mathcal{L}} \cap B_{\mathcal{L}}(x,\delta)|}{|B_{\mathcal{L}}(x,\delta)|}.$$

The steady state distribution $Q_{\mathcal{L}}$ of $\mathcal{SW}$ is given by

$$Q_{\mathcal{L}}(x) = \frac{\ell_{\mathcal{L}}(x)}{\ell_{\mathcal{L}}(K_{\mathcal{L}})}.$$

This is just the usual degree formula (1.14). The *average local conductance* is

$$\lambda_{\mathcal{L}} = \frac{\ell_{\mathcal{L}}(K_{\mathcal{L}})}{|K_{\mathcal{L}}|}.$$

**Theorem 4.2.1** *If $K \supseteq B$ then*

$$\lambda_{\mathcal{L}} \geq 1 - \delta\sqrt{n}.$$

**Proof** (Deferred to Section 4.2.1) □

It will be useful to note that $U_{\mathcal{L}}$ is close in distrbution to $Q_{\mathcal{L}}$.

**Lemma 4.2.1**

$$D_{\mathrm{tv}}(Q_{\mathcal{L}}, U_{\mathcal{L}}) \leq \frac{1-\lambda_{\mathcal{L}}^2}{\lambda_{\mathcal{L}}}.$$

**Proof** Let $A \subseteq K_{\mathcal{L}}$. Then

$$|Q_{\mathcal{L}}(A) - U_{\mathcal{L}}(A)| = \left| \sum_{x \in A} \left( \frac{\ell_{\mathcal{L}}(x)}{\ell_{\mathcal{L}}(K_{\mathcal{L}})} - \frac{1}{|K_{\mathcal{L}}|} \right) \right| \leq$$
$$\sum_{x \in A} \left( \frac{\ell_{\mathcal{L}}(x)}{\ell_{\mathcal{L}}(K_{\mathcal{L}})} - \frac{\ell_{\mathcal{L}}(x)}{|K_{\mathcal{L}}|} \right) + \sum_{x \in A} \left( \frac{1}{|K_{\mathcal{L}}|} - \frac{\ell_{\mathcal{L}}(x)}{|K_{\mathcal{L}}|} \right)$$
$$\leq \frac{|A|(|K_{\mathcal{L}}| - \ell_{\mathcal{L}}(K_{\mathcal{L}}))}{\ell_{\mathcal{L}}(K_{\mathcal{L}})|K_{\mathcal{L}}|} + \frac{|K_{\mathcal{L}}| - \ell_{\mathcal{L}}(K_{\mathcal{L}})}{|K_{\mathcal{L}}|} \leq \frac{1 - \ell_{\mathcal{L}}}{\ell_{\mathcal{L}}} + 1 - \ell_{\mathcal{L}}.$$

□

Our choice of $\delta = o(1/\sqrt{n})$ is such that $\lambda_{\mathcal{L}} = 1 - o(1)$ for all bodies discussed in this and later sections.

**Theorem 4.2.2** *Let $K \subseteq dB$ be a convex body, $d > 32\delta$. Then the mixing time $\tau_{\mathcal{SW}}(\epsilon)$ of the speedy walk satisfies*

$$\tau_{\mathcal{SW}}(\epsilon) \leq \kappa n d^2 \delta^{-2} \log(1/\epsilon)$$

*for some absolute constant $\kappa > 0$.*

**Proof** (Deferred to Section 4.5.3) □

We will refer later to two sampling algorithms. First we need the following nesting of convex sets: Let $K$ be a convex body where $B \subseteq K \subseteq dB$. Let $K^{(i)} = (2^{i/n}B) \cap K$ for $0 \leq i \leq m = n \log_2 d$. Then $K^{(0)} = B$ and $K^{(m)} = K$. In general quantities superscripted by $i$ will refer to $K^{(i)}$ e.g. $\lambda_{\mathcal{L}}^{(i)}$ denotes the average local conductance of $K^{(i)}$.

**Algorithm** NESTED SAMPLING:
**begin**
  Choose $u_0$ uniformly from $B_{\mathcal{L}}$.
  **For** $i = 1$ **to** $m$ **do**
  $v \leftarrow u_{i-1}$
  **begin**
    **A**: Carry out a $t$-step speedy walk on $K_{\mathcal{L}}^{(i)}$ starting at $v$ and ending at $w$.
    **If** $u = \frac{2n}{2n-1} w \in K_{\mathcal{L}}^{(i)}$ **then** $u_i \leftarrow u$ and go to the next $i$.
    **Otherwise** $v \leftarrow w$, **goto** A.
  **end**
**end**

**Algorithm** ORDINARY SAMPLING:

**begin**
  Run Algorithm NESTED SAMPLING to obtain $v_1 = u_m$.
  **For** $i = 1$ **to** $p$ **do**
  $w \leftarrow v_{i-1}$
  **begin**
    **A**: Carry out a $t$-step speedy walk on $K_{\mathcal{L}}$ starting at $w$ and ending at $x$.
    **If** $v = \frac{2n}{2n-1}x \in K_{\mathcal{L}}^{(i)}$ **then** $v_i \leftarrow v$ and go to the next $i$.
    **Otherwise** $w \leftarrow x$, **goto** A.
  **end**
**end**

Let $U_{\mathcal{L}}$ denote the uniform distribution on $K_{\mathcal{L}}$ and let $U_{\mathcal{L}}^{(i)}$ denote the uniform distribution on $K_{\mathcal{L}}^{(i)}$ for $i = 1, 2, \dots, m$.

**Theorem 4.2.3** *Let* $0 < \alpha < \frac{1}{50}$ *and*

$$\delta \leq \frac{\alpha}{\sqrt{n}} \text{ and } t = \left\lceil \kappa n d^2 \delta^{-2} \ln(10/\alpha) \right\rceil.$$

*Let* $U^{(i)}$ *denote the distribution of* $u_i$ *in* NESTED SAMPLING, given $u_j, j \neq i$ *and let* $V^{(i)}$ *denote the distribution of* $v_i$ *in* ORDINARY SAMPLING, given $v_j, j \neq i$. *Then*

*(a)* $D_{\text{tv}}(U_{\mathcal{L}}^{(i)}, U^{(i)}) \leq 4\alpha$ *for* $i = 1, 2, \dots, m$ *and conditional on an event* $\mathcal{G}$ *of probability at least* $1 - 5m\alpha$, *the expected number of oracle calls in* NESTED SAMPLING *is at most* $10mt$.

*(b)* $D_{\text{tv}}(U_{\mathcal{L}}, V^{(i)}) \leq 4\alpha$ *for* $i = 1, 2, \dots, p$ *and conditional on an event* $\mathcal{G}$ *of probability at least* $1 - 5m\alpha$, *the expected number of oracle calls in* ORDINARY SAMPLING *is at most* $10(m + p)t$.

**Proof** Deferred to Section 4.5.1 □

## 4.3 Volume Algorithm

$0 < \epsilon < 1$ be given. Let

$$\delta = \frac{1}{\sqrt{8n \ln(n/\epsilon)}}, \; p = \left\lceil \frac{120m}{\epsilon^2} \right\rceil, \; \epsilon_0 = \left\lceil \frac{\epsilon}{12800^2 mnp} \right\rceil \text{ and } t = \left\lceil \kappa n \left(\frac{d}{\delta}\right)^2 \ln \frac{1}{\epsilon_0} \right\rceil. \tag{4.2}$$

We write

$$|K_{\mathcal{L}}| = |K_{\mathcal{L}}^{(0)}| \prod_{i=1}^{m} \frac{|K_{\mathcal{L}} i|}{K_{\mathcal{L}}^{(i-1)}|}.$$

We need to be sure that an estimate of $|K_{\mathcal{L}}|$ yields a good estimate of vol$(K)$.

**Theorem 4.3.1** *Let $S \supseteq \alpha B$ be a convex set in $\mathbb{R}^n$ where $\alpha \geq \delta$. Then*

$$|vol(S) - \eta^n |S_{\mathcal{L}}|| \leq \frac{\epsilon}{10} vol(S).$$

**Proof** (Deferred to Section 4.2.1) □

We note that $K^{(i)} \subseteq 2^{1/n} K^{(i-1)}$ and so

$$\frac{1}{2} \leq \rho_i = \frac{|K_{\mathcal{L}}^{(i-1)}|}{|K^{(i)}|} \leq 1. \tag{4.3}$$

We use sampling to estimate the ratios $\rho_i$, $i = 1, 2, \ldots, m$. Since $|K_{\mathcal{L}}^{(0)}| = |B_{\mathcal{L}}|$ can be computed to arbitrary accuracy, we see that this will give us an estimate of $|K_{\mathcal{L}}|$ and hence of vol$(K)$.

**Algorithm** VOLUME COMPUTATION
Run Algorithm NESTED SAMPLE $p$ times with $t$ as in (4.2) and $\alpha = \epsilon_0$ to obtain $u_{i,r}$, $i = 0, 1, \ldots, m$, $r = 1, 2, \ldots, p$.

Now for $r = 1, 2, \ldots, p$ define

$$a_{i,r} = \begin{cases} 1 & u_{i,r} \in K_{\mathcal{L}}^{(i-1)} \\ 0 & \text{otherwise} \end{cases}$$

and let $b_i = a_{i,1} + \cdots + a_{i,p}$ for $i = 1, 2, \ldots, m$.

Then put

$$\zeta = \eta^n \frac{|B_{\mathcal{L}}| p^m}{b_1 b_2 \cdots b_m}.$$

**Theorem 4.3.2** *Assume that $B \subseteq K \subseteq dB$. Then*

$$\mathbf{Pr}(\zeta \in [(1-\epsilon) vol(K), (1+\epsilon) vol(K)]) \geq \frac{3}{4} - o(1).$$

*Furthermore, conditional on an event $\mathcal{G}$ of probability $1 - O(\frac{\epsilon}{n})$ the expected number of oracle calls for Algorithm* VOLUME COMPUTATION *is $O(n^4 d^2 \ln d (\ln n/\epsilon)^2 \epsilon^{-2})$.*

**Proof** Let $\beta_i = p\rho_i$ and

$$X = \sum_{i=1}^{m+1} \ln\left(\frac{b_i}{\beta_i}\right).$$

We show

$$\mathbf{Pr}(|X| \geq \epsilon/2) \leq \frac{1}{4} + o(1). \tag{4.4}$$

Included in this calculation is the assumption that $\mathcal{G}$ occurs. Now, $\mathbf{Pr}(\mathcal{G}) = 1 - O(\frac{\epsilon}{n})$ and so what we actually prove is

$$\mathbf{Pr}(|X| < \epsilon/2) \geq \mathbf{Pr}(|X| < \epsilon/2 \mid \mathcal{G})\mathbf{Pr}(\mathcal{G}) \geq \frac{3}{4}\left(1 - O\left(\frac{\epsilon}{n}\right)\right).$$

Now $e^X \zeta = \eta^n |K_{\mathcal{L}}|$ and so $|X| < \epsilon/2$ implies

$$\mathrm{vol}(K) \in \left[\left(1 - \frac{\epsilon}{10}\right) e^{-\epsilon/2}\zeta, \left(1 + \frac{\epsilon}{10}\right) e^{\epsilon/2}\zeta\right].$$

Hence for $i = 1, 2, \ldots, m$, $b_i$ has the binomial distribution $B(p, \alpha_i)$ where

$$\alpha_i = \mathbf{Pr}(u_i \in K_{\mathcal{L}}^{(i-1)}) = \frac{|K_{\mathcal{L}}^{(i-1)}|}{|K_{\mathcal{L}}^{(i)}|} + \epsilon_1 = \rho_i + \epsilon_1$$

where $|\epsilon_1| \leq \epsilon_0$.

It follows from (4.3) that $\alpha_i \geq 1/3, 1 \leq i \leq m$.

Applying the Chernoff bounds we obtain that with probability $1 - o(1)$

$$b_i \geq \frac{\beta_i}{\sqrt{2}} \qquad i = 1, 2, \ldots, m. \tag{4.5}$$

Set

$$A = \sum_{i=1}^{m} \frac{b_i - \beta_i}{\beta_i}, \; C = \sum_{i=1}^{m} \left(\frac{b_i - \beta_i}{\beta_i}\right)^2 \text{ and } D = \sum_{i<j} \frac{(b_i - \beta_i)(b_j - \beta_j)}{\beta_i \beta_j}.$$

Using the formula for the variance of the binomial distribution we get

$$\mathbf{E}(C) = \sum_{i=1}^{m} \frac{1}{\beta_i}\left(1 - \frac{\beta_i}{p}\right) + O(m\epsilon_0) \leq \frac{\epsilon^2}{150}.$$

Now $a_{i,r}, a_{j,s}$ are independent for $r \neq s$ and Lemma**??** implies $|\mathbf{Pr}(a_{i,r} = 1 \mid a_{j,r} = 1) - \alpha_i| \leq \epsilon_0$ for arbitrary $i, j, r$ and so

$$\mathbf{E}\left(\frac{(b_i - \beta_i)(b_j - \beta_j)}{\beta_i \beta_j}\right) \leq \frac{4p^2 \epsilon_0}{\beta_i \beta_j} \leq 40\epsilon_0$$

and hence

$$\mathbf{E}(D) < 20m^2\epsilon_0 \leq \frac{\epsilon^2}{640}.$$

**Claim 4.3.1** *Whenever (4.5) holds, $C < \epsilon^2/30$ and $D < \epsilon^2/64$ then we have $|X| < \epsilon/2$.*

**Proof of Claim** Since $A^2 = C + 2D$ we see that $|A| < \epsilon/\sqrt{15}$. If $X \geq 0$ then using the inequality $\ln x \leq x - 1$, we obtain $X \leq A < \epsilon/2$. If $X < 0$ then using (4.5) and the inequality $\ln x \geq x - 1 - (x-1)^2$ (for $x \geq 1/\sqrt{2}$) we get $X \geq A - C \geq -\epsilon/2$.
**End of proof of claim**

By Markov's inequality, the probability that either $C > \epsilon^2/30$ or $D > \epsilon^2/64$ is at most .25 and (4.4) follows.

We now consider the expected number of steps in the algorithm. We first remark that the algorithm requires

$$t(m+1)p = O(n^4 d^2 \ln d (\ln n/\epsilon)^2 \epsilon^{-4}) \text{ proper steps.}$$

Given $\mathcal{G}$, the expected total number of steps, proper and improper is say $\leq 5Ctmp$ for some $C > 0$. We stop the algorithm if fewer than $Ctmp$ proper steps have been made after $50Ctmp$ steps in total. Then we succeed in producing an estimate with probability at least $\frac{9}{10} - o(1)$.

We have not said anything about the size of $d$. Using the Ellipsoid Algorithm one can in $O^*(n^4)$ steps[5] find an affine transformation $K' = AK + b$ of $K$ such that $B \subseteq K' \subseteq O(n^{3/2})B$. Here $A$ is an $n \times n$ matrix and $\mathrm{vol}(K') = \det(A)\mathrm{vol}(K)$. So, applying the above theorem we obtain an $O^*(n^7)$ algorithm. In Section 4.4 we show how to reduce $d$ to $O^*(n^{1/2})$ and obtain an $O^*(n^5)$ algorithm.

## 4.4 Putting a body into isotropic position

For a convex body $K$ and real function $f$ we let

$$\mathbf{E}_K(f) = \int_K f(x)dx.$$

The definition extends naturally to vectors of functions.

A body $K$ is in *isotropic position* if its *centre of gravity*

$$b = b(K) = \mathbf{E}_K(x) = 0 \qquad \text{and} \qquad \mathbf{E}_K((v^T x)^2) = 1 \text{ for all } |v| = 1.$$

If a body is in isotropic position then it contains $B$ and most of its volume is contained
reference? in $O(\sqrt{n})B$. This makes it a useful concept for volume approximation. It is known that for any convex body $K$ there is an affine transformation $T$ such that $TK$ is in isotropic position. We only manage to obtain $\theta$-isotropic position in this section, i.e.

$$|b(K)| \leq \theta \qquad \text{and} \qquad 1 - \theta \leq \mathbf{E}_K((v^T x)^2) \leq 1 + \theta \text{ for all } |v| = 1.$$

[5] $O^*$ notation ignores all factors other than powers of $n$.

**Theorem 4.4.1** *If $K$ is in $\theta$-nearly isotropic position then*

**(a)** $(1-2\theta)B \subseteq K \subseteq (1+2\theta)(n+1)B$.

**(b)** $vol(K \cap d_{\epsilon,\theta}B) \geq (1-\epsilon)vol(K)$ *where* $d_{\epsilon,\theta} = \left(\frac{(1+\theta)n}{\epsilon}\right)^{1/2}$.

**Proof** (a) . Can't find a simple proof of (a)
(b) Let $x$ be chosen randomly from $K$. By assumption, $\mathbf{E}(|x|^2) \leq (1+\theta)n$. Therefore, for any $d > 0$,

$$1 - \frac{\text{vol}(K \cap dB)}{\text{vol}(K)} = \mathbf{Pr}(|x| > d) \leq \frac{\mathbf{E}(|x|^2)}{d^2} \leq \frac{(1+\theta)n}{d^2}$$

and (b) follows. □

Our aim is to desribe an algorithm for finding an affine transformation $A$ such that $AK$ is in $\theta$-nearly isotropic position for some small $\theta$.

**Algorithm** ISOTROPY 1
Suppose $0 < \theta, \gamma \leq 1/4$ and let

$$\epsilon_1 = \frac{\gamma^2\theta^2}{32(n+1)^4} \text{ and } m_1 = \left\lceil \frac{80n^2}{\theta^2\gamma^2} \right\rceil.$$

use Algorithm ORDINARY SAMPLE with $p = m_1$ and $\alpha = \epsilon_1$ to obtain $y^{(1)}, y^{(2)}, \ldots, y^{(m_1)} \in K_{\mathcal{L}}$. Compute the vector

$$\bar{y} = \frac{1}{m_1}\sum_{i=1}^{m_1} y^{(i)}$$

and the matrix

$$Y = \frac{1}{m_1}\sum_{i=1}^{m_1}(y^{(i)} - \bar{y})(y^{(i)} - \bar{y})^T.$$

If $Y$ is singular, declare failure and repeat. Otherwise, output $K' = Y^{-1/2}(K - \bar{y})$.

**Theorem 4.4.2** *With probability $\geq 1-\gamma$, the body $K'$ is in $\theta$-nearly isotropic position.*

**Proof** (Deferred to Section 4.6.2) □

The rest of this section describes how to obtain $y^{(1)}, y^{(2)}, \ldots, y^{(m_1)}$ with expected number $O^*(n^5)$ of oracle calls. Having done this, the $O^*(n^5)$ volume algorithm is immediate. It follows from Theorem 4.4.1 that $B \subseteq (1-2\theta)^{-1}(K' \cap d_{\epsilon,\theta}B) \subseteq (1-2\theta)^{-1}d_{\epsilon,\theta}B$. Applying the algorithm of Section 4.3 we obtain a good approximation to $(1-2\theta)^{-n}\text{vol}(K' \cap d_{\epsilon,\theta}B)$ which by Theorem 4.4.1 is a good approximation to $(1-2\theta)^{-n}\text{vol}(K') = (1 -$

$2\theta)^{-n}\det(Y^{-1/2})\mathrm{vol}(K)$. The expected number of oracle calls for the volume computation is $O(n^4 d_{\epsilon,\theta}^2 \ln d_{\epsilon,\theta}(\ln n/\epsilon)^2\epsilon^{-4}) = O^*(n^5)$.

Let us assume for the moment that

$$B \subseteq K \subseteq 10nB. \tag{4.6}$$

We will need to compute an approximation to $b(K)$.

**Algorithm** BARYCENTRE
Let $0 < \phi, \gamma < 1$ be given and let

$$m_2 = \left\lceil \frac{8n}{\phi\gamma} \right\rceil \text{ and } \epsilon_2 = \frac{\phi\gamma}{20(n+1)^2}.$$

Apply Algorithm ORDINARY SAMPLE with $p = m_2$ and $\alpha = \epsilon_2$ to obtain $z^{(1)}, z^{(2)}, \ldots, z^{(m_2)}$ and compute their centre of gravity $g$.

**Theorem 4.4.3**

**(a)** *With probability* $\geq 1-\gamma$, $g - b(K) \in -\phi(K - b(K))$.

**(b)** *Assume that* $K$ *is in isotropic position. Then with probability* $\geq 1-\gamma$, $|g - b(K)| \leq \phi$.

**Proof** (Deferred to Section 4.6.1). □

We may assume therefore that we have carried out Algorithm BARYCENTRE and that

$$b(K) \in -\frac{1}{10}K. \tag{4.7}$$

The algorithm for generating $y^{(1)}, y^{(2)}, \ldots, y^{(m_1)}$ in Algorithm ISOTROPY is then

**(Step 1)** Apply Algorithm ORDINARY SAMPLE with $p = 1$ and $\alpha = \gamma/6$ to get a single point $u$. This needs $O^*(n^5)$ oracle calls, in expectation.

**(Step 2)** Rescale $K$ so that

$$\frac{1}{\sqrt{n}}B \subseteq K \subseteq 10\sqrt{n}B, \tag{4.8}$$

which implies

$$\mathbf{E}_K(|x|^2) \leq 100n. \tag{4.9}$$

Apply Algorithm LOCAL CONDUCTANCE (Section 4.4.1 below) to increase the local conductance of $K$ to at least $1 - 100\theta \geq .999$. This algorithm needs $O^*(n^5)$ oracle calls, in expectation.

**(Step 3)** Apply Algorithm ORDINARY SAMPLE to generate $y^{(1)}, y^{(2)}, \ldots, y^{(m_1)}$. The expected number of oracle calls is $O^*(n^5)$, because we have reduced the diameter to $O(\sqrt{n})$ having used Algorithm LOCAL CONDUCTANCE to get the local conductance up to .999.

This completes the description for getting $K$ into $\theta$-nearly isotropic position when (4.8) is satisfied. We now show how to eliminate this assumption. Assume that $K \subseteq dB$.

**Algorithm** ISOTROPY 2
**begin**
$p = \lceil \log_2 d \rceil$ , $K_0 = K$.
**For** $i = 0$ **to** $p$ **do**
**begin**
$K_i' = K_i \cap 10nB$.
Use Algorithm ISOTROPY 1 to find with probabilty $\geq 1 - \frac{\gamma}{\log_2 d}$,
a map $T_i$ which takes $K_i'$ into $\theta$-nearly isotropic position.
$K_{i+1} = T_i K_i$.
**end**
**Output** $K_p$.
**end**

**Theorem 4.4.4** *With probability* $\geq 1-\gamma$, *the body* $K_p$ *produced by Algorithm* ISOTROPY 2 *is in* $\theta$*-nearly isotropic position.*

**Proof** Define $d_i = \max\left\{\frac{d}{2^i}, 10n\right\}$. It suffices to prove by induction on $i$ that if all iterations are successful (which happens with probability at least $1-\gamma$) then

$$K_i \subseteq d_i B. \tag{4.10}$$

In which case, since $d_{p-1} = 10n$, $K_p = T_p T_{p-1} \cdots T_0 K$ is in $\theta$-nearly isotropic position.

The case $i = 0$ is trivial and so let $i > 0$. Let $v \in K_i$ be the image under $T_{i-1}$ of $u \in K_{i-1}$. If $u \in K_{i-1}'$ then $v$ lies in $T_{i-1}K_{i-1}'$ which is in $\theta$-nearly isotropic position and so by Theorem 4.4.1 $|v| \leq (1+2\theta)(n+1) \leq 2n$. So assume that $u \in K_{i-1} \setminus K_{i-1}'$. Let $q$ be the point where the line segnment $[0,u]$ meets the boundary of $K_{i-1}'$. Let $x = T_{i-1}(0)$ and $y = T_{i-1}(q)$ where $|x|, |y| \leq 2n$. Now $u = \tau q$ where $\tau = |u|/(10n) > 1$. Since $T_{i-1}$ is affine, it follows that $v = \tau y + (1-\tau)x$ and hence

$$|v| \leq \tau|y| + |1-\tau|\,|x| < 4n\tau < \frac{|u|}{2} \leq \frac{d}{2^i},$$

which proves (4.10). □

### 4.4.1 Improving Local Conductance

We assume that (4.7), (4.8) hold. We now define a *flat step*. Its aim is to improve local conductance. Suppose we carry out $\mathcal{SW}$ and we do an improper step from $u \in K$ to $v \notin K$. We find by binary search a point $u' \in [u, v]$ such that $u' \in (2^{1/n}K) \setminus K$. We assume now that we have a *separation oracle* for $K$. It returns a hyperpalne $H$ through $u'$ such that $K$ is contained in one of the open half-spaces of $H$. Let $h$ be the unit normal of $H$ whhich is directed away from the origin. Let

$$U_h = \left(1 - \frac{1}{2n}\right)(I + hh^T). \tag{4.11}$$

A flat step replaces $K$ by $U_h K$ if $H$ is at distance $< 1/2$ from the origin.

**Algorithm** LOCAL CONDUCTANCE
Let $0 < \phi, \gamma < 1$ be given and let

$$\delta_0 = \frac{\{\theta, \gamma\}}{24\sqrt{n}},\ d_0 = \sqrt{\frac{2n}{\theta}},\ M = \left\lceil \frac{32}{\theta} n \log n \right\rceil,\ T = \left\lceil \kappa n^2 d^2 \delta^{-2} \ln(\delta_0 \sqrt{n}) \right\rceil.$$

**Step 1** Select an integer $N$ uniformly at random from $\{0, 1, \ldots, M-1\}$.

**Step 2** Using ORDINARY SAMPLE generate a point $u \in K_{\mathcal{L}}$ whose distribution is closer than $\lambda/6$ in variation distance to $U_{\mathcal{L}}$.

**Step 3** Let $K_0 = K$.

> **for** $i = 0$ **to** $N - 1$ **do**
> **begin**
> Starting at $u$ execute $\mathcal{SW}$ on $K_i' = K_i \cap d_0 B$ until either
> (i) $T$ proper steps are made, (ii) a flat step is made.
> In case (i) $K_{i+1} = K_i$ and in case (ii) $K_{i+1} = U_h K_i$.
> **end**

**Step 4** **Output** $K_N$.

**Theorem 4.4.5** *Assume that $K$ satisfies (4.7), (4.8) and (4.9). Then Algorithm* LOCAL CONDUCTANCE *produces a convex body $K_N$ which also satisfies (4.7), (4.8) and (4.9). The expectation of the average local conductance of $K_N$ is at least $1 - 100\theta$. With probability $\geq 1 - \gamma$, the number of calls to the oracle is at most $3MT = O^*(n^5)$.*

**Proof** [Deferred to Section 4.6.3]

# 4.5 Deferred proofs of Section 4.3

## 4.5.1 Proof of Theorem 4.2.3

(a) It follows from Theorem 4.2.2 that given $v$, the conditional distribution $W^{(i)}$ of $w$ is within variation distance $\alpha$ of $Q_{\mathcal{L}}^{(i)}$. We must examine the distribution of $u = \frac{2n}{2n-1}w$. Let $c = \frac{2n-1}{2n}$.

$$Q_{\mathcal{L}}^{(i)}(cK_{\mathcal{L}}^{(i)}) = \sum_{x \in cK_{\mathcal{L}}^{(i)}} \frac{|(x+\delta B_{\mathcal{L}}) \cap K_{\mathcal{L}}^{(i)}|}{|x+\delta B_{\mathcal{L}}|} = \sum_{x \in cK_{\mathcal{L}}^{(i)}} \frac{|(x+c\delta B_{\mathcal{L}}) \cap cK_{\mathcal{L}}^{(i)}|}{|x+c\delta B_{\mathcal{L}}|}$$
$$\geq (1-c^{-1}\alpha)|cK_{\mathcal{L}}^{(i)}| \geq (1-c^{-1}\alpha)c^n Q_{\mathcal{L}}^{(i)}(K_{\mathcal{L}}^{(i)}) \geq \frac{8}{15} Q_{\mathcal{L}}^{(i)}(K_{\mathcal{L}}^{(i)})$$

The first inequality in the above comes from Theorem 4.2.1. It follows that

$$W^{(i)}(cK_{\mathcal{L}}^{(i)}) \geq Q_{\mathcal{L}}^{(i)}(cK_{\mathcal{L}}^{(i)}) - \alpha \geq \frac{8}{15} Q_{\mathcal{L}}^{(i)}(K_{\mathcal{L}}^{(i)}) - \alpha \geq \frac{8}{15} - \frac{23}{15}\alpha \geq \frac{1}{2}$$

and so the expected number of executions of A to find $u \in K_{\mathcal{L}}^{(i)}$ is at most 2.

The distribution of the first point $u$ such that $u \in cK_{\mathcal{L}}^{(i)}$ is proportional to the restriction of $W^{(i)}$ to $cK_{\mathcal{L}}^{(i)}$. Therefore, for $S \subseteq K_{\mathcal{L}}^{(i)}$,

$$V^{(i)}(S) - U_{\mathcal{L}}^{(i)}(S) = \frac{W^{(i)}(cS)}{W^{(i)}(cK_{\mathcal{L}}^{(i)})} - U_{\mathcal{L}}^{(i)}(S) \leq \frac{Q_{\mathcal{L}}^{(i)}(cS) + \alpha}{Q_{\mathcal{L}}^{(i)}(cK_{\mathcal{L}}^{(i)}) - \alpha} - U_{\mathcal{L}}^{(i)}(S).$$

Here

$$Q_{\mathcal{L}}^{(i)}(cS) \leq \frac{|cS|}{\ell_{\mathcal{L}}^{(i)}(K_{\mathcal{L}}^{(i)})} \text{ and } Q_{\mathcal{L}}^{(i)}(cK_{\mathcal{L}}^{(i)}) = \frac{\ell_{\mathcal{L}}^{(i)}(cK_{\mathcal{L}}^{(i)})}{\ell_{\mathcal{L}}^{(i)}(K_{\mathcal{L}}^{(i)})} \geq \frac{(1-\alpha)|cK_{\mathcal{L}}^{(i)}|}{\ell_{\mathcal{L}}^{(i)}(K_{\mathcal{L}}^{(i)})} > \frac{1}{2}.$$

Hence,

$$V^{(i)}(S) - U_{\mathcal{L}}^{(i)}(S) \leq \frac{|cS| + \alpha \ell_{\mathcal{L}}^{(i)}(K_{\mathcal{L}}^{(i)})}{(1-\alpha)|cK_{\mathcal{L}}^{(i)}| - \alpha \ell_{\mathcal{L}}^{(i)}(K_{\mathcal{L}}^{(i)})} - \frac{|cS|}{|cK_{\mathcal{L}}^{(i)}|} < 4\alpha.$$

To complete the proof of (a) we need to discuss the average number of actual steps for each speedy step. Consider a walk in $K_{\mathcal{L}}^{(i+1)}$ starting from $u_i$. Let $W^{(i)}$ be the distribution of $u_i$. Let

$$B^{(i)} = \left\{ x \in K_{\mathcal{L}}^{(i)} : \frac{W^{(i)}(x)}{Q_{\mathcal{L}}^{(i+1)}(x)} > 3 \right\}.$$

Now for $x \in K_{\mathcal{L}}^{(i)}$ we have

$$Q_{\mathcal{L}}^{(i+1)}(x) = \frac{\ell_{\mathcal{L}}^{(i+1)}(x)}{\ell_{\mathcal{L}}^{(i+1)}(K_{\mathcal{L}}^{(i+1)})} \geq \frac{\ell_{\mathcal{L}}^{(i)}(x)}{\ell_{\mathcal{L}}^{(i+1)}(K_{\mathcal{L}}^{(i+1)})} \geq \frac{2}{5} Q_{\mathcal{L}}^{(i)}(x) \tag{4.12}$$

where the last inequality comes from (**??**). (4.12) clearly holds for $x \in K_{\mathcal{L}}^{(i+1)} \setminus K_{\mathcal{L}}^{(i)}$ and so $Q_{\mathcal{L}}^{(i+1)}(V^{(i)}) \geq 2Q_{\mathcal{L}}^{(i)}(V^{(i)})/5$. Hence

$$B^{(i)} \subseteq \hat{B}^{(i)} = \left\{ x \in K_{\mathcal{L}}^{(i)} : \frac{W^{(i)}(x)}{Q_{\mathcal{L}}^{(i)}(x)} > \frac{6}{5} \right\}.$$

So,

$$\frac{1}{5} W^{(i)}(\hat{B}^{(i)}) \leq |W^{(i)}(\hat{B}^{(i)}) - Q_{\mathcal{L}}^{(i)}(\hat{B}^{(i)})| \leq \alpha. \tag{4.13}$$

We show next that if $Z$ denotes the number of oracle calls per speedy step in the walk from $u_i$ then

$$\mathbf{E}(Z \mid v_i \notin B^{(i)}) \leq \frac{4}{\ell_{\mathcal{L}}^{(i)}} \leq 5. \tag{4.14}$$

Let $u_i = x_0, x_1, \ldots,$ denote the sequence of points reached by proper steps. Let $W_k$ denote the distribution of $x_k$ conditional on $u_i \notin B^{(i)}$. Then for $x \in K_{\mathcal{L}}^{(i+1)} \setminus B^{(i)}$

$$W_0(x) = \frac{P^{(i)}(x)}{P^{(i)}(K_{\mathcal{L}}^{(i+1)} \setminus B^{(i)})} \leq \frac{3Q_{\mathcal{L}}^{(i+1)}(x)}{1 - 5\epsilon_0} \leq 4Q_{\mathcal{L}}^{(i+1)}(x).$$

It follows by induction that

$$W_k(x) \leq 4Q_{\mathcal{L}}^{(i+1)}(x) \qquad \text{for } k = 1, 2, \ldots$$

Given $\ell_{\mathcal{L}}^{(i+1)}(x_k)$, the expected number of steps (proper and improper) between $x_k$ and $x_{k+1}$ is $1/\ell_{\mathcal{L}}^{(i+1)}(x_k)$. The expected number conditional only on $v_i \notin B^{(i)}$ is thus

$$\sum_{x \in K_{\mathcal{L}}^{(i+1)}} \frac{W_k(x)}{\ell_{\mathcal{L}}^{(i+1)}(x)} \leq 4 \sum_{x \in K_{\mathcal{L}}^{(i+1)}} \frac{Q_{\mathcal{L}}^{(i+1)}(x)}{\ell_{\mathcal{L}}^{(i+1)}(x)} = \frac{4}{\lambda_{\mathcal{L}}^{(i)}}$$

proving (4.14).

We therefore define $\mathcal{G} = \{u_i \notin B^{(i)} : i = 1, 2, \ldots, m\}$. The proof of (b) is similar to (a). $\square$

### 4.5.2 Proof of Theorem 4.2.1

We start by considering a *continuous local conductance*. For $x \in K$ we let

$$\ell(x) = \frac{\mathrm{vol}(B(x,\delta) \cap K)}{\mathrm{vol}(B(x,\delta))}$$

and the *average continuous local conductance*

$$\lambda = \frac{\lambda(K)}{\mathrm{vol}(K)}.$$

**Lemma 4.5.1** *Let $L$ be a measurable subset of the surface of a convex set $K$ in $\mathbb{R}^n$ and let $S$ be the set of pairs $(x, y)$ with $x \in K, y \notin K, |x - y| \leq \delta$, and such that the line segment $xy$ meets $L$. Then the $(2n)$-dimensional volume of $S$ is at most*

$$\delta\, vol_{n-1}(L)\frac{c_{n-1}}{(n+1)c_n} vol_n(\delta B) \tag{4.15}$$

*where $c_l$ denotes the volume of the unit ball in $\mathbb{R}^l$.*

**Proof** It suffices to prove the assertion for the case when $L$ is "infinitesimally small". In this case, the measure of $S$ is maximised when the surface of $K$ is a hyperplane $H$ in a larger neighbourhood of $L$. Then the measure of $S$ is independent of $K$ and is given by

$$\mathrm{vol}_{n-1}(L)\int_{\alpha=0}^{\delta}\int_{\theta=0}^{\pi}\frac{\alpha\sin\theta}{n+1}\left(\left(\frac{\delta}{\alpha}\right)^{n-1}-1\right)(n-1)c_{n-1}(\alpha\cos\theta)^{n-2}\alpha d\alpha d\theta$$

Fix the distance $\alpha$ of $x$ from $L$ and the angle $\theta$ between the line joining $x$ and $L$ and the hyperplane $H$. The volume of the corresponding $y$'s is $\mathrm{vol}_{n-1}(L)\frac{\alpha\sin\theta}{n+1}\left(\left(\frac{\delta}{\alpha}\right)^{n-1}-1\right)$ – the volume of the part of a cone on the $y$-side of $H$. Now multiply by $(n-1)c_{n-1}(\alpha\cos\theta)^{n-2}$ – the $(n-2)$-volume swept out by $x$ the surface of an $n-1$ dimensional ball of radius $\alpha\cos\theta$. Then integrate over $\alpha, \theta$. □

**Corollary 4.5.1** *Let $K$ and $L$ be as in Lemma 4.5.1. Choose $x$ uniformly from $K$ and choose $u$ uniformly from $\delta B$. The probability that $[x, x+u]$ intersects $L$ is at most*

$$\frac{\delta\, vol_{n-1}(L)}{2\sqrt{n}\, vol(K)}.$$

**Proof** Divide (4.15) by $\mathrm{vol}(K) \times \mathrm{vol}(\delta B)$ and use the fact that $c_n/c_{n-1} > 2/\sqrt{n}$ for $n > 2$. □

The average local conductance $\lambda$ thus satisfies

$$\lambda \geq 1 - \frac{\delta}{2\sqrt{n}} \frac{\mathrm{vol}_{n-1}(\partial K)}{\mathrm{vol}(K)}.$$

If $K \supseteq B$ then $\mathrm{vol}(K) \geq \mathrm{vol}_{n-1}(\partial K)/n$ and so

$$\lambda \geq 1 - \frac{\delta\sqrt{n}}{2}. \tag{4.16}$$

We now need to relate $\lambda$ to $\lambda_{\mathcal{L}}$. We first prove Theorem 4.3.1.

**Proof of Theorem 4.3.1.**

Let $S_{\mathcal{L}}^I = \{x \in S_{\mathcal{L}} : C(x) \subseteq S\}$ and let $S_{\mathcal{L}}^B = S_{\mathcal{L}} \setminus S_{\mathcal{L}}^I$ denote the interior and border points of $S_{\mathcal{L}}$ respectively. $S \supseteq \alpha B$ implies that

$$(1 + 2\alpha^{-1}\eta\sqrt{n})^{-1} \bigcup_{x \in S_{\mathcal{L}}} C(x) \subseteq S \subseteq (1 + \alpha^{-1}\eta\sqrt{n}) \bigcup_{x \in S_{\mathcal{L}}^I} C(x). \tag{4.17}$$

The theorem follows easily from this. □

Note that (4.17) implies

$$\begin{aligned} |S_{\mathcal{L}}^B| &\leq \eta^{-n}\mathrm{vol}(S)((1 + 2\alpha^{-1}\eta\sqrt{n})^n - (1 + \alpha^{-1}\eta\sqrt{n})^{-n}) \\ &\leq 4\alpha^{-1}\eta^{-(n-1)}\sqrt{n}\mathrm{vol}(S). \end{aligned} \tag{4.18}$$

**Lemma 4.5.2**

**(a)** $x \in K$ *implies that* $\ell(x) \geq \left(\frac{\delta}{10d}\right)^n$.

**(b)** $x, x' \in K$ *and* $|x - x'| \leq \eta\sqrt{n}$ *implies* $|\ell(x) - \ell(x')| \leq \frac{\epsilon}{n}\ell(x)$.

**(c)** $x \in K_{\mathcal{L}}$ *implies* $|\ell(x) - \ell_{\mathcal{L}}(x)| \leq \frac{\epsilon}{n}\ell(x)$.

*Note: the estimates* $\frac{\epsilon}{n}$ *are much larger than we will actually prove.*

**Proof**

(a) Consider the finite cone $C$ with point $x$ and base the intersection of $B$ with the hyperplane through the origin $O$ which is perpendicular to the line $L$ joining $x$ to $O$. $C$ contains a ball of radius $\frac{\delta}{10}$ with centre on $L$, at distance $\frac{\delta}{2}$ from $x$.

(b)

$$\mathrm{vol}(B(x,\delta) \setminus B(x',\delta)) \leq (1 - (1 - \delta^{-1}\eta\sqrt{n})^n)\mathrm{vol}(B(x,\delta) \leq 2\delta^{-1}n^{3/2}\eta\delta^n.$$

Now use (a).

(c) This follows from (b) and Theorem 4.3.1, taking account of the note prior to the proof. □

Theorem 4.2.1 follows.

### 4.5.3 Proof of Theorem 4.2.2

**Geometric lemmas related to local condutance**

The following classic theorem is basic to the study of convexity:

**Theorem 4.5.1 Brunn-Minkowski Theorem** *Let $K_1, K_2$ be convex bodies in $\mathbb{R}^n$. Then*

$$vol(K_1 + K_2)^{1/n} \geq vol(K_1)^{1/n} + vol(K_2)^{1/n}.$$

**Corollary 4.5.2** *Let $K_1, K_2$ be convex bodies. Then the function $f(x) = vol((x + K_1) \cap K_2)^{1/n}$ is concave.*

**Proof** This follows from Theorem 4.5.1 and

$$(\lambda x + (1-\lambda)y + K_1) \cap K_2 \supseteq \lambda((x + K_1) \cap K_2) + (1-\lambda)((y + K_1) \cap K_2).$$

□

In the remainder of this subsection $x, y$ are members of $K$ and $|x - y| < \delta/\sqrt{n}$. Let

$$C = (x + \delta B) \cap (y + \delta B),\ M_x = (x + \delta B) \setminus C \text{ and } M_y = (y + \delta B) \setminus C,$$

$$R_x = M_x \cap (x - y + C) \text{ and } R_y = M_y \cap (y - x + C).$$

Let $C'$ be obtained by blowing up $C$ from its centre $(x + y)/2$ by a factor $\rho = 1 + \frac{4}{4n-1}$.

**Lemma 4.5.3**

$$M_x \setminus R_x \subseteq C'.$$

**Proof** Assume w.l.o.g. that $x = -y$ and let $z = \mu x + w \in M_x \setminus R_x$ where $w$ is orthogonal to $x$. It can be seen that $0 < \mu < 2$. Now $|\rho^{-1}z - x| \leq |\rho^{-1}z - y|$ and so it is enough to show that $|\rho^{-1}z - y| \leq \delta$. This follows by straightforward calculation. □

**Lemma 4.5.4**

$$vol(K \cap (M_x \setminus R_x)) \leq 3vol(K \cap C).$$

**Proof** By Lemma 4.5.3, blowing up $C$ by a factor $\rho$ covers both $K \cap C$ and $K \cap (M_x \setminus R_x)$. Hence

$$\mathrm{vol}(K \cap (C \cup (M_x \setminus R_x))) \leq \mathrm{vol}(K \cap C') \leq \left(1 + \frac{4}{4n-1}\right)^n \mathrm{vol}(K \cap C) \leq 3\mathrm{vol}(K \cap C)$$

and the lemma follows. □

We say that a real-valued function $f(x)$ defined on the convex set $K \subseteq \mathbb{R}^n$ is *log-concave* if $\log f(x)$ is concave on $K$. This clearly entails $f(x) > 0$ on $K$.

In particular:

$$\text{If } f(x)^\alpha \text{ is concave, for some } \alpha > 0, \text{ then } f \text{ is itself log-concave.} \tag{4.19}$$

**Lemma 4.5.5**

$$vol(K \cap C)^2 \geq vol(K \cap R_x) vol(K \cap R_y).$$

**Proof** Corollary 4.5.2 and (4.19) imply that the function $g(u) = \text{vol}(K \cap (u + C))$ is log-concave. Therefore

$$\begin{aligned} g(0)^2 \geq g(x-y)g(y-x) &= \text{vol}(((x-y)+C) \cap K)\text{vol}((((y-x)+C) \cap K) \\ &\geq \text{vol}(K \cap R_x)\text{vol}(K \cap R_y). \end{aligned}$$

□

**Lemma 4.5.6**

$$vol(K \cap C) \geq \frac{1}{5} \min \{vol(K \cap (x + \delta B)), vol(K \cap (y + \delta B))\}.$$

**Proof** We have

$$\begin{aligned} \text{vol}(K \cap R_x) &= \text{vol}(K \cap M_x) - \text{vol}(K \cap (M_x \setminus R_x)) \\ &\geq \text{vol}(K \cap (x + \delta B)) - \text{vol}(K \cap C) - 3\text{vol}(K \cap C) \end{aligned}$$

by Lemma 4.5.4. We also get a symmetric lower bound for $\text{vol}(K \cap R_y)$. Then by Lemma 4.5.5 we have

$$\begin{aligned} \text{vol}(K \cap C) &\geq \min \{\text{vol}(K \cap R_x), \text{vol}(K \cap R_y)\} \\ &\geq \min \{\text{vol}(K \cap (x + \delta B)), \text{vol}(K \cap (y + \delta B))\} - 4\text{vol}(K \cap C). \end{aligned}$$

The lemma follows. □

**Lemma 4.5.7** *Suppose $S_1, S_2$ is a partition of $K$ into two measurable sets where $x \in S_1$, $y \in S_2$ and $|x - y| \leq \delta/\sqrt{n}$. Then*

$$\frac{vol((x + \delta B) \cap S_2)}{vol(\delta B)} + \frac{vol((y + \delta B) \cap S_1)}{vol(\delta B)} \geq \frac{1}{5} \min \{\ell(x), \ell(y)\}. \tag{4.20}$$

**Proof** The LHS $\Lambda$ of (4.20) is at least

$$\frac{1}{\mathrm{vol}(\delta B)}(\mathrm{vol}(S_1 \cap C) + \mathrm{vol}(S_2 \cap C)) = \frac{\mathrm{vol}(K \cap C)}{\mathrm{vol}(\delta B)}.$$

Thus by Lemma 4.5.6,

$$\Lambda \geq \frac{1}{5\mathrm{vol}(\delta B)} \min\{\mathrm{vol}(K \cap (x + \delta B)), \mathrm{vol}(K \cap (y + \delta B))\} = \frac{1}{5} \min\{\ell(x), \ell(y)\}.$$

□

### Geometric lemmas for the main argument

Now, for any $a \in \mathbb{R}^n$, $|a| = 1$, consider the set of hyperlanes $H(s) = \{ax = s\}$ orthogonal to $a$, and half-spaces $H^+(s) = \{ax \leq s\}$, $H^-(s) = \{ax \geq s\}$ they define. If $K$ is any convex body, let $K(s) = K \cap H(s)$, $K^+(s) = K \cap H^+(s)$, $K^-(s) = K \cap H^-(s)$. (We call $K(s)$ a "cross section" of $K$ in "direction" $a$.) Let $\beta_1 = \inf_s\{K(s) \neq \emptyset\}$, $\beta_2 = \sup_s\{K(s) \neq \emptyset\}$. Then $w = \beta_2 - \beta_1$ is the *width* of $K$ in direction $a$, and we will write $w = W(K, a)$. Note that

**Lemma 4.5.8** *diameter* $K = \max_a W(K, a)$.

**Proof**

$$\begin{aligned}\text{diameter } K &= \max\{|x - y| : x, y \in K\} = \max\{|z| : z \in K - K\} \\ &= \max_{z \in K-K} \max_{|a|=1} az = \max_{|a|=1} \max_{z \in K-K} az = \max_a W(K, a).\end{aligned}$$

□

We will also need the following technical result.

**Lemma 4.5.9** *Let $a_1, a_2, \ldots, a_{n-1}$ be mutually orthogonal unit vectors. Then diameter $K(s) < n^{1/2} \max_i W(K, a_i)$ for all $s$.*

**Proof** If $a, |a| = 1$ is in the subspace generated by the $a_i$ then $W(K(s), a) \leq W(K, a)$. But $W(K, a) \leq \sqrt{n-1} \max_i W(K, a_i)$, since $K$ can clearly be contained in an (infinite) cubical cylinder of side $\max_i W(K, a_i)$. Applying Lemma 4.5.8 now gives the conclusion. □

Let $\alpha(s) = \mathrm{vol}_{n-1}(K(s))$ and $V(s) = \mathrm{vol}_n(K^+(s))$, and assume, without loss, that $\beta_1 = 0$ and $\beta_2 = w$. Note then $V(w) = \mathrm{vol}_n(K)$. It is a consequence of the Brunn-Minkowski theorem, that $\alpha(s)^{1/(n-1)}$ is a concave function of $s$ in $[0, w]$. Then we have

**Lemma 4.5.10** *$V(s)/V(w) \leq ns/w$.*

**Proof** First we show that if $0 < u < s$, $\alpha(u)/\alpha(s) \geq (u/s)^{n-1}$. This follows since if $u = \lambda 0 + (1-\lambda)s$, then Brunn-Minkowski implies

$$\begin{aligned}\alpha(u)^{1/(n-1)} &\geq \lambda\alpha(0)^{1/(n-1)} + (1-\lambda)\alpha(s)^{1/(n-1)}\\ &\geq (1-\lambda)\alpha(s)^{1/(n-1)} = (u/s)\alpha(s)^{1/(n-1)}.\end{aligned}$$

Thus

$$V(s) \geq \int_0^s (u/s)^{n-1}\alpha(s)\,du = (s/n)\alpha(s), \tag{4.21}$$

$$V(w) - V(s) \leq \int_s^w (u/s)^{n-1}\alpha(s)\,du = (w^n - s^n)/(ns^{n-1})\alpha(s). \tag{4.22}$$

Dividing (4.22) by (4.21) gives $V(s)/V(w) \geq (s/w)^n$. By symmetry, this inequality in turn implies

$$(V(w) - V(s))/V(w) \geq ((w-s)/w)^n = (1 - s/w)^n \geq 1 - ns/w,$$

since $(1-x)^n \geq 1 - nx$ for $x \in [0,1]$. This gives the result. □

We will need the following simple lemma asserting the existence of a hyperplane simultaneously "bisecting the measure" of two arbitrary sets.

**Lemma 4.5.11** *Let $S_1, S_2 \subseteq \mathbb{R}^n$, be measurable and $L$ a two-dimensional linear subspace of $\mathbb{R}^n$. Let $f$ be continuous on $K$. Then there exists a hyperplane $H$, with normal $a \in L$, such that the half-spaces $H^+$, $H^-$ determined by $H$ satisfy $f(S_i \cap H^+) = f(S_i \cap H^-)$ for $i = 1, 2$.*

**Proof** Let $\alpha_1, \alpha_2$ be a basis for $L$. For each $\theta \in [-1, +1]$, let $b_i(\theta)$ be such that the hyperplane $(\theta\alpha_1 + (1 - |\theta|)\alpha_2)x = b_i(\theta)$ bisects the $f$-measure of $S_i$ for $i = 1, 2$. (If $S_i$ is disconnected in such a way that the possible $b_i$ form an interval, $b_i(\theta)$ will be its midpoint.) It clearly suffices to show that $b_1(\theta_0) = b_2(\theta_0)$ for some $\theta_0$. If $b_1(-1) = b_2(-1)$ we are done, so suppose w.l.o.g. that $b_1(-1) > b_2(-1)$. We clearly have $b_i(1) = -b_i(-1)$ for $i = 1, 2$, so $b_1(1) < b_2(1)$. But since $f$ is a continuous measure, it follows easily that $b_i(\theta)$ is a continuous function of $\theta$. The existence of $\theta_0 \in (-1, 1)$ now follows. □

### Three lemmas on logconcavity

**Lemma 4.5.12** *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be log-concave and let $F(x) = \int_0^x f(t)dt$. Then $F$ is also log-concave.*

**Proof** It suffices to show that for $0 < a < b$ and $c = (a+b)/2$ that $F(c)^2 \geq F(a)F(b)$. Now

$$\Delta = F(c)^2 - F(a)F(b) = B(A+B) - AC$$

where $A = \int_0^a f(t)dt$, $B = \int_a^c f(t)dt$ and $C = \int_c^b f(t)dt$.

Let $g(x) = Ge^{\theta x}$ where $\theta = (c-a)^{-1}\log(f(c)/f(a))$ and $G = f(a)e^{-\theta a}$ so that $g(x) = f(x)$ for $x = a, c$. If $f(c) \leq f(a)$ then $C \leq B$ and then clearly $\Delta \geq 0$. We can therefore assume that $f(c) > f(a)$ and hence that $\theta > 0$.

The log-concavity of $f$ implies that

$$f(x) \geq g(x) \text{ for } x \in [a,c] \text{ and } f(x) \leq g(x) \text{ for } x \notin [a,c].$$

Thus

$$\Delta \geq \hat{B}(A + \hat{B}) - A\hat{C}$$

where $\hat{B} = \int_a^c g(t)dt = G\theta^{-1}(e^{\theta c} - e^{\theta a})$, $\hat{C} = \int_c^b g(t)dt = G\theta^{-1}(e^{\theta b} - e^{\theta c})$.

We can therefore prove the lemma by showing that

$$\hat{B} + \frac{\hat{B}^2}{\hat{A}} \geq \hat{C} \tag{4.23}$$

where $\hat{A} = \int_0^a g(t)dt = G\theta^{-1}(e^{\theta a} - 1) \geq A$. But (4.23) is equivalent to

$$(e^{\theta a} - 1)(e^{\theta b} - e^{\theta c}) \leq (e^{\theta a} - 1)(e^{\theta c} - e^{\theta a}) + (e^{\theta c} - e^{\theta a})^2$$

or after simplification,

$$e^{\theta a}e^{\theta b} + 2e^{\theta c} \leq e^{2\theta c} + e^{\theta a} + e^{\theta b}$$

which follows from $e^{\theta a}e^{\theta b} = e^{2\theta c}$ and the convexity of $e^{\theta x}$. □

**Corollary 4.5.3** *Let $f, F$ be as in Lemma 4.5.12. Let $0 \leq x \leq d$ and $0 \leq t \leq d - x$. Then*

$$F(x+t) - F(x) \geq \frac{t}{d}F(x)\log\left(\frac{F(d)}{F(x)}\right).$$

**Proof** Let $\tilde{F}(x) = F(x)/F(d)$. Lemma 4.5.12 implies that $\tilde{F}$ is log-concave. Write $x + t = \lambda x + (1-\lambda)d$ where $\lambda = \frac{d-x-t}{d-x}$. Then the log-concavity of $\tilde{F}$ implies

$$\tilde{F}(x+t) \geq \tilde{F}(x)^{1-t/(d-x)}$$

and so

$$\begin{aligned}
\tilde{F}(x+t) - \tilde{F}(x) &\geq \tilde{F}(x)(\tilde{F}(x)^{-t/(d-x)} - 1) \\
&= \tilde{F}(x)\left(\exp\left\{\frac{t}{d-x}\log\left(\frac{1}{\tilde{F}(x)}\right)\right\} - 1\right) \\
&\geq \tilde{F}(x) \cdot \frac{t}{d-x}\log\left(\frac{1}{\tilde{F}(x)}\right)
\end{aligned}$$

and the lemma follows. □

**Lemma 4.5.13** *The local conductance $\ell$ is a log-concave function on $K$.*

**Proof** This follows from Corollary 4.5.2 and (4.19). $\square$

### The main argument

Let $S_1, S_2$ be a partition of $K$ into two measurable sets. Let

$$h(x) = \begin{cases} \frac{\mathrm{vol}((x+\delta B)\cap S_2)}{\mathrm{vol}(\delta B)} & x \in S_1 \\ \\ \frac{\mathrm{vol}((x+\delta B)\cap S_1)}{\mathrm{vol}(\delta B)} & x \in S_2 \end{cases}$$

Then

$$\frac{h(S_1)}{\ell(K)} = \int_{x\in S_1} \frac{\ell(x)}{\ell(K)} \cdot \frac{\mathrm{vol}((x+\delta B)\cap S_2)}{\mathrm{vol}((x+\delta B)\cap K)} dx$$

is the probability $Q(S_1, S_2)$ that $x \in S_1$ and $y \in S_2$ where $x$ is a point chosen from $K$ with distribution $\ell(x)/\ell(K)$ and $y$ is obtained from $x$ by one *continouous* speedy step. It follows as in (2.2) that $h(S_1) = h(S_2)$. The conclusion of Lemma 4.5.7 is that

$$h(x) + h(y) \geq \frac{1}{5} \min\{\ell(x), \ell(y)\} \text{ for } x \in S_1, y \in S_2, |x-y| \leq \delta/\sqrt{n}. \tag{4.24}$$

Theorem 4.2.2 will follow from

**Theorem 4.5.2** *Let $B \subseteq K \subseteq dB$ be a closed convex set in $\mathbb{R}^n$. Let $K$ be partitioned into two measurable subsets $S_1, S_2$. Let $\ell$ be a log-concave function which is strictly positive on $K$. Let $h$ be a non-negative integrable function which satisfies (4.24). Then*

$$\frac{h(K)\ell(K)}{2\ell(S_1)\ell(S_2)} \geq \frac{1}{5000\sqrt{n}} \min\left\{\frac{\delta}{d}\log\left(\frac{\ell(K)^2}{\ell(S_1)\ell(S_2)}\right), 1\right\}. \tag{4.25}$$

**Proof** We first prove

$$h(K) \geq \frac{1}{2500\sqrt{n}} \min\left\{\frac{\delta}{d}\max_{i=1,2}\left\{\ell(S_i)\log\left(\frac{\ell(K)}{\ell(S_i)}\right)\right\}, \ell(S_1), \ell(S_2)\right\}. \tag{4.26}$$

Let $\ell_0 = \min\{\ell(x) : x \in K\} > 0$. We let $\epsilon = \min\left\{\frac{\ell_0}{100}, \frac{\delta}{\sqrt{n}}\right\}$ and then let $\delta_1 \leq \delta/(10\sqrt{n})$ be such that

$$|\ell(x) - \ell(x')| \leq \epsilon \text{ whenever } x, x' \in K, |x - x'| \leq \delta_1.$$

We fix a line $L = \{x_s = a + su : s \in \mathbb{R}\}$ in direction $u$, where $a, u \in \mathbb{R}^n$, $|u| = 1$.

Let $\alpha(s), K(s)$ be as in the previous section and let $I = \{s : K(s) \neq \emptyset\} = [\beta_1, \beta_2]$. Let $\bar{h}(s) = h(K(s))/\alpha(s)$ be the average of $h$ over $K(s)$.

We consider first the case where $K$ is *needle-like* i.e. each $K(s)$ has diameter at most $\delta_1/2$.

It follows that $\ell(x) \in [.99\ell(x_s), 1.01\ell(x_s)]$ for $x \in K(s)$.

Let $H = \left\{s \in I : \bar{h}(s) \geq \frac{1}{30}\ell(x_s)\right\}$. Let $J_i = \{s \in I \setminus H : \text{vol}_{n-1}(K(s) \cap S_i) \geq 2\alpha(s)/5\}$.

**Claim 4.5.1**

**(a)** *If* $s \in I \setminus H$ *then* $\min_{i=1,2}\{vol_{n-1}(K(s) \cap S_i)\} \leq \frac{1}{3}\alpha(s)$.

**(b)** $s \in J_i$ *and* $y \in S_{3-i}$ *implies* $h(y) \geq \frac{1}{12}\ell(x_s)$.

**Proof** (a) Suppose $s \in I \setminus H$ and that $\text{vol}_{n-1}(K(s) \cap S_i) \geq 2\alpha(s)/5$ for $i = 1, 2$. Choose $x \in K(s) \cap S_1$, $y \in K(s) \cap S_2$. Then we have

$$h(x) + h(y) \geq \frac{1}{5}\min\{\ell(x), \ell(y)\} \geq \frac{1}{6}\ell(x_s). \tag{4.27}$$

If $h(x) \geq \frac{1}{12}\ell(x_s)$ for all $x \in K(s) \cap S_1$ then $\bar{h}(s) \geq \frac{1}{30}\ell(x_s)$, contradiction. Otherwise it follows that $h(y) \geq \frac{1}{12}\ell(x_s)$ for all $y \in K(s) \cap S_2$ and we get a similar contradiction.
(b) If $h(y) < \frac{1}{12}\ell(x_s)$ then (4.27) implies $h(x) \geq \frac{1}{12}\ell(x_s)$ for all $x \in S_i \cap K(s)$, which implies $\bar{h}(s) \geq \frac{1}{30}\ell(x_s)$, contradiction.
**End of proof of Claim 4.5.1**

It is clear that $J_1 \cup J_2 = I \setminus H$ and it follows from Claim 4.5.1 that $J_1$ and $J_2$ are disjoint.

Now let $\mu(s) = \ell(x_s)\alpha(s)$ for $s \in I$. Then

$$\int_{s\in H} \bar{h}(s)\alpha(s)ds \geq \frac{1}{30}\int_{s\in H} \ell(x_s)\alpha(s)ds = \frac{1}{30}\mu(H). \tag{4.28}$$

On the other hand

$$\int_{s\in H} \bar{h}(s)\alpha(s)ds \leq h(K). \tag{4.29}$$

Now for $i = 1, 2$,

$$\begin{aligned}\ell(S_i) = \int_{s\in J_i}\int_{x\in K(s)\cap S_i} \ell(x)dx + \int_{s\in J_{3-i}}\int_{x\in K(s)\cap S_i} \ell(x)dx + \int_{s\in H}\int_{x\in K(s)\cap S_i} \ell(x_s)dx \\ \leq 1.01\mu(J_i) + 12h(K) + 1.01\mu(H). \quad (4.30)\end{aligned}$$

The term $12h(K)$ is a consequence of Claim 4.5.1.
Similarly,

$$\ell(S_i) \geq \frac{99}{100}\mu(J_i) \text{ and } \ell(K) \leq \frac{101}{100}\mu(I). \tag{4.31}$$

If (4.26) fails then (4.30) gives

$$\ell(S_i) \leq 1.02(\mu(J_i) + \mu(H))$$

and then together with (4.28), (4.29) we obtain

$$\mu(H) < \frac{31}{2500\sqrt{n}} \min\left\{\frac{\delta}{n}\max_{i=1,2}\left\{(\mu(J_i)+\mu(H))\log\left(\frac{101\mu(I)}{99\mu(J_i)}\right)\right\},\right.$$
$$\mu(J_1)+\mu(H), \mu(J_2)+\mu(H).\}$$

We see immediately that $\mu(H) = o(\mu(J_i))$ so that $\mu(J_i) + \mu(H) = (1 + o(1))\mu(J_i)$ for $i = 1, 2$ and then dividing through by $1 - o(1)$ we can write

$$\mu(H) < (1+o(1))\frac{31}{2500\sqrt{n}} \min\left\{\frac{\delta}{d}\max_{i=1,2}\left\{\mu(J_i)\log\left(\frac{101\mu(I)}{99\mu(J_i)}\right)\right\}, \mu(J_1), \mu(J_2)\right\} \quad (4.32)$$

Assuming $\mu(J_1) \leq \mu(J_2)$ we have

$$\mu(J_1)\log\left(\frac{101\mu(I)}{99\mu(J_1)}\right) \leq \mu(J_1)\log\left(\frac{\mu(I)}{\mu(J_1)}\right) \times \frac{\log(202/99)}{\log 2} \leq 1.03\mu(J_1)\log\left(\frac{\mu(I)}{\mu(J_1)}\right) \quad (4.33)$$

and

$$\mu(J_2)\log\left(\frac{101\mu(I)}{99\mu(J_2)}\right) \leq \mu(J_2)\log(202/99) < \mu(J_2). \quad (4.34)$$

Using (4.33), (4.34) in (4.32), we obtain

$$\mu(H) < \frac{1}{80\sqrt{n}} \min\left\{\frac{\delta}{d}\max_{i=1,2}\left\{\mu(J_i)\log\left(\frac{\mu(I)}{\mu(J_i)}\right)\right\}, \mu(J_1), \mu(J_2)\right\}. \quad (4.35)$$

We first dismiss the *degenerate* case where, say, $\mu(J_1) = 0$. It follows from (4.28–4.30) that $h(K) \geq \frac{1}{43}\ell(S_1)$ and the theorem is clearly true.

**Claim 4.5.2** *If $s \in J_1$ and $t \in J_2$ then $|s - t| > \delta_1$.*

**Proof** $\bar{h}(s) < \frac{1}{30}\ell(x_s)$ implies that there exists $x \in S_1 \cap K(s)$ such that $h(x) \leq \frac{1}{18}\ell(x_s)$. Similarly there exists $y \in S_2 \cap K(t)$ such that $h(y) \leq \frac{1}{18}\ell(x_t)$. If $|s - t| \leq \delta_1$ then $|x - y| \leq \frac{\delta}{\sqrt{n}}$ and so $h(x) + h(y) \geq \frac{1}{5}\ell(x) \geq \frac{1}{6}\ell(x_s)$, assuming that $\ell(x) \leq \ell(y)$. It follows that $\ell(x_t) \geq 2\ell(x_s)$ and so $|\ell(x_s) - \ell(x_t)| \geq \ell_0$, contradicting $|x_s - x_t| \leq \delta_1$. **End of proof of Claim 4.5.2**

We now show that we can assume w.l.o.g. the existence of an interval $(\sigma, \tau) \subseteq H$ such that if $A_1 = [\beta_1, \sigma]$ and $A_2 = [\tau, \beta_2]$ then

$$\mu(A_i \cap J_i) \geq \frac{1}{2}\mu(J_i),\ i = 1, 2. \quad (4.36)$$

Let

$$b_1 = \inf\left\{s:\ \mu(J_i \cap [\beta_1, s]) \geq \frac{1}{2}\mu(J_i),\ i = 1, 2\right\}.$$

Assume w.l.o.g. that

$$\mu(J_1 \cap [\beta_1, s]) \geq \frac{1}{2}\mu(J_1) \text{ and } \mu(J_2 \cap [s, \beta_2]) \geq \frac{1}{2}\mu(J_2).$$

It follows from Claim 4.5.2 that $b_1 \in J_1 \cup H$. Let

$$b_2 = \inf([b_1, \beta_2] \cap J_2).$$

If $b_2 \geq \beta_2$ we are in the degenerate case dealt with following (4.35). Let

$$b_3 = \sup([b_1, b_2] \cap J_1)$$

and let $\sigma = b_3$ and $\tau = b_2$. Equation (4.36) is satisfied and Claim 4.5.2 implies that $\tau - \sigma \geq \delta_1$.

Suppose now that $\tau - \sigma \geq \frac{2\delta}{3\sqrt{n}}$. The Brunn-Minkowski Theorem implies that $\alpha^{1/(n-1)}$ is concave and so $\alpha$ is log-concave. It follows that $\mu$ is also log-concave. Applying Corollary 4.5.3 (twice) we have

$$\mu(H) \geq \max_{i=1,2}\left\{\frac{2\delta}{3d\sqrt{n}}\mu(A_i)\log\left(\frac{\mu(I)}{\mu(A_i)}\right)\right\}. \tag{4.37}$$

Suppose next that $\xi = \tau - \sigma < \frac{2\delta}{3\sqrt{n}}$. Let $\lambda(s) = \ell(x_\sigma)e^{-\theta(s-\sigma)/\xi}$ where $e^\theta = \ell(x_\sigma)/\ell(x_\tau)$, so that $\lambda(s) = \ell(x_s)$ for $s = \sigma, \tau$. The log-concavity of $\ell$ implies that $\lambda(s) \leq \ell(x_s)$ for $s \in H$ and $\lambda(s) \geq \ell(x_s)$ for $s \notin H$. Our aim is to find a contradiction to (4.35) and so we can assume in fact that $\ell(x_s) = \lambda(s)$ for $s \in I$.

Suppose that $\mu(\tau) \geq (1+\zeta)\mu(\sigma)$ for some $\zeta > 0$. The log-concavity of $\mu$ implies that $\mu(s) \geq \mu(\sigma)$ for $s \in H$ and that $\mu(\sigma - t) \leq (1+\zeta)^{-t/\xi}\mu(\sigma)$ for $t > 0$. But then

$$\mu(A_1) \leq \mu(\sigma)(\log(1+\zeta))^{-1}\xi \text{ and } \mu(H) \geq \mu(\sigma)\xi.$$

This implies

$$\mu(H) \geq \frac{1}{10\sqrt{n}}\mu(A_1) \tag{4.38}$$

if $\zeta \geq \frac{1}{5\sqrt{n}}$. Using the same argument when $\mu(\sigma) \geq (1+\zeta)\mu(\tau)$ we can now assume that

$$\left|\frac{\mu(\sigma)}{\mu(\tau)} - 1\right| \leq \frac{1}{10\sqrt{n}}. \tag{4.39}$$

It follows immediately that

$$\mu(H) \geq \frac{\mu(\sigma)\xi}{2}. \tag{4.40}$$

We choose $u \in J_1 \cap [\sigma - \delta_1, \sigma]$ and $v \in J_2 \cap [\tau, \tau + \delta_1]$ and argue as in Claim 4.5.2 to prove that

$$\frac{\ell(x_\sigma)}{\ell(x_\tau)} > 3/2, \tag{4.41}$$

which implies

$$\theta \geq \log 3/2.$$

Now let $\eta$ be a super-gradient of $\alpha^{1/(n-1)}$ at the point $\sigma$ i.e. $\alpha(\sigma - t)^{1/(n-1)} \leq \alpha(\sigma)^{1/(n-1)} - \eta t$ for $t \in \mathbb{R}$. Then

$$\mu(\sigma - t) \leq (\alpha(\sigma)^{1/(n-1)} - \eta t)^{n-1} \ell(x_\sigma) e^{\theta t/\xi}$$

for $t \in \mathbb{R}$.

(4.39) and (4.41) imply that $\alpha(\tau) > \alpha(\sigma)$ and so $\eta > 0$. Now, putting $\alpha_0 = \alpha(\sigma)^{1/(n-1)}$, using $1 - x \leq e^{-x-x^2}$ for $0 \leq x \leq 1$ and noting that necessarily $\alpha_0 \geq (\sigma - \beta_1)\eta$, we obtain

$$\begin{aligned} \mu(A_1) &\leq \ell(x_\sigma) \int_0^{\sigma-\beta_1} (\alpha_0 - \eta t)^{n-1} e^{\theta t/\xi} dt \\ &\leq \mu(\sigma) \int_0^{\sigma-\beta_1} \exp\left\{\frac{\theta t}{\xi} - (n-1)\left(\frac{\eta t}{\alpha_0} + \frac{\eta^2 t^2}{\alpha_0^2}\right)\right\} dt. \end{aligned} \tag{4.42}$$

Now

$$\begin{aligned} \eta\xi &\geq \alpha(\tau)^{1/(n-1)} - \alpha(\sigma)^{1/(n-1)} \\ &= \alpha_0 \left(\left(\frac{\alpha(\tau)}{\alpha(\sigma)}\right)^{1/(n-1)} - 1\right) \geq \alpha_0 \left(\left(\frac{\ell(x_\sigma)}{\ell(x_\tau)}\left(1 - \frac{1}{4\sqrt{n}}\right)\right)^{1/(n-1)} - 1\right) \\ &\geq \alpha_0 \left(\exp\left\{\frac{\theta}{n-1} - \frac{1}{3n^{3/2}}\right\} - 1\right) \geq \alpha_0 \frac{\theta}{n-1}\left(1 - \frac{1}{2\sqrt{n}}\right). \end{aligned}$$

So

$$\frac{\eta}{\alpha_0} \geq \frac{\theta}{(n-1)\xi}\left(1 - \frac{5\delta}{\sqrt{n}}\right).$$

Going back to (4.42) we obtain

$$\mu(A_1) \leq \mu(\sigma) \int_{-\infty}^{\infty} \exp\left\{-\frac{\theta^2 t^2}{2\xi^2(n-1)} + \frac{\theta t}{2\xi\sqrt{n}}\right\} dt = (1 + o(1))\xi\mu(\sigma)\sqrt{\pi n/(2\theta^2)},$$

Comparing with (4.40) we see that

$$\mu(H) \geq (1-o(1))\frac{\theta}{\sqrt{\pi n}}\mu(A_1). \tag{4.43}$$

It follows from (4.36), (4.37), (4.38) and (4.43) that

$$\begin{aligned}\mu(H) &\geq \frac{1}{10\sqrt{n}}\min\left\{\frac{\delta}{d}\max_{i=1,2}\left\{\mu(A_i)\log\left(\frac{\mu(I)}{\mu(A_i)}\right)\right\},\mu(A_1),\mu(A_2)\right\}\\ &\geq \frac{1}{20\sqrt{n}}\min\left\{\frac{\delta}{d}\max_{i=1,2}\left\{\mu(A_i)\log\left(\frac{\mu(I)}{\mu(A_i)}\right)\right\},\mu(J_1),\mu(J_2)\right\} \qquad (4.44)\end{aligned}$$

Now for $i=1,2$,

$$\mu(A_i)\leq\mu(J_i) \text{ implies } \mu(A_i)\log\left(\frac{\mu(I)}{\mu(A_i)}\right)\geq\frac{1}{2}\mu(J_i)\log\left(\frac{\mu(I)}{\mu(J_i)}\right). \tag{4.45}$$

$x\log x^{-1}$ has a unique maximum over $x\in[0,1]$ at $e^{-1}$ and so

$$\mu(J_i)\leq\mu(A_i)\leq\frac{9}{10}\mu(I) \text{ implies } \mu(A_i)\log\left(\frac{\mu(I)}{\mu(A_i)}\right)\geq\frac{1}{4}\mu(J_i)\log\left(\frac{\mu(I)}{\mu(J_i)}\right). \tag{4.46}$$

Finally, if $\mu(J_i)\leq\mu(A_i)=(1-\alpha)\mu(I)$, $\alpha\leq\frac{1}{10}$ then $\mu(J_{3-i})=\beta\mu(I)$ where $\alpha\leq\beta\leq 2\alpha$ and $\mu(J_i)=(1-\beta)\mu(I)-\mu(H)$. Now (4.35) implies $\mu(H)=o(\alpha)$ and then we have

$$\mu(J_i)\log\left(\frac{\mu(I)}{\mu(J_i)}\right)\leq\mu(I)(\beta+\beta^2) \text{ and } \mu(A_i)\log\left(\frac{\mu(I)}{\mu(A_i)}\right)\geq\frac{9}{10}\alpha\mu(I)$$

and so

$$\mu(A_i)\log\left(\frac{\mu(I)}{\mu(A_i)}\right)\geq\frac{1}{3}\mu(J_i)\log\left(\frac{\mu(I)}{\mu(J_i)}\right). \tag{4.47}$$

It follows from (4.44) – (4.47) that

$$\mu(H)\geq\frac{1}{80\sqrt{n}}\min\left\{\frac{\delta}{d}\max_{i=1,2}\left\{\mu(J_i)\log\left(\frac{\mu(I)}{\mu(A_i)}\right)\right\},\mu(J_1),\mu(J_2)\right\}$$

contradicting (4.35) and completing the proof of the needle-like case.

We now turn to the general case where $K$ is not necesarily needle-like. Let $\ell_1 = \max\{\ell(x):\ x\in K\}$ and $M=\max\left\{\ell_0^{-1},\ell_1\right\}$. Suppose there is a convex body $K$ with sets $S_1,S_2$ such that (4.26) fails. Suppose that there exist mutually orthogonal directions $a_1,\dots,a_j$ such that $\max_{1\leq i\leq j}W(K,a_i)<\delta_1/\sqrt{n}$. If $j\geq n-1$, by Lemma 4.5.9 the needle-like case applies and we have a contradiction. Thus suppose $j\leq n-2$ is maximal such that a counter-example can be found. Let $L$ be a two-dimensional linear subspace orthogonal to $a_1,\dots,a_j$. By Lemma 4.5.11 there is a hyperplane $P$ with normal $a\in L$,

$|a| = 1$, which bisects the $\ell$-measure of both $S_1, S_2$. We choose $P^+$ to be the half-space such that $h(K \cap P^+)$ is smaller. Let us write $K'$ for $K \cap P^+$ etc. If the theorem fails for $K$, $S_1$, $S_2$, then it follows that it must also fail for $K'$, $S_1'$, $S_2'$. (The diameter can only decrease, and the value of $\ell_0$ increase, so the same $d$, $\delta_1$, $\epsilon$ will apply.) Also,

$$\mathrm{vol}(K') \geq \frac{\ell(K')}{M} = \frac{\ell(K)}{2M} \geq \frac{\ell(K)}{2M^2}.$$

Thus, by Lemma 4.5.10, $W(K', a) \leq \rho W(K, a)$ where $\rho = 1 - \frac{1}{2nM^2}$.

Suppose we iterate this bisection, obtaining a sequence of bodies

$$K = K^{(1)} \supset K^{(2)} \supset \cdots K^{(m)} \supset \cdots,$$

where $K^{(m)} = P^{(m)} \cap K^{(m-1)}$, containing sets for which the theorem fails. Now $K^{(m)}$ clearly converges to a compact convex set $K^*$. If $a^{(m)}$ is the normal to $P^{(m)}$, by compactness $a^{(m)}$ has a cluster point $a^* \in L$. By continuity, taking the limit in $0 \leq W(K^{(m+1)}, a^{(m)}) \leq \rho W(K^{(m)}, a^{(m)})$ gives $0 \leq W(K^*, a^*) \leq \rho W(K^*, a^*)$. Thus $W(K^*, a^*) = 0$, and hence for some $m$, $W(K^{(m)}, a^{(m)}) < \delta_1/\sqrt{n}$, contradiction.

Assuming that $\ell(S_1) \leq \ell(S_2)$ and using (4.26) we obtain

$$\begin{aligned}
\frac{h(K)\ell(K)}{2\ell(S_1)\ell(S_2)} &\geq \frac{1}{2500\sqrt{n}} \min\left\{\frac{\delta}{d}\log\left(\frac{\ell(K)}{\ell(S_1)}\right), 1\right\} \\
&\geq \frac{1}{2500\sqrt{n}} \min\left\{\frac{\delta}{d}\log\left(\frac{\ell(K)^2}{2\ell(S_1)\ell(S_2)}\right), 1\right\} \\
&\geq \frac{1}{2500\sqrt{n}} \min\left\{\frac{\delta}{2d}\log\left(\frac{\ell(K)^2}{\ell(S_1)\ell(S_2)}\right), 1\right\}
\end{aligned}$$

and the theorem follows. □

We now complete the proof of Theorem 4.2.2. Suppose that we have a partition $S_{\mathcal{L}}^{(1)}, S_{\mathcal{L}}^{(2)}$ of $K_{\mathcal{L}}$ with $S_{\mathcal{L}}^{(1)} \leq S_{\mathcal{L}}^{(2)}$. We need to bound the following quantity from below:

$$\Phi_{\mathcal{L}}(S_{\mathcal{L}}^{(1)}) = \frac{\ell_{\mathcal{L}}(K_{\mathcal{L}})}{\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)})\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(2)})} \sum_{x \in S_{\mathcal{L}}^{(1)}} \frac{|B_{\mathcal{L}}(x,\delta) \cap S_{\mathcal{L}}^{(2)}|}{|B_{\mathcal{L}}(x,\delta)|}.$$

We define

$$\delta' = \delta - \eta\sqrt{n},\ K^* = (1 - \eta\sqrt{n})K,\ S_1^* = K^* \cap \bigcup_{x \in S_{\mathcal{L}}^{(1)}} C(x) \text{ and } S_2^* = K^* \setminus S_1^*.$$

Then denoting the local conductance of $K^*$ by $\ell^*$ we see that Theorem 4.26 implies

$$\frac{\ell^*(K^*)}{\ell^*(S_1^*)\ell^*(S_2^*)} \int_{x \in S_1^*} \frac{\mathrm{vol}(B(x,\delta') \cap S_2^*)}{\mathrm{vol}(B(x,\delta'))}\, dx \geq \frac{1}{5000\sqrt{n}} \min\left\{\frac{\delta'}{d}\log\left(\frac{\ell(K^*)}{\ell^*(S_1^*)}\right), 1\right\} \tag{4.48}$$

Arguing as in Lemma 4.5.2 we get

$$\ell^*(K^*) \leq \left(1+\frac{\epsilon}{n}\right)\eta^n \ell_{\mathcal{L}}(K_{\mathcal{L}}), \ell^*(S_i^*) \geq \left(1+\frac{\epsilon}{n}\right)^{-1}\eta^n \ell_{\mathcal{L}}(S_{\mathcal{L}}^{(i)*}),\ i=1,2 \tag{4.49}$$

where $S_{\mathcal{L}}^{(i)*} = \left\{x \in S_{\mathcal{L}}^{(i)}:\ C(x)\cap K^* \neq \emptyset\right\}$.

Furthermore,

$$\int_{x\in S_1^*} \frac{\mathrm{vol}(B(x,\delta')\cap S_2^*)}{\mathrm{vol}(B(x,\delta'))}\,dx = \int_{x\in S_1^*}\int_{x\in S_2^*} 1_{|x-y|\leq\delta'}\,dxdy \leq \\ \sum_{x\in S_{\mathcal{L}}^{(1)*}}\sum_{x\in S_{\mathcal{L}}^{(2)*}} \eta^{2n} 1_{|x-y|\leq\delta} \leq \sum_{x\in S_{\mathcal{L}}^{(1)*}} |B_{\mathcal{L}}(x,\delta)\cap S_{\mathcal{L}}^{(2)}|. \tag{4.50}$$

It follows from (4.49), (4.50) and $\eta^n|B_{\mathcal{L}}(x,\delta)| \leq \left(1+\frac{\epsilon}{n}\right)\mathrm{vol}(B(x,\delta'))$ – Lemma 4.3.1 – that

$$\Phi_{\mathcal{L}}(S_{\mathcal{L}}^{(1)}) \geq \frac{\left(1+\frac{\epsilon}{n}\right)^{-4}}{5000\sqrt{n}} \frac{\ell_{\mathcal{L}}^*(S_{\mathcal{L}}^{(1)*})}{\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)})}\cdot\frac{\ell_{\mathcal{L}}^*(S_{\mathcal{L}}^{(2)*})}{\ell_{\mathcal{L}} S_{\mathcal{L}}^{(2)})}\min\left\{\frac{\delta'}{d}\log\left(\frac{\left(1+\frac{\epsilon}{n}\right)^{-1}\ell_{\mathcal{L}}(K_{\mathcal{L}})}{\left(1+\frac{\epsilon}{n}\right)\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)*})}\right),1\right\} \tag{4.51}$$

Arguing as in Lemma 4.5.2 we get

$$\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(2)*}) \geq \ell_{\mathcal{L}}(S_{\mathcal{L}}^{(2)}) - 2\frac{\epsilon}{n}\ell_{\mathcal{L}}(K_{\mathcal{L}}) \geq \ell_{\mathcal{L}}(S_{\mathcal{L}}^{(2)})\left(1-4\frac{\epsilon}{n}\right)$$

since $\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(2)}) \geq \frac{1}{2}\ell_{\mathcal{L}}(K_{\mathcal{L}})$.

**Case 1:** $\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)*}) \geq \frac{1}{2}\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)})$.
It follows from (4.51) that

$$\Phi_{\mathcal{L}}(S_{\mathcal{L}}^{(1)}) \geq \frac{1}{10001\sqrt{n}}\min\left\{\frac{\delta}{d}\log\left(\frac{\ell_{\mathcal{L}}(K_{\mathcal{L}})}{\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)})}\right),1\right\}. \tag{4.52}$$

**Case 1:** $\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)*}) < \frac{1}{2}\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)})$.
We show that

$$x \in S_{\mathcal{L}}^{(1)B} = S_{\mathcal{L}}^{(1)} \setminus S_{\mathcal{L}}^{(1)*} \text{ implies } |B_{\mathcal{L}}(x,\delta)\cap S_{\mathcal{L}}^{(2)}| \geq \frac{1}{2}|B_{\mathcal{L}}(x,\delta)\cap K_{\mathcal{L}}|. \tag{4.53}$$

As a consequence

$$\Phi_{\mathcal{L}}(S_{\mathcal{L}}^{(1)}) \geq \frac{\ell_{\mathcal{L}}(K_{\mathcal{L}})}{\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)})\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(2)})}\sum_{x\in S_{\mathcal{L}}^{(1)B}}\frac{|B_{\mathcal{L}}(x,\delta)\cap K_{\mathcal{L}}|}{2|B_{\mathcal{L}}(x,\delta)|} = \frac{\ell_{\mathcal{L}}(K_{\mathcal{L}})\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)B})}{2S_{\mathcal{L}}^{(1)})\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(2)})} \geq \frac{1}{4}.$$

Suppose that in contradiction to (4.53) that $\exists x \in S_{\mathcal{L}}^{(1)B}$ such that

$$|B_{\mathcal{L}}(x,\delta) \cap S_{\mathcal{L}}^{(1)}| > \frac{1}{2}|B_{\mathcal{L}}(x,\delta) \cap K_{\mathcal{L}}| \geq \frac{1}{3}\eta^{-n}(10d/\delta)^{-n} \tag{4.54}$$

where the last inequality is from Theorem 4.3.1 and Lemma 4.5.2.

Now

$$\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)B}) \leq |S_{\mathcal{L}}^{(1)B}| \leq 3\eta^{-(n-1)}d^n n^{1/2} \tag{4.55}$$

and (4.53) implies

$$\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)*}) \geq (\eta^{-n}(2d)^{-n} - 3\eta^{-(n-1)}d^n\sqrt{n})(2d)^{-n}. \tag{4.56}$$

(4.55) and (4.56) contradict $\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)*}) < \ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)B})$.

Thus (4.52) holds in general and so

$$\begin{aligned}
\Phi_{\mathcal{L}}(S_{\mathcal{L}}^{(1)}) &\geq \frac{1}{10001\sqrt{n}}\min\left\{\frac{\delta}{d}\log\left(\frac{\ell_{\mathcal{L}}(K_{\mathcal{L}})^2}{2\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)})\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(2)})}\right),1\right\} \\
&\geq \frac{1}{10001\sqrt{n}}\min\left\{\frac{\delta}{2d}\log\left(\frac{\ell_{\mathcal{L}}(K_{\mathcal{L}})^2}{\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(1)})\ell_{\mathcal{L}}(S_{\mathcal{L}}^{(2)})}\right),1\right\}
\end{aligned}$$

Thus the conductance function $\Phi_{\mathcal{L}}(x)$ for the speedy chain satisfies the conditions of Theorem 2.6.4 with $A = \frac{\delta}{20002d\sqrt{n}}$ and $B = \frac{1}{20002\sqrt{n}}$ and Theorem 4.2.2 follows. □

## 4.6 Deferred proofs of Section 4.4

### 4.6.1 Proof of Theorem 4.4.3

(a) The claim is invariant under affine transformations and so we can assume that $K$ is in isotropic position. We therefore have to show that $g \in -\phi K$ which is implied by $|g| \leq \frac{\phi}{n}$. Now

$$\mathbf{E}(|g|^2) = \frac{1}{m_2^2}\left(\sum_i \mathbf{E}(|z^{(i)}|^2) + \sum_{i\neq j}\mathbf{E}({z^{(i)}}^T z^{(j)})\right). \tag{4.57}$$

Let $\xi_i = W^{(i)} - U_{\mathcal{L}}$ where $W^{(i)}$ will be the conditional distribution of $z^{(i)}$ given $z^{(j)}, j \neq i$. Then

$$\mathbf{E}(|z^{(i)}|^2) = \sum_{x\in K_{\mathcal{L}}}|x|^2 U_{\mathcal{L}}(x) + \sum_{x\in K_{\mathcal{L}}}|x|^2\xi(x) \leq n + 2n^2\epsilon_2.$$

Also, for $i < j$,

$$\begin{aligned}\mathbf{E}(z^{(i)^T} z^{(j)}) &= \sum_{x \in K_{\mathcal{L}}} \mathbf{E}(x^T z^{(j)} \mid z^{(i)} = x) W^{(i)}(x) \\ &= \sum_{x \in K_{\mathcal{L}}} \sum_{y \in K_{\mathcal{L}}} x^T y \mathbf{Pr}(z^{(j)} = y \mid z^{(i)} = x) W^{(i)}(x) \\ &= \left( \sum_{x \in K_{\mathcal{L}}} x \xi_i(x) \right)^T \left( \sum_{y \in K_{\mathcal{L}}} y \xi_j(y) \right) \leq 4n^2 \epsilon_2^2.\end{aligned}$$

Hence

$$\mathbf{E}(|g|^2) \leq \frac{1}{m_2^2}(n^2 + 2n^2\epsilon_2 + 4n^4\epsilon_2^2) \leq \frac{\gamma\phi^2}{4}$$

and so we can use the Markov inequality to complete the proof of (a).

The proof of (b) is similar. □

### 4.6.2 Proof of Theorem 4.4.2

Replacing $K$ by $TK$ for some non-singular affine transformation $T$ yields the same value for $K'$ and so we can assume that $K$ is in isotropic position.

We start by proving the second condition of $\theta$-isotropy. We want to prove that with probability at least $1 - \eta$ every $w \in \mathbb{R}^n$ satisfies

$$(1 - \theta)|w|^2 \leq \frac{1}{\text{vol}(K')} \int_{K' - b(K')} (w^T y)^2 dy \leq (1 + \theta)|w|^2. \tag{4.58}$$

By change of variables, $y \to Y^{1/2}y + \bar{y}$, (4.58) can be written as

$$(1 - \theta)|v|^2 \leq \frac{1}{\text{vol}(K)} \int_K (v^T y)^2 dy \leq (1 + \theta)|v|^2.$$

We can assume that $|v| = 1$ in (4.59) and so the middle term is 1. So we have to prove

$$\frac{1}{1 + \theta} \leq v^T Y v \leq \frac{1}{1 - \theta}. \tag{4.59}$$

Putting $Y = Z - \bar{y}\,\bar{y}^T$ where $Z = \frac{1}{m} \sum_{i=1}^m y^{(i)} y^{(i)^T}$ we see that we have to show that with probability at least $1 - \eta$, for every $v \in \mathbb{R}^n, |v| = 1$,

$$\frac{1}{1 + \theta} + (v^T \bar{y})^2 \leq v^T Z v \leq \frac{1}{1 - \theta} + (v^T \bar{y})^\cdot$$

Now by Theorem 4.4.3 we have that with probability at least $1 - \gamma$, we have

$$|\bar{y}| \leq \theta/4 \tag{4.60}$$

and so it suffices to prove that for all $|v| = 1$,

$$|v^T \mathbf{E}(Z) v - 1| \le \frac{\theta}{4}. \tag{4.61}$$

Indeed,

$$\begin{aligned} v^T \mathbf{E}(Z) v &= \frac{1}{m_1} \sum_{i=1}^{m_1} v^T \mathbf{E}(y^{(i)} {y^{(i)}}^T) v = \frac{1}{m_1} \sum_{i=1}^{m_1} \sum_{x \in K_{\mathcal{L}}} (v^T x)^2 W^{(i)}(x) \\ &= \frac{1}{m_1} \sum_{i=1}^{m_1} (v^T x)^2 (U_{\mathcal{L}}(x) + \xi^{(i)}(x)) = 1 + \frac{1}{m_1} \sum_{i=1}^{m_1} (v^T x)^2 \xi^{(i)}(x) \end{aligned}$$

and (4.61) follows from

$$\left| \frac{1}{m_1} \sum_{i=1}^{m_1} (v^T x)^2 \xi^{(i)}(x) \right| \le n^4 \epsilon_1.$$

Next we prove that with probability at least $1 - \frac{\gamma}{2}$

$$||Z - \mathbf{E}(Z)|| \le \frac{\theta}{4}. \tag{4.62}$$

To prove this we use

$$||Z - \mathbf{E}(Z)||^2 \le \mathrm{Tr}((Z - \mathbf{E}(Z))^2).$$

We compute the expectation of this trace.

$$\begin{aligned} m_1^2 (Z - \mathbf{E}(Z))^2 &= \sum_{i=1}^{m_1} (y^{(i)} {y^{(i)}}^T - \mathbf{E}(y^{(i)} {y^{(i)}}^T)^2 + \\ &\sum_{i \ne j} (y^{(i)} {y^{(i)}}^T - \mathbf{E}(y^{(i)} {y^{(i)}}^T))(y^{(j)} {y^{(j)}}^T - \mathbf{E}(y^{(j)} {y^{(j)}}^T)). \end{aligned} \tag{4.63}$$

The first term is handled as follows: Fix any $i$; then

$$\mathbf{E}((y^{(i)} {y^{(i)}}^T - \mathbf{E}(y^{(i)} {y^{(i)}}^T)^2) = \mathbf{E}((y^{(i)} {y^{(i)}}^T)^2) - \mathbf{E}(y^{(i)} {y^{(i)}}^T)^2$$

and hence

$$\begin{aligned} \mathbf{E}(\mathrm{Tr}((y^{(i)} {y^{(i)}}^T - \mathbf{E}(y^{(i)} {y^{(i)}}^T)^2))) &\le \mathbf{E}(\mathrm{Tr}((y^{(i)} {y^{(i)}}^T)^2)) = \mathbf{E}(|y^{(i)}|^4) \\ &= \sum_{x \in K_{\mathcal{L}}} |x|^4 U_{\mathcal{L}}(x) + \sum_{x \in K_{\mathcal{L}}} |x|^4 \xi^{(i)}(x) \le 8n^2 + \epsilon_1 n^4, \end{aligned}$$

where we have used the inequality $\mathbf{E}_K(|x|^4) \le 8\mathbf{E}_K(|x|^2)$ by a theorem of Gromov and Milman.

which theorem?

The second term in (4.63)is handled as follows: Fix any $i \neq j$; then

$$\mathrm{Tr}((y^{(i)}y^{(i)^T} - \mathbf{E}(y^{(i)}y^{(i)^T}))(y^{(j)}y^{(j)^T} - \mathbf{E}(y^{(j)}y^{(j)^T}))) = \sum_{k=1}^{n}\sum_{l=1}^{n}(y_k^{(i)}y_l^{(i)} - \mathbf{E}(y_k^{(i)}y_l^{(i)}))(y_k^{(j)}y_l^{(j)} - \mathbf{E}(y_k^{(j)}y_l^{(j)})).$$

The expectation of each term here can be bounded by $\epsilon_1(n+1)^4$ and the expectation of the second term in (4.63)c an be bounded by $\epsilon_1 m_1^2(n+1)^4$. So,

$$\mathbf{E}(||Z - \mathbf{E}(Z)||^2) \leq m_1^{-1}(8n^2 + \epsilon_1 n^4) + \epsilon_1(n+1)^4 \leq \frac{\gamma^2\phi^2}{8}$$

and (4.61) follows.

To complete the proof it suffices to remark that $b(K') = -Y^{1/2}\bar{y}$ and hence if (4.59) and (4.60) hold, then

$$|b(K')| = \sqrt{\bar{y}^T Y \bar{y}} \leq \sqrt{\frac{\theta^2}{16(1-\theta)}} < \theta.$$

□

### 4.6.3 Proof of Theorem 4.4.5

We first prove some preliminary lemmas.

**Lemma 4.6.1** *If $b(K) \in -\alpha K$ for $\alpha > 0$ and $h^T x \leq c$ for $h \in \mathbb{R}^n, |h| = 1$ and $x \in K$ then $\mathbf{E}_K((h^T x)^2) \leq (1 + 2\alpha + 2\alpha^2)c^2$.*

**Proof** We first observe that if $f = \rho^T x + \sigma$ is an arbitrary linear function on $K$ then

$$\mathbf{E}_K(f^2) \leq \mathbf{E}_K(f)^2 + \left(\max_K f - \mathbf{E}_K(f)\right)^2. \tag{4.64}$$

Indeed, (4.64) is clear if $K$ is in isotropic position for then $\mathbf{E}_K(f) = \sigma$, $\mathbf{E}_K(f^2) = |\rho|^2$ and $\max_K f \geq |\rho|$ since $K \supseteq B$. Now note thet if (4.64) is true for some $K$ and all $f$ then it remains true for $AK$ and all $f$, where $A$ is an affine transformation.

Now let $f = h^T x$. Then $\mathbf{E}_K(h^T x) = h^T b \geq -\alpha c$ since $b \in -\alpha K$. So, applying (4.64), we get

$$\mathbf{E}_K((h^T x)^2) \leq c^2 + (c + \alpha c)^2.$$

□

**Lemma 4.6.2** *Let $h \in \mathbb{R}^n, |h| = 1$ for which $h^T x < \frac{1}{2}$ for all $x \in K$. Let $U_h$ be as in (4.11). Then*

**(a)** *If $K$ satifies (4.7) and (4.9) then so does $U_h K$.*

**(b)** *$vol(U_h K) \geq \frac{9}{8} vol(K)$.*

**Proof** For (4.7) we use

$$b(U_h K) = U_h b(K) \in U_h \left( -\frac{1}{10} K \right) = -\frac{1}{10} U_h K.$$

For (4.9) we see that Lemma 4.6.1 implies $\mathbf{E}_K((h^T x)^2) \leq \frac{1}{3}$. Hence

$$\mathbf{E}_{U_h K}(|x|^2) = \mathbf{E}_K(|U_h x|^2) = \left(1 - \frac{1}{2n}\right)^2 E_K(|x + (x^T h)h|^2) =$$
$$\left(1 - \frac{1}{2n}\right)^2 E_K(|x|^2 + 3(h^T x)^2) < \left(1 - \frac{1}{2n}\right)^2 (100n + 1) < 100n.$$

This completes the proof of (a).

Since $\det(U_h) = 2\left(1 - \frac{1}{2n}\right)^n \geq \frac{9}{8}$ we have (b). □

**Lemma 4.6.3** *Let $K$ be a convex body containing the origin and let $v$ be chosen uniformly from $K$. Make one step of a lazy random walk starting from $v$. Then the probability that this step is a nonflat improper step is at most $4\delta_0\sqrt{n}$.*

**Proof** Put $K_1 = conv(K \cup \frac{1}{2}B)$. Assume that the attempted step $v \to u$ is nonflat improper. Then trivially $u \notin K_1$. We prove that the $(2n)$-dimensional measure of the set $S$ of pairs $v, u$ with $v \in K$, $u \in \mathbb{R}^n \setminus K_1$ and $|v - u| \leq \delta_0$ is at most $4\delta_0\sqrt{n}\mathrm{vol}(K)\mathrm{vol}(\delta_0 B)$ and this will prove the lemma.

Let $q'$ be the point of intersection of the segment $[v, u]$ and $\partial K_1$. Then clearly $q' \in F' = \partial K_1 \cap (2^{1/n} K)$. Applying Lemma 4.5.1 to $K_1$ we get that

$$\mathrm{vol}_{2n}(S) \leq \delta_0 \mathrm{vol}_{n-1}(F') \frac{c_{n-1}}{(n+1)c_n} \mathrm{vol}(\delta_0 B) < \frac{\delta_0}{\sqrt{n}} \mathrm{vol}_{n-1}(F')\mathrm{vol}(\delta_0 B).$$

The hyperplane supporting $K_1$ at any point of $F'$ has distance at least $1/2$ from the origin. Hence the union $U$ of segments connecting 0 to $F'$ has volume at least $\mathrm{vol}_{n-1}(F')/(2n)$. On the other hand, clearly $U \subseteq 2^{1/n} K$. This implies that

$$\mathrm{vol}_{n-1}(F') \leq 4n \mathrm{vol}(K)$$

and so

$$\mathrm{vol}_{2n}(S) \leq 4\delta_0\sqrt{n}\mathrm{vol}(K)\mathrm{vol}(\delta_0 B).$$

□

**Lemma 4.6.4** *Let $K \subseteq d_0 B$, $d_0 \geq 1$ be a convex body with average local conductance $\lambda$ with respect to $\delta_0$ moves where $0 < \delta_0 < \frac{1}{32}$. Let $u \in K$. Starting from $u$, do a lazy random walk in $K$ until at least*

$$T = \lceil \kappa n^2 d_0^2 \delta_0^{-2} \log(\delta_0 \sqrt{n}) \rceil$$

*proper steps are made. Then the probability that no flat steps were attempted is at most $\lambda + 6\delta_0\sqrt{n}$.*

**Proof** We may assume that $\delta_0 < 1/(6\sqrt{n})$. Consider a random walk in the body $K_1 = conv(K \cup \frac{1}{2}B)$ starting at $u$. Until this walk hits $K_1 \setminus K$ it can be considered a random walk in $K$. Conversely, a random walk in $K$ can be considered a random walk in $K_1$ until the first flat step is attempted, because until then, any time we attempt to step out of $K$, we are actually stepping out of $K_1$. Hence the probability that a random walk of length $T$ in $K$ attempts a flat step is at least as large as the probability that a random walk in $K_1$ of length $T$ hits $K_1 \setminus K$.

Now $(1/2)B \subseteq K_1 \subseteq d_0 B$ and so (4.16) implies that the average local conductance of $K_1$ is at least $1 - \delta_0\sqrt{n}$. Theorem 4.2.2 and Lemma 4.2.1 imply that the distribution of the point $w$ at the $(T-1)$th proper step is within variation distance $2\delta_0\sqrt{n}$ of uniform. Lemma 4.6.3 then implies that the probability the step from $w$ is proper or nonflat is at most $\lambda + 2\delta_0\sqrt{n} + 4\delta_0\sqrt{n}$. □

We can now prove Theorem 4.4.5. The first assertion is clear by Lemma 4.6.2. This implies that the volume of every $K_i$ is at most that of $B(0, 10n)$ which is $(10n)^{n/2} c_n$. plausible
On the other hand (4.9) implies that $\mathrm{vol}(K_0) \geq n^{-n/4} c_n$. Since each flat step increase the volume by at least 9/8 we see that at most $8n \log n$ flat steps can occur.

Consider the algorithm going on for $M$ rather than $N$ iterations. Let $L_i, L_i'$ be the average local conductances of $K_i, K_i'$ respectively. It follows from (4.9), $K_i' \subseteq dB$ and the Markov inequality that

$$\frac{\mathrm{vol}(K_i')}{\mathrm{vol}(K_i)} \geq 1 - 50\theta.$$

So,

$$L_i \geq L_i' \frac{\mathrm{vol}(K_i')}{\mathrm{vol}(K_i)} \geq L_i'(1 - 50\theta).$$

Let $\lambda_i = \mathbf{E}(L_i)$ ($L_i$ is a random variable). Let $X_i$ be the indicator variable of the event that the $i$th random walk ended with a flat step. Then $\sum_i X_i$ is the number of such walks and hence

$$\sum_i X_i \leq 8n \log n. \tag{4.65}$$

On the other hand, from Lemma 4.6.4, we get that

$$\mathbf{Pr}(X_{i+1} = 1 \mid \textit{previous history}) \geq 1 - 6\delta_0\sqrt{n} - L_i' \geq 1 - \frac{\theta}{4} - \frac{1}{1 - 50\theta} L_i$$

and so

$$\sum_{i=0}^{M-1} \mathbf{E}(X_i) \geq \sum_{i=0}^{M-1} \left(1 - \frac{\theta}{4} - \frac{1}{1-50\theta} L_i\right)$$

and so by (4.65)

$$\frac{1}{M} \sum_{i=0}^{M-1} \lambda_i \geq (1-50\theta)\left(1 - \frac{8n\log n}{M} - \frac{\theta}{4}\right) \geq 1 - 100\theta.$$

Since $N$ is chosen randomly from $\{0, 1, \dots, M-1\}$ we see that

$$\mathbf{E}(L_N) = \lambda_N \geq 1 - 100\theta.$$

For simplicity imagine that the last walk goes on if necessary until a total of at least $3MT$ steps are made. If the number of nonflat improper steps during the algorithm is larger than $MT$ then their number among the first $3MT$ steps is larger than $MT$. Since $u$ and therefore evey given point in the sequence has a distribution that is closer to uniform than $\gamma/6$ (in total variation distance), the probability that a given step is nonflat improper is at most $\gamma/6 + 4\delta_0\sqrt{n} < \gamma/3$ by Lemma isolem4. Thus the expected number of nonflat improper steps is at most $\gamma MT$. The probability bound on the number of steps then follows from the Markov inequality. □