On Randomly Generated Intersecting Hypergraphs II

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Abstract

Let c be a positive constant. Suppose that $r = o(n^{5/12})$ and the members of $\binom{[n]}{r}$ are chosen sequentially at random to form an intersecting hypergraph $\mathcal{H}$. We show that **whp**[1] $\mathcal{H}$ consists of a simple hypergraph $\mathcal{S}$ of size $\Theta(r/n^{1/3})$, a distinguished vertex v and all r-sets which contain v and meet every edge of $\mathcal{S}$. This is a continuation of the study of such random intersecting systems started in [2] where the case $r = O(n^{1/3})$ was considered. To obtain the stated result we continue to investigate this question in the range $\omega(n^{1/3}) \leq r \leq o(n^{5/12})$.

1 Introduction

The study of random combinatorial structures has emerged as an important component of Discrete Mathematics. The most intensely studied area is that of random graphs [4], [10] and many of the results of this area have been extended to hypergraphs or set systems. There are (at least) two ways to study random structures. One can set up a probability space and sample directly from it or one can define a random process and study its outcome. (One can argue that there is no formal difference between the two approaches, but one cannot deny a qualitative difference between them). As examples of the first approach, one can study the distribution

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[1] A sequence of events $\mathcal{E}_1, \ldots, \mathcal{E}_n, \ldots$ is said to occur **with high probability** (**whp**) if $\lim_{n\to\infty} \Pr(\mathcal{E}_n) = 1$.

of the number of triangles in the random graph $G_{n,p}$, or one can estimate the probability that the random graph $G_{n,1/2}$ is triangle-free [6]. As an example of the second approach, one can consider a process where we add random edges, but avoid introducing triangles [7]. In this paper, which is a continuation of [2], we follow this idea and consider intersecting hypergraphs.

A **hypergraph** is a family of subsets of a given ground set. The subsets are called **edges**. The **degree** of a vertex is the number of edges that contain it. When we deal with more than one hypergraph, the appropriate edges and degrees should be clear from the context. In the proofs, we will refer to auxiliary **graphs**, which are simple graphs in the ordinary sense. That is, 2-uniform hypergraph. An **intersecting hypergraph** is one in which each pair of edges has a non-empty intersection. Here, we consider **r-uniform hypergraphs** which are those for which all edges contain r vertices.

The motivating idea for this paper is the classical Erdős-Ko-Rado theorem [5] which states that a maximum size r-uniform intersecting hypergraph has $\binom{n-1}{r-1}$ edges if $r \leq n/2$ and $\binom{n}{r}$ edges if $r > n/2$. Furthermore, for $r < n/2$ any maximum-sized family must have the property that all edges contain a common vertex.

In the last four decades this theorem has attracted the attention of many researchers and it has been generalized in many ways. It is worth mentioning, for example, the famous conjecture of Frankl on the structure of maximum t-intersecting families in a certain range of $n(t, r)$ which was investigated by Frankl and Füredi [9] and completely solved only a few years ago by Ahlswede and Khachatrian [1]. Another type of generalization can be found in [3].

The first attempt to 'randomize' this topic was given by Fishburn, Frankl, Freed, Lagarias and Odlyzko [8]. Also note that other random hypergraph structures were considered already by Rényi e.g., in [11], he identified the anti-chain threshold. The paper [2] began the study of randomly generated intersecting systems. More precisely, the edges were taken on-line; that is, one at a time, ensuring that at each stage, the resulting hypergraph remained intersecting. I.e., we considered the following random process:

CHOOSE RANDOM INTERSECTING SYSTEM
Choose $E_1 \in \binom{[n]}{r}$. Given $\mathcal{H}_t := \{E_1, \ldots, E_t\}$, let $\mathcal{A}(\mathcal{H}_t) = \{E \in \binom{[n]}{r} : E \notin \mathcal{H}_t$ and $E \cap E_\tau \neq \emptyset$ for $1 \leq \tau \leq t\}$. Choose E_{t+1} uniformly at random from $\mathcal{A}(\mathcal{H}_t)$. The procedure halts when $\mathcal{A}(\mathcal{H}_t) = \emptyset$ and $\mathcal{H} = \mathcal{H}_t$ is then output by the procedure.

It should be made clear that sets are chosen *without* replacement.

The paper [2] studied the case where $r = O(n^{1/3})$. The main result of that paper was the following theorem which determines the threshold for the event that edges chosen randomly online to form an intersecting hypergraph will attain the Erdős-Ko-Rado bound:

We say that $\mathcal{H}$ **fixes** x if every member of $\mathcal{H}$ contains x.

Theorem 1 *[2] Let $\mathcal{E}_{n,r}$ be the event that $\left\{|\mathcal{H}| = \binom{n-1}{r-1}\right\}$. For $r < n/2$, this is equivalent to*

$\mathcal{F}$ *fixing some* $x \in [n]$. *Then if* $r = c_n n^{1/3} < n/2$,

$$\lim_{n\to\infty} \Pr(\mathcal{E}_{n,r}) = \begin{cases} 1 & c_n \to 0 \\ \frac{1}{1+c^3} & c_n \to c \\ 0 & c_n \to \infty \end{cases}.$$

For $k \geq 3$ let

$$t_k = \min\{t : \Delta(\mathcal{H}_t) = k\}$$

where for $v \in [n]$, $\deg_t(v)$ is the degree of v in $\mathcal{H}_t$ and $\Delta(\mathcal{H}_t) = \max_{v\in V} \{\deg_t(v)\}$.

As r grows beyond $n^{1/3}$ the structure of $\mathcal{H}$ grows more complex. In this paper we are able to analyze a small part of the range where $r/n^{1/3} \to \infty$: Hypergraph $\mathcal{H}_t$ is **simple** if $|E_i \cap E_j| \leq 1$ for all $1 \leq i \neq j \leq t$. In the range investigated, we show that **whp** $\mathcal{H}_t$ remains simple and has maximum degree two for the first $\Theta(r/n^{1/3})$ edges. Then at time t_3 a vertex v of degree three is created, then $o(t_3)$ edges are added that avoid v until, at time t_4, v achieves degree four. Then, **whp**, the process finishes with all the subsequent edges containing v and intersecting the $t_4 - 4$ edges of $\mathcal{H}_{t_4}$ not containing v.

Theorem 2 *Suppose that* $\omega \to \infty$ *and*

$$\omega n^{1/3} \leq r \leq n^{5/12}/\omega. \tag{1}$$

Then

1. $nt_4^3/(6r^3)$ *converges to the exponential distribution with mean 1 i.e.*

$$\lim_{n\to\infty} \Pr(t_4 \geq cr/n^{1/3}) = e^{-c^3/6}.$$

2. $\mathcal{H}_{t_4}$ *is simple* **whp**.

3. $\mathcal{H}_{t_4}$ *has a unique vertex* v *of degree 4 and no vertex of degree 3,* **whp**.

4. $\mathcal{H}$ *is the unique hypergraph consisting of all edges that (a) contain* v *and (b) meet every edge of* $\mathcal{H}_{t_4}$ *which does not contain* v.

Furthermore, given $\mathcal{H}_{t_4}$ *satisfying the above,*

$$|\mathcal{H}| = \sum_{i=0}^{t_4-4} (-1)^i \binom{t_4-4}{i} \binom{n-1-ri+i(i-1)/2}{r-1}. \tag{2}$$

$$= (1+o(1)) \left(\frac{r^2}{n}\right)^{t_4-4} \binom{n-1}{r-1} \tag{3}$$

We break the proof of Theorem 2 into stages. In Section 2.1 we study the growth of $\mathcal{H}$ until the time t_3 when the first vertex of degree at least three is created. As part of the analysis we show that **whp** $\mathcal{H}$ remains simple up to this point. Subsequent sections inch along until the maximum degree is $\Delta_0 = \lfloor (r/n^{1/3})^{1/2} \rfloor$. At this time, **whp**, $\mathcal{H}$ is still simple and v is the unique vertex of degree more than two. Section 2.4 then shows that things finish as described in the theorem. Section 3.2 provides estimates for some quantities used in the proofs.

In this and the previous work [2] we have tried to examine the structure of a typical intersecting family, in particular the one grown by a natural sequential process. Our proof requires long and careful computation, but more than this, we needed strong insight into the final goal to guide us. We hope to continue this study for other ranges of r, but we do not at the moment see what the typical structure is like for larger r.

2 Proof of Theorem 2

2.1 The first degree-three vertex

In this section we study the growth of $\mathcal{H}$ until the first time t_3 that $\mathcal{H}$ contains a vertex of degree 3. We define

$$t_\cap = \min\{t : \exists s < t : |E_s \cap E_t| \geq 2\}$$

to be the first time that $\mathcal{H}_t$ is not simple.

Our aim now is to show that the following behavior is typical: For any constant $c > 0$, the probability that $t_3 \geq cr/n^{1/3}$ falls off like $e^{-c^3/6}$ and that **whp** at time t_3, $\mathcal{H}_{t_3}$ is simple.

So we split the possibilities into various events. We consider:

- $\mathcal{A}_t \stackrel{\text{def}}{=} \{\Delta(\mathcal{H}_t) \leq 2\}$
- $\mathcal{A}_t^v \stackrel{\text{def}}{=} \{\deg_t(w) \leq 2, \forall w \neq v \text{ and } \deg_t(v) \geq 3\}$
- $\mathcal{B}_t \stackrel{\text{def}}{=} \{|E_i \cap E_j| = 1, \forall 1 \leq i < j \leq t\}$

In words, $\mathcal{A}_t$ is the event that the hypergraph $\mathcal{H}_t$ has maximum degree 2, $\mathcal{A}_t^v$ is the event that $\deg_t(v) \geq 3$ and that the maximum degree of the remaining vertices $w \neq v$ is 2 and $\mathcal{B}_t$ is the event that $\mathcal{H}_t$ is simple.

We partition the possibilities for $\mathcal{H}_t$ into three disjoint events and later we will express them in terms of the $\mathcal{A}_i, \mathcal{A}_i^v, \mathcal{B}_i$.

$$\begin{aligned}
\mathcal{E}_1^{(t)} &= \{t_3 > t\} \wedge \{t_\cap > t\} \\
\mathcal{E}_2^{(t)} &= \{t_3 \leq t\} \wedge \{t_3 < t_\cap\} \wedge \{\exists v : \mathcal{A}_{t_3}^v\} \\
\mathcal{E}_3 &= \bigcup_t \left(\overline{\mathcal{E}_1^{(t)}} \cap \overline{\mathcal{E}_2^{(t)}} \right).
\end{aligned}$$

Lemma 3 *Assume that*

$$t \leq \frac{Kr}{n^{1/3}} \tag{4}$$

where K is some arbitrarily large positive constant.

(a) $\Pr(\mathcal{E}_1^{(t)}) = \exp\left\{-\frac{t^3 n}{6r^3}\right\} + O(t^5n^2/r^6 + t^2r^2/n + t^2n/r^3)$.

(b) $\Pr(\mathcal{E}_2^{(t)}) = 1 - \exp\left\{-\frac{t^3 n}{6r^3}\right\} + O(t^5n^2/r^6 + t^2r^2/n + t^2n/r^3)$.

(c) $\Pr(\mathcal{E}_3) = o(1)$.

Here the hidden constant depends on K.

Proof We write

$$\Pr(\mathcal{E}_1^{(t)}) = \prod_{i=1}^{t-1} \Pr\left(\mathcal{A}_{i+1} \wedge \mathcal{B}_{i+1} \mid \mathcal{A}_i \wedge \mathcal{B}_i\right) \tag{5}$$

$$\Pr(\mathcal{E}_2^{(t)}) = \sum_{i=1}^{t-1} \Pr\left(\left\{\bigvee_v \mathcal{A}_{i+1}^v\right\} \wedge \mathcal{B}_{i+1} \mid \mathcal{A}_i \wedge \mathcal{B}_i\right) \Pr(\mathcal{A}_i \wedge \mathcal{B}_i) \tag{6}$$

Note $\Pr(\mathcal{A}_1 \wedge \mathcal{B}_1) = 1$.

Consider the following quantities:

- $\nu_{\text{all}}(t)$ is the number of r-sets that intersect every edge of $\mathcal{H}_t$.
- Assume that $\mathcal{H}_t$ is a simple hypergraph and that $E_i \cap E_j = \{x_{i,j}\}$ for $1 \leq i < j \leq t$. Let $G = (S, F)$ be a graph with s vertices and f edges where $S \subseteq [t]$.
  - $\nu_G^*(t)$ is the number of r-sets E such that $x_{i,j} \in E$ iff $(i,j) \in F$.
  - $\nu_G(t)$ is the number of r-sets E such that $x_{i,j} \in E$ iff $(i,j) \in F$ and which meet each E_i, $i \notin S$ in exactly one vertex.
  - $\mathcal{M}$ denotes the set of G which are matchings.
- As we will see, the dominant term $\nu_\emptyset(t)$ is the number of r-sets that (i) intersect every edge of $\mathcal{H}_t$, (ii) keep $\mathcal{H}_{t+1}$ simple, given that $\mathcal{H}_t$ is simple and (iii) keep $\Delta(\mathcal{H}_{t+1}) \leq 2$, given $\Delta(\mathcal{H}_t) \leq 2$.
  (This is the case $f = 0$.)

Note that $s \leq t$ and $f \leq \binom{s}{2}$ and $f = 0$ implies $s = 0$.

$$\begin{aligned}
\nu_G(t) &= (r-t+1)^{t-s}\binom{n - t(r-t+1) - \binom{t}{2}}{r-t+s-f} \\
&= (1 + O(tr^2/n))r^{t-s}\frac{n^{r-t+s-f}}{(r-t+s-f)!} \qquad (7) \\
&= (1 + O(tr^2/n))\left(\frac{n}{r^2}\right)^s \left(\frac{r}{n}\right)^f \nu_\emptyset(t).
\end{aligned}$$

We have taken care to remove extraneous error terms in our "big O" notation, based on the bound (4) that we have given for t.
Furthermore, we obtain an expression

$$\nu_{\emptyset}(t) = (1 + O(tr^2/n))n^{r-t}r^t/(r-t)!$$

by putting $s = f = 0$ into (7).

Continuing our estimates,

$$\nu^*_G(t) - \nu_G(t) \leq \sum_{p=t-s+1}^{r-f} \binom{n}{r-f-p} \sum_{\substack{a_j \geq 1, j \in [t-s] \\ \sum a_j = p}} \prod_{j=1}^{t-s} \binom{r}{a_j}. \tag{8}$$

Explanation: The integer p denotes the number of elements of E that belong to $\mathcal{H}_t$ but do not lie in an edge corresponding to a vertex of G. The quantity a_j is the number of elements of E which lie in the j^{th} such edge. Having chosen these p elements and the f elements of $\mathcal{H}_t$ corresponding to the edges of G, we have at most $\binom{n}{r-f-p}$ choices for the remaining elements of E.

Increasing p by 1 reduces the first binomial coefficient of (8) by a factor of $\sim r/n$. We can get all choices satisfying $a_1 + \cdots + a_t = p+1$ by choosing $a'_1 + \cdots + a'_t = p$, choosing a j and adding one to a'_j. Thus the number of choices increases by at most tr and so the ratio of $p+1$ terms to p terms is $O(tr^2/n)$.

So,

$$\begin{aligned}
\nu^*_G(t) - \nu_G(t) &\leq (1 + O(tr^2/n))\frac{tr^{t-s+1}}{2}\binom{n}{r-f-t+s-1} \\
&\leq (1 + O(tr^2/n))\frac{tr^{t-s+1}}{2}\frac{n^{r-t+s-f-1}}{(r-t+s-f-1)!} \\
&= (1 + O(tr^2/n))\frac{tr^2}{2n}\nu_G(t) \\
&\leq (1 + O(tr^2/n))\frac{tr^2}{2n}\left(\frac{n}{r^2}\right)^s\left(\frac{r}{n}\right)^f \nu_{\emptyset}(t).
\end{aligned}$$

Applying Proposition 7 (see Section 3) with $x = \frac{r}{n}$, $y = \frac{n}{r^2}$ we get:

$$\begin{aligned}
\nu_{\emptyset}(t)^{-1} \sum_{\substack{G \in \mathcal{M} \\ f=1}} \nu_G(t) &= (1 + O(tr^2/n))\binom{t}{2}\frac{n}{r^3} \\
\nu_{\emptyset}(t)^{-1} \sum_{\substack{G \in \mathcal{M} \\ f \geq 2}} \nu_G(t) &= O(t^4n^2/r^6) \\
\nu_{\emptyset}(t)^{-1}\left(\nu^*_{\emptyset}(t) - \nu_{\emptyset}(t)\right) &= O(tr^2/n) \\
\nu_{\emptyset}(t)^{-1} \sum_{\substack{G \in \mathcal{M} \\ f \geq 1}} (\nu^*_G(t) - \nu_G(t)) &= O(t^3/r) \\
\frac{\nu_{rest}(t)}{\nu_{\emptyset}(t)} &= O(t^4n/r^4).
\end{aligned}$$

The quantity

$$\nu_{rest}(t) = \sum_{\substack{G \notin \mathcal{M} \\ f \neq 0}} \nu_G^*(t)$$

accounts for every possibility not specifically mentioned in the previous four ratios.

We remark for future use that this implies that if (4) holds then

$$\nu_{\text{all}}(t) = \left(1 + O(tr^2/n + t^2n/r^3)\right) \nu_\emptyset(t) = \left(1 + O(tr^2/n + t^2n/r^3)\right) r^t \binom{n}{r-t}. \tag{9}$$

Continuing, it further follows that

$$\begin{aligned}
\Pr\left(\mathcal{A}_{t+1} \wedge \mathcal{B}_{t+1} \mid \mathcal{A}_t \wedge \mathcal{B}_t\right) &= \frac{\nu_\emptyset(t)}{\nu_{\text{all}}(t)} \\
&= 1 - \frac{t^2n}{2r^3} + O(t^4n^2/r^6 + tr^2/n + tn/r^3). \qquad (10) \\
\Pr\left(\left\{\bigvee_v \mathcal{A}_{t+1}^v\right\} \wedge \mathcal{B}_{t+1} \mid \mathcal{A}_t \wedge \mathcal{B}_t\right) &= \frac{\nu_\emptyset(t)}{\nu_{\text{all}}(t)} \sum_{\substack{G \in \mathcal{M} \\ f=1}} \frac{\nu_G(t)}{\nu_\emptyset(t)} \\
&= \frac{t^2n}{2r^3} + O(t^4n^2/r^6 + tr^2/n + tn/r^3). \qquad (11) \\
\Pr\left(\mathcal{A}_{t+1} \wedge \overline{\mathcal{B}_{t+1}} \mid \mathcal{A}_t \wedge \mathcal{B}_t\right) &= \frac{\nu_\emptyset(t)}{\nu_{\text{all}}(t)} \frac{\nu_\emptyset^*(t) - \nu_\emptyset(t)}{\nu_\emptyset(t)} \\
&= O(tr^2/n) \qquad (12) \\
\Pr\left(\bigwedge_v \overline{\mathcal{A}_{t+1}^v} \wedge \mathcal{B}_{t+1} \mid \mathcal{A}_t \wedge \mathcal{B}_t\right) &= \frac{\nu_\emptyset(t)}{\nu_{\text{all}}(t)} \sum_{\substack{G \in \mathcal{M} \\ f \geq 2}} \frac{\nu_G(t)}{\nu_\emptyset(t)} \\
&= O(t^4n^2/r^6) \qquad (13) \\
\Pr\left(\overline{\mathcal{A}_{t+1}} \wedge \overline{\mathcal{B}_{t+1}} \mid \mathcal{A}_t \wedge \mathcal{B}_t\right) &= \frac{\nu_\emptyset(t)}{\nu_{\text{all}}(t)} \left[\frac{\nu_{\text{rest}}^*(t)}{\nu_\emptyset(t)} + \sum_{\substack{G \in \mathcal{M} \\ f \geq 1}} \frac{\nu_G^*(t) - \nu_G(t)}{\nu_\emptyset(t)}\right] \\
&= O(t^3/r). \qquad (14)
\end{aligned}$$

Thus, if (4) holds,

$$\begin{aligned}
\Pr(\mathcal{E}_1^{(t)}) = \Pr(\mathcal{A}_t \wedge \mathcal{B}_t) &= \prod_{j=1}^{t-1}\left(1 - \frac{j^2 n}{2r^3} + O(t^4n^2/r^6 + tr^2/n + tn/r^3)\right) \\
&= \prod_{j=1}^{t-1} \exp\left\{-\frac{j^2 n}{2r^3} + O(t^4n^2/r^6 + tr^2/n + tn/r^3)\right\} \\
&= \exp\left\{-\frac{t^3 n}{6r^3}\right\} + O(t^5n^2/r^6 + t^2r^2/n + t^2n/r^3),
\end{aligned}$$

proving (a).

For $\mathcal{E}_2^{(t)}$ we have

$$\begin{aligned}
\Pr(\mathcal{E}_2^{(t)}) &= \sum_{j=1}^{t} \frac{j^2 n}{2r^3} e^{-j^3 n/6r^2} + O(t^5n^2/r^6 + t^2r^2/n + t^2n/r^3) \\
&= 1 - \exp\left\{-\frac{t^3 n}{6r^3}\right\} + O(t^5n^2/r^6 + t^2r^2/n + t^2n/r^3)
\end{aligned}$$

and (b) follows.

To prove (c) expand

$$\overline{\mathcal{E}_1^{(t)}} \cap \overline{\mathcal{E}_2^{(t)}} = (\{t_3 \le t\} \vee \{t_\cap \le t\}) \cap (\{t_3 > t\} \vee \{t_3 \ge t_\cap\} \vee \{\not\exists v : \mathcal{A}_{t_3}^v\}).$$

We see then that

$$\mathcal{E}_3 = \{t_3 \ge t_\cap\} \vee \{\not\exists v : \mathcal{A}_{t_3}^v\}).$$

Now let $t_K = Kr/n^{1/3}$. From parts (a),(b) we see that

$$\begin{aligned}
\Pr(t_3 \ge t_\cap) &\le \Pr(t_3 \ge t_K) + \Pr(t_\cap \le t_K) \\
&\le \Pr(\overline{\mathcal{E}_2^{(t_K-1)}}) + \Pr(\overline{\mathcal{E}_1^{(t_K)}}) \\
&= o(1). \\
\Pr(\not\exists v : \mathcal{A}_{t_3}^v) &\le \Pr(\overline{\mathcal{E}_2^{(t_K)}}) \\
&= e^{-K^3/6} + o(1).
\end{aligned}$$

We can make K as large as we like and (c) follows. □

The following summarizes what we have proved so far:

Claim 4 *With high probability there* $v \in V(\mathcal{H}_{t_3})$ *such that* $\mathcal{H}_{t_3}$ *is simple and* v *is the unique vertex of degree more than 2. Furthermore,*

$$\lim_{n\to\infty} \Pr\left(t_3 \ge \frac{\alpha r}{n^{1/3}}\right) = e^{-\alpha^3/6}.$$

Proof This follows from $\Pr(\mathcal{E}_1^{(t_\alpha)}) = \Pr(t_3 \ge t_\alpha) - o(1)$ where $t_\alpha = \frac{\alpha r}{n^{1/3}}$. □

2.2 A Useful Lemma

In this section, we develop a lemma that can be used in each of the next two sections. It analyzes the behavior after t_3 when the maximum degree is still relatively small. We will be given $\mathcal{H}_t$, an intersecting hypergraph that satisfies $\mathcal{A}_t^v \wedge \mathcal{B}_t$ with maximum degree $\Delta \geq 3$.

Denote the following:

- $\nu_{\text{all}}^A(t)$ is the number of remaining r-sets that have a non-empty intersection with all edges of $\mathcal{H}_t$ and contain v.
- $\nu_{\text{all}}^B(t)$ is the number of remaining r-sets that have a non-empty intersection with all edges of $\mathcal{H}_t$ but do not contain v.
- $\nu_{\emptyset}^A(t)$ is the number of r-sets that intersect the edges of $\mathcal{H}_t$ containing v in v *only* and intersect the remaining $\tau_\Delta = t - \Delta$ edges in exactly one vertex, never creating any vertex of degree greater than 2 (other than v).
- $\nu_{\emptyset}^B(t)$ is the number of r-sets that do not contain v, but do intersect each of the t edges in exactly one vertex, also never creating any vertex of degree greater than 2.

Lemma 5 *Let $\mathcal{H}_t$ be an intersecting hypergraph that satisfies $\mathcal{A}_t^v \wedge \mathcal{B}_t$ with maximum degree Δ, $3 \leq \Delta < \Delta_0$, where $\Delta_0 = \lfloor (r/n^{1/3})^{1/2} \rfloor$. Furthermore, assume that*

$$t = \Theta\left(\frac{r}{n^{1/3}}\right). \tag{15}$$

With the above notation,

$$\frac{\nu_{\text{all}}^A(t)}{\nu_{\emptyset}^A(t)} = 1 + O\left(\frac{t^2 n}{r^3}\right) \tag{16}$$

$$\frac{\nu_{\text{all}}^B(t)}{\nu_{\emptyset}^B(t)} = 1 + O\left(\frac{t^2 n}{r^3}\right) \tag{17}$$

$$\frac{\nu_{\emptyset}^B(t)}{\nu_{\emptyset}^A(t)} = \left(1 + O\left(\frac{tr^2}{n}\right)\right) \frac{n}{r} \left(\frac{r^2}{n}\right)^{\Delta} \tag{18}$$

Proof

First we compute the two main expressions.

$$\begin{aligned}
\nu_\emptyset^A(t) &= (r-t+1)^{\tau_\Delta}\binom{n-(rt-\binom{\tau_\Delta}{2})-\Delta\tau_\Delta-(\Delta-1))}{r-\tau_\Delta-1} \\
&= \big(1+O(tr^2/n)\big)\frac{r^{\tau_\Delta}n^{r-\tau_\Delta-1}}{(r-\tau_\Delta-1)!} \qquad (19)\\
\nu_\emptyset^B(t) &= (r-t+1)^{\tau_\Delta}(r-\tau_\Delta-1)^{\Delta}\binom{n-(rt-\binom{\tau_\Delta}{2})-\Delta\tau_\Delta-(\Delta-1))}{r-t} \\
&= \big(1+O(tr^2/n)\big)\frac{r^t n^{r-t}}{(r-t)!} \\
&= \big(1+O(tr^2/n)\big)\frac{n}{r}\left(\frac{r^2}{n}\right)^{\Delta}\nu_\emptyset^A(t) \qquad (20)
\end{aligned}$$

Equation (18) follows immediately.

Re-number the edges so that $v\in E_{\tau_\Delta+i}, 1\le i\le\Delta$. Suppose that $E_i\cap E_j=\{x_{i,j}\}$ for $1\le i<j\le\tau_\Delta$ and that $E_i\cap E_{\tau_\Delta+j}=\{y_{i,j}\}$ for $1\le i\le\tau_\Delta$ and $1\le j\le\Delta$.

Let $G=(S,F)$ be a graph with $s\ge 2$ vertices and $f\ge 1$ edges where $S\subseteq[\tau_\Delta]$ and F spans S. We define $\nu_G^{A_*}(t)$ to be the number of r-sets E such that (i) $v\in E$, (ii) $x_{i,j}\in E$ iff $(i,j)\in F$. For this count we are dropping the condition that the new edge intersects the old edges in exactly one vertex.

$$\begin{aligned}
\nu_G^{A_*}(t) &\le \sum_{p\ge\tau_\Delta-s}\binom{n}{r-1-f-p}\sum_{\substack{a_j\ge1, j\in[\tau_\Delta-s]\\ \sum a_j=p}}\prod_{j=1}^{\tau_\Delta-s}\binom{r}{a_j} \qquad (21)\\
&= \big(1+O(tr^2/n)\big)r^{\tau_\Delta-s}\binom{n}{r-1-f-\tau_\Delta+s},
\end{aligned}$$

since increasing p by 1 reduces the first binomial coefficient in (21) by a factor of $\sim r/n$ and then we gain at most a further tr factor in the number of choices for the a_j.

So,

$$\begin{aligned}
\nu_G^{A_*}(t) &\le \big(1+O(tr^2/n)\big)r^{\tau_\Delta-s}\frac{n^{r-1-f-\tau_\Delta+s}}{(r-1-f-\tau_\Delta+s)!} \\
&= \big(1+O(tr^2/n+f^2/r)\big)\left(\frac{r}{n}\right)^f\left(\frac{n}{r^2}\right)^s\nu_\emptyset^A(t).
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
\nu_\emptyset^{A_*}(t)-\nu_\emptyset^A(t) &\le \sum_{p\ge\tau_\Delta+1}\binom{n}{r-1-p}\sum_{\substack{a_j\ge1, j\in[\tau_\Delta-s]\\ \sum a_j=p}}\prod_{j=1}^{\tau_\Delta}\binom{r}{a_j} \\
&= \big(1+O(tr^2/n)\big)\binom{n}{r-\tau_\Delta-2}\frac{\tau_\Delta}{2}r^{\tau_\Delta}(r-1) \\
&= \big(1+O(tr^2/n)\big)\frac{\tau_\Delta r^2}{2n}\nu_\emptyset^A(t)
\end{aligned}$$

So, from Proposition 7 of Section 3,

$$\begin{aligned}\frac{\nu^A_{\text{all}}(t)}{\nu^A_{\emptyset}(t)} - 1 &\le \left(1+O(tr^2/n)\right)\left[\sum_{\emptyset\neq G}\left(\frac{r}{n}\right)^f\left(\frac{n}{r^2}\right)^s + \frac{\tau_\Delta r^2}{2n}\right]\\ &= O(t^2n/r^3)\end{aligned}$$

proving (16).

Now for graph $G=(S,F)$ and bipartite graph $H\subseteq([\tau_\Delta]\setminus S)\times[\Delta]$ define $\nu^{B_*}_{G,H}(t)$ to be the number of r-sets E such that (i) $v\notin E$, (ii) $x_{i,j}\in E$ iff $(i,j)\in F$ and (iii) $y_{i,j}\in E$ iff $(i,j)\in H$. Also let $u=u_H=|\{x:\ \exists y\ s.t.\ (x,y)\in H\}|$ and $\ell=\ell_H=|\{y:\ \exists x\ s.t.\ (x,y)\in H\}|$ and $h=|E(H)|$. Suppose either G or H is non-empty.

$$\begin{aligned}&\nu^{B_*}_{G,H}(t)\\ &\le \sum_{p\ge\tau_\Delta-s-u}\sum_{q\ge\Delta-\ell}\binom{n}{r-f-h-p-q}\sum_{\substack{a_j\ge1,j\in[\tau_\Delta-s-u]\\ \sum a_j=p}}\prod_{j=1}^{\tau_\Delta-s-u}\binom{r}{a_j}\sum_{\substack{b_j\ge1,j\in[\Delta-\ell]\\ \sum b_j=q}}\prod_{j=1}^{\Delta-\ell}\binom{r}{b_j}\\ &= \left(1+O(tr^2/n)\right)\binom{n}{r-f-h-t+s+u+\ell}\sum_{\substack{i_j\ge1,j\in[\tau_\Delta-s-u]\\ \sum a_j=\tau_\Delta-s-u}}\prod_{j=1}^{\tau_\Delta-s-u}\binom{r}{a_j}\sum_{\substack{b_j\ge1,j\in[\Delta-\ell]\\ \sum b_j=\Delta-\ell}}\prod_{j=1}^{\Delta-\ell}\binom{r}{b_j}\\ &= \left(1+O(tr^2/n)\right)r^{t-s-u-\ell}\frac{n^{r-f-h-\tau_\Delta+s+u-\Delta+\ell}}{(r-f-h-t+s+u+\ell)!}\\ &= \left(1+O(tr^2/n)\right)\left(\frac{r}{n}\right)^{f+h}\left(\frac{n}{r^2}\right)^{s+u+\ell}\nu^B_\emptyset(t)\end{aligned}$$

Finally,

$$\begin{aligned}\nu^{B_*}_{\emptyset,\emptyset}(t)-\nu^B_\emptyset(t) &\le \sum_{p\ge\tau_\Delta+1}\sum_{q\ge\Delta+1}\binom{n}{r-p-q}\sum_{\substack{a_j\ge1,j\in[\tau_\Delta]\\ \sum a_j=p}}\prod_{j=1}^{\tau_\Delta}\binom{r}{a_j}\sum_{\substack{b_j\ge1,j\in[\tau_\Delta]\\ \sum b_j=q}}\prod_{j=1}^{\Delta}\binom{r}{b_j}\\ &\quad+\sum_{q\ge\Delta+1}\binom{n}{r-\tau_\Delta-q}\left(\frac{\tau_\Delta r^{\tau_\Delta}(r-1)}{2}\right)\sum_{\substack{b_j\ge1,j\in[\tau_\Delta]\\ \sum b_j=q}}\prod_{j=1}^{\Delta}\binom{r}{b_j}\\ &\quad+\sum_{p\ge\tau_\Delta+1}\binom{n}{r-p-\Delta}\sum_{\substack{a_j\ge1,j\in[\tau_\Delta]\\ \sum a_j=p}}\prod_{j=1}^{\tau_\Delta}\binom{r}{a_j}\left(\frac{\Delta r^\Delta(r-1)}{2}\right)\\ &= \left(1+O(tr^2/n)\right)\frac{tr^t(r-1)n^{r-t-1}}{2(r-t-1)!}\\ &= \left(1+O(tr^2/n)\right)\frac{tr^2}{2n}\,\nu^B_\emptyset(t).\end{aligned}$$

So,
$$\frac{\nu_{\text{all}}^B(t)}{\nu_\emptyset^B(t)} - 1 \leq \left(1 + O(tr^2/n)\right)\left[\frac{tr^2}{2n} + \sum_G^* \sum_H^* \left(\frac{r}{n}\right)^f \left(\frac{n}{r^2}\right)^s \left(\frac{r}{n}\right)^h \left(\frac{n}{r^2}\right)^{u+\ell}\right].$$

(The notation $\sum^* \sum^*$ means that the summation avoids $(G, H) = (\emptyset, \emptyset)$)

Now from Propositions 7 and 8 of Section 3, ($\tau = \tau_\Delta - s$, $\tau\Delta xy^2 = \tau\Delta(r/n)(n^2/r^4)) \leq \tau_\Delta \Delta n/r^3 = o(1)$),

$$\begin{aligned}
&\frac{\nu_{\text{all}}^B(t)}{\nu_\emptyset^B(t)} - 1 \\
&\leq \left(1 + O(tr^2/n)\right)\left[\frac{tr^2}{2n} + \sum_{G \neq \emptyset} \left(\frac{r}{n}\right)^f \left(\frac{n}{r^2}\right)^s \left(1 + (1 + o(1))\frac{t^2 n}{r^3}\right) + (1 + o(1))\frac{\tau_\Delta \Delta n}{r^3}\right] \\
&= O(t^2 n/r^3)
\end{aligned}$$

proving (17). □

2.3 From t_3 to t_4

Assume that $t_3 = \Theta(r/n^{1/3})$. Let t_4 be the first time there is a vertex of degree 4.

It follows from Lemma 5 that as long as (15) holds and $\Delta \leq \Delta_0$,

$$\frac{\nu_\emptyset^A(t) + \nu_\emptyset^B(t)}{\nu_{\text{all}}(t)} = 1 - O(t^2 n/r^3). \tag{22}$$

$$\frac{\nu_\emptyset^B(t)}{\nu_\emptyset^A(t)} = \left(1 + O(tr^2/n)\right)\frac{r^5}{n^2}, \qquad \Delta = 3. \tag{23}$$

$$\frac{\nu_\emptyset^B(t)}{\nu_\emptyset^A(t)} = O(r^7/n^3). \qquad \Delta \geq 4. \tag{24}$$

Let $t = t_3 + \xi$ where $\xi \geq 0$ and $\xi = O(r^5/n^2) = o(r/n^{1/3})$. Note that $r = o(n^{5/12})$ implies that $r^5/n^2 = o(r/n^{1/3})$. Then (22) implies

$$\Pr(\mathcal{A}_t^v \wedge \mathcal{B}_t) = 1 - O(t^2 r^2/n) = 1 - o(1).$$

It follows from (22) and (23) that

$$\begin{aligned}
\Pr(t_4 > t) &= \prod_{\tau = t_3 + 1}^{t} \frac{1 - O(\tau^2 n/r^3)}{1 + (1 - O(\tau r^2/n))n^2/r^5} \\
&= e^{O(\xi t_3^2 n/r^3)} e^{O(t_3 n/r^5)} \left(\frac{r^5}{n^2 + r^5}\right)^\xi \\
&= e^{o(1)} \left(\frac{r^5}{n^2 + r^5}\right)^\xi.
\end{aligned} \tag{25}$$

If $r = o(n^{2/5})$ then the RHS of (25) is $o(1)$ for all $\xi \geq 1$ and we deduce that in this case $t_4 = t_3 + 1$ **whp**.

If $r = an^{2/5}$ where $a = a(n) \to \infty$ is allowed and $\xi = cr^5/n^2$ where $c > 0$ is constant then

$$\Pr(t_4 > t) = e^{o(1)}(1 + a^{-5})^{-ca^5}.$$

We deduce that

$$\Pr(\neg(\mathcal{A}_{t_4}^v \wedge \mathcal{B}_{t_4})) \leq o(1) + e^{o(1)}(1 + a^{-5})^{-ca^5}.$$

Since c can be made arbitrarily large, we deduce that at time t_4 there is **whp**, a unique vertex v of degree > 2. The following summarizes what we have proved in this section:

Claim 6 *With high probability $t_4 = (1 + o(1))t_3$ and there exists $v \in V(\mathcal{H}_{t_4})$ such that $\mathcal{H}_{t_4}$ is simple and v is the unique vertex of degree more than two.*

□

2.4 Finishing the proof

Assume now that $\mathcal{A}_{t_4}^v \wedge \mathcal{B}_{t_4}$ holds and that $t_4 = (1 + o(1))t_3$. The probability that in the next $\Delta_0 - 4$ steps we either (i) add an edge not containing v or (ii) that we make a new vertex of degree 3 is at most $O(\Delta_0(t_3^2 n/r^3 + r^7/n^3)) = o(1)$.

Assume then that $\mathcal{A}_{t_{\Delta_0}}^v \wedge \mathcal{B}_{t_{\Delta_0}}$ holds. Recall that $\Delta_0 = \lfloor (r/n^{1/3})^{1/2} \rfloor$. For time $t \geq t_{\Delta_0}$, let $\nu_{\text{all}}^A(t)$ be the number of r-sets which meet every edge of $\mathcal{H}_t$ and contain v and let $\nu_{\text{all}}^B(t)$ be the number of r-sets which meet every edge of $\mathcal{H}_t$ and do not contain v. Lemma 5 and $r \leq n^{5/12}$ imply that

$$\frac{\nu_{\text{all}}^B(t_{\Delta_0})}{\nu_{\text{all}}^A(t_{\Delta_0})} \leq n^{2/3 - \Delta_0/6}.$$

Now if $t = t_{\Delta_0} + \sigma$ and every edge added between t_{Δ_0} and t contains v then $\nu_{\text{all}}^A(t) \geq \nu_{\text{all}}^A(t_{\Delta_0}) - \sigma$ and $\nu_{\text{all}}^B(t) \leq \nu_{\text{all}}^B(t_{\Delta_0})$. So,

$$\Pr(v \notin E_{t+1} \mid v \in E_j, t_{\Delta_0} \leq j \leq t) \leq \frac{\nu_{\text{all}}^B(t_{\Delta_0})}{\nu_{\text{all}}^A(t_{\Delta_0}) - \sigma} \leq \frac{n^{2/3 - \Delta_0/6}\nu_{\text{all}}^A(t_{\Delta_0})}{\nu_{\text{all}}^A(t_{\Delta_0}) - \sigma} \leq 2n^{2/3 - \Delta_0/6},$$

as long as $\sigma \leq \nu_{\text{all}}^A(t_{\Delta_0})/2$.

Now $\nu_{\text{all}}^A(t_{\Delta_0}) \geq \binom{n - t_{\Delta_0} r}{r - t_{\Delta_0}} \gg n^{\Delta_0/7}$ and so we see that if $t_\infty = t_{\Delta_0} + n^{\Delta_0/7}$ then **whp**

$$v \in E_t,\ t_{\Delta_0} \leq t \leq t_\infty. \tag{26}$$

Let Ω denote the set of $(r-1)$-subsets of $[n] \setminus \{v\}$ which meet $E_1, \ldots, E_{\tau_4}$, $\tau_4 = t_4 - 4$. (Assume a re-numbering so that these are the edges of $\mathcal{H}_{t_4}$ which do not contain v). We know from (9) that

$$|\Omega| \leq (1 + o(1))r^{\tau_4}\binom{n}{r - \tau_4}. \tag{27}$$

If we condition on (26), the sets $E_t \setminus \{v\}$, $t_{\Delta_0}+1 \le t \le t_\infty$ will be chosen uniformly at random from Ω without replacement. For a fixed t and r-set $E \subseteq [n]\setminus\{v\}$ which meets $E_1, E_2, \ldots, E_{\tau_4}$, let

$$\pi_E = \Pr((E_t \setminus \{v\}) \cap E \neq \emptyset)$$

and let

$$\hat{\pi} = \max_E\{\pi_E\}.$$

Then

$$\Pr(\exists E:\ v \notin E, E \cap E_t \neq \emptyset, t_{\Delta_0}+1 \le t \le t_\infty) \le \binom{n-1}{r}\hat{\pi}^{n^{\Delta_0/7}}. \tag{28}$$

(Without replacement there are fewer choices that will meet E).

Now fix E and let $a_i = |E \cap E_i|$, $i = 1, 2, \ldots, t$ where $a_1 \ge a_2 \ge \cdots \ge a_t$. Also let $s = \max\{j : a_j \ge r/3\}$ and note that $s \in \{0, 1, 2, 3\}$. Then if $\Omega_E = \{Y \in \Omega :\ Y \cap E = \emptyset\}$,

$$\begin{aligned}
|\Omega_E| &\ge \left(\prod_{j=s+1}^{t}(r-a_j-j)\right)(n-2r)_{r-t}((r-t)!)^{-1} \\
&\ge (1-o(1))r^{t-s}n^{r-t}((r-t)!)^{-1}\exp\left\{-2r^{-1}\sum a_j\right\} \\
&\ge (1-o(1))e^{-2}r^{t-s}n^{r-t}((r-t)!)^{-1} \\
&\ge \frac{|\Omega|}{10r^s}
\end{aligned}$$

Thus,

$$\pi_E \le 1 - \frac{1}{10r^3}.$$

It follows from (28) that

$$\Pr(\exists E:\ v \notin E, E \cap E_t \neq \emptyset, t_{\Delta_0}+1 \le t \le t_\infty) \le \binom{n-1}{r-1}\left(1-\frac{1}{10r^3}\right)^{n^{\Delta_0/7}} = o(1)$$

and this together with Claim 4 finishes the proof of Theorem 2 (except for (2)) since now we see that **whp** every edge chosen from time t_∞ onwards will contain v.

To prove (2) we re-label the edges of $\mathcal{H}_{t_4}$ as $E_1, E_2, \ldots, E_{t_4}$ so that $v \in E_{t_4-3}, E_{t_4-2}, E_{t_4-1}, E_{t_4}$. Recall $\tau_4 = t_4 - 4$. Then for $S \subseteq [\tau_4]$ we let

$$\mathcal{E}_S = \left\{F \in \binom{[n]\setminus\{v\}}{r-1}\ :\ F \cap E_j = \emptyset, j \in S\right\}.$$

Since $\mathcal{H}$ is simple, if $|S| = i$ then we have $|\mathcal{E}_S| = \binom{n-ri+i(i-1)/2}{r-1}$ and (2) follows directly from the inclusion-exclusion formula.

To obtain (3) we use (16) and (19) with $t = t_4$ and $t_\Delta = \tau_4$ (after observing that **whp** from t_4 on, all the edges added contain v). We have two asymptotic expressions:

$$\frac{r^{\tau_4}n^{r-\tau_4-1}}{(r-\tau_4-1)!}$$

and

$$\left(\frac{r^2}{n}\right)^{t_4-4}\binom{n-1}{r-1} = \left(\frac{r^2}{n}\right)^{\tau_4}\binom{n-1}{r-1} \sim \frac{r^{2\tau_4}n^{r-1}}{n^{\tau_4}(r-1)!} \qquad \text{using } r^2 = o(n).$$

Thus the ratio of these two expressions is asymptotically

$$\frac{(r-1)!}{r^{\tau_4}(r-\tau_4-1)!} = \frac{r^{\tau_4}}{(r-1)_{\tau_4}} \sim 1$$

because $\tau_4^2/r = O(r/n^{2/3}) = o(1)$. □

3 Functionals

3.1 Graph functionals

We introduce a certain type of graph functional. Here G is a graph with s vertices and f edges. $\mathcal{M}$ denotes set of graphs which are matchings.

Proposition 7 *Let t be a positive integer and x and y be nonnegative quantities such that $x = o(1)$, $y = \omega(1)$ and $t^2xy^2 = O(1)$. Then,*

$$\sum_{\emptyset \neq G \notin \mathcal{M}} x^f y^s = O(t^4x^2y^3) \tag{29}$$

$$\sum_{G \in \mathcal{M}, f \geq 2} x^f y^s = O(t^4x^2y^4) \tag{30}$$

$$\sum_{G \in \mathcal{M}, f = 1} x^f y^s = \binom{t}{2}xy^2. \tag{31}$$

Proof Let $n_t(f,s)$ count the number of subgraphs of $K_{[t]}$ that have exactly f edges and exactly $t-s$ isolated vertices. $n_t(f,s) \leq \binom{\binom{t}{2}}{f}$. Let $f_0 = \lceil (s+1)/2 \rceil$.

$$\begin{aligned}
\sum_{\emptyset \neq G \notin \mathcal{M}} x^f y^s &= \sum_{s=3}^{t} \sum_{f=f_0}^{\binom{s}{2}} n_t(f,s) x^f y^s \\
&\leq \sum_{s=3}^{t} \sum_{f=f_0}^{\binom{s}{2}} \frac{\binom{t}{2}^f}{f!} x^f y^s \\
&\leq \sum_{s=3}^{t} y^s \frac{(t^2x/2)^{f_0}}{f_0!} \sum_{f=0}^{\binom{s}{2}-f_0} \frac{\binom{t}{2}^f}{f!} x^f \\
&\leq \sum_{s=3}^{t} y^s \frac{(t^2x/2)^{f_0}}{f_0!} \sum_{f=0}^{\binom{s}{2}-f_0} \frac{1}{f!} \left(\frac{t^2x}{2}\right)^f \\
&\leq \exp\left\{\frac{t^2x}{2}\right\} \left(\sum_{t=0}^{\lfloor t/2 \rfloor -2} \frac{(t^2x/2)^{t+2} y^{2t+3}}{t!} + \sum_{t=0}^{\lfloor t/2 \rfloor -2} \frac{(t^2x/2)^{t+3} y^{2t+4}}{t!} \right) \\
&= (1+o(1))(t^4x^2y^3/4 + t^6x^3y^4/8) \sum_{t=0}^{\lfloor t/2 \rfloor -2} \frac{(t^2xy^2/2)^t}{t!} \\
&\leq t^4x^2y^3 \exp\left\{\frac{t^2xy^2}{2}\right\} \\
&= O(t^4x^2y^3).
\end{aligned}$$

In the case where F is a matching, $n_t(f,s) = \frac{(t)_{2f}}{2^f} \frac{1}{f!}$ because $s = 2f$. Thus

$$\begin{aligned}
\sum_{G \in \mathcal{M}, f \geq 2} x^f y^s &= \sum_{f=2}^{\lfloor t/2 \rfloor} \frac{(t)_{2f}}{2^f} \frac{1}{f!} x^f y^{2f} \\
&\leq \frac{1}{2} \left(\frac{t^2xy^2}{2}\right)^2 \exp\left\{\frac{t^2xy^2}{2}\right\}
\end{aligned}$$

and (30) follows. Equation (31) is clear. □

3.2 Grid functionals

Now we introduce a functional on a grid.

Proposition 8 *Suppose that τ is a positive integer and that x, y are positive reals such that*

$$x = o(1),\ y = \omega(1), \tau \Delta x y^2 = o(1).$$

Then

$$\sum_{\substack{\emptyset \neq H \subseteq [\tau]\times[\Delta] \\ u=\left|H|_{[\tau]}\right| \\ \ell=\left|H|_{[\Delta]}\right|}} x^h y^{u+\ell} = (1+o(1))\tau\Delta x y^2.$$

Proof Let $n(h,u,\ell)$ denote the number of sets $H \subseteq u \times \ell$ with $|H| = h$, $\left|H|_{[u]}\right| = u$ $\left|H|_{[\ell]}\right| = \ell$

$$\sum_{\substack{H \subseteq [\tau]\times[\Delta] \\ u=\left|H|_{[\tau]}\right| \\ \ell=\left|H|_{[\Delta]}\right|}} x^h y^{u+\ell} = \sum_{h,u,\ell} \binom{\tau}{u}\binom{\Delta}{\ell} n(h,u,\ell) x^h y^{u+\ell}$$

We use the bounds

$$n(h,u,\ell) \leq \begin{cases} \ell^u \binom{u\ell}{h-\ell}, & \text{if } u \geq \ell; \\ u^\ell \binom{\ell u}{h-u}, & \text{if } \ell \geq u. \end{cases}$$

Indeed, if $u \geq \ell$ then

$$\begin{aligned} n(h,u,\ell) &\leq \sum_{t=1}^{\ell} \binom{u}{t} \sum_{\substack{d_1+\cdots+d_t=\ell \\ d_1\geq 1,\ldots,d_t\geq 1}} \frac{\ell!}{d_1!\cdots d_t!} \ell^{u-t} \binom{u\ell-\ell-u+t}{h-\ell-u+t} \\ &\leq \sum_{t=1}^{\ell} \binom{u}{t} \ell^{u-t} t\ell \binom{u\ell-\ell-u+t}{h-\ell-u+t} \\ &\leq \sum_{t=1}^{\ell} \binom{u}{u-t} \ell^u \binom{u\ell-u}{h-\ell-u+t} \\ &\leq \ell^u \binom{u\ell}{h-\ell} \qquad \text{by the Vandermonde identity.} \end{aligned}$$

Similarly, if $\ell \geq u$ then $n(h,u,\ell) \geq u^\ell \binom{u\ell}{h-u}$.

Therefore, we can bound the summation by four other summations.

$$\sum_{\substack{\emptyset \neq H \subseteq [\tau]\times[\Delta] \\ u=\left|H|_{[\tau]}\right| \\ \ell=\left|H|_{[\Delta]}\right|}} x^h y^{u+\ell}$$

$$\leq \sum_{\ell \geq 1} \binom{\tau}{\ell}\binom{\Delta}{\ell} \ell^\ell \left(xy^2\right)^\ell \qquad Case:\ h=u=\ell \tag{32}$$

$$+ \sum_{\ell \geq 2}\sum_{h>\ell} \binom{\tau}{\ell}\binom{\Delta}{\ell} \ell^\ell \binom{\ell^2}{h-\ell} x^h y^{2\ell} \qquad Case:\ h>u=\ell \tag{33}$$

$$+ \sum_{\ell \geq 1}\sum_{u>\ell}\sum_{h\geq u} \binom{\tau}{u}\binom{\Delta}{\ell} \ell^u \binom{u\ell}{h-\ell} x^h y^{u+\ell} \qquad Case:\ h\geq u>\ell \tag{34}$$

$$+ \sum_{u \geq 1}\sum_{\ell>u}\sum_{h\geq \ell} \binom{\tau}{u}\binom{\Delta}{\ell} u^\ell \binom{\ell u}{h-u} x^h y^{u+\ell} \qquad Case:\ h\geq \ell>u \tag{35}$$

Consecutive terms in (32) increase by a factor of size $O(\tau\Delta xy^2) = o(1)$ and so the sum is dominated by the first term i.e.

$$RHS(32) = (1+o(1))\tau\Delta xy^2.$$

Let $m = \min\{\Delta, \tau\}$.

$$\begin{aligned} RHS(33) &\leq \sum_{\ell=1}^{m} \binom{\tau}{\ell}\binom{\Delta}{\ell} \ell^\ell y^{2\ell} \sum_{h'\geq 0} \binom{\ell^2}{h'+1} x^{h'+\ell+1} \\ &\leq (1+o(1)) \sum_{\ell=1}^{m} \binom{\tau}{\ell}\binom{\Delta}{\ell} \left(\ell xy^2\right)^\ell \ell^2 x e^{\ell^2 x} \\ &= (1+o(1))\tau\Delta x^2 y^2. \end{aligned}$$

since successive terms increase by a factor of size $O(\tau\Delta xy^2)$.

Now we bound summation (34).

$$\begin{aligned} RHS(34) &\leq \sum_{\ell\geq 1}\sum_{u>\ell}\sum_{h\geq u} \binom{\Delta}{\ell}\binom{\tau}{u} \ell^u \binom{u\ell}{h-u} x^h y^{u+\ell} \\ &\leq \sum_{\ell\geq 1}\sum_{u>\ell} \binom{\Delta}{\ell}\binom{\tau}{u} (\ell xy^2)^u \sum_{h'\geq 0} \binom{u\ell}{h'} x^{h'} \\ &\leq (1+o(1)) \sum_{\ell\geq 1}\sum_{u>\ell} \binom{\Delta}{\ell}\binom{\tau}{u} (\ell xy^2)^u \\ &= (1+o(1)) \sum_{\ell\geq 1} \binom{\Delta}{\ell}\binom{\tau}{\ell+1} (\ell xy^2)^{\ell+1} \\ &\leq (1+o(1))\tau^2\Delta x^2 y^4 \end{aligned}$$

By symmetry, (35) is also bounded by $(1+o(1))\tau\Delta^2 x^2 y^4$.

The sum of (32), (33), (34) and (35) is therefore $(1+o(1))\tau\Delta xy^2$. $\square$

4 Remarks

The thresholds, $r = \Theta(n^{1/3})$ and $r = \Theta(n^{5/12})$ each present their own unique difficulties. When $r = cn^{5/12}$ there is a probability, $p_1 = p_1(c) > 0$ that the first intersection of size greater than 1 will occur before a degree three vertex. In our notation, $t_\cap < t_3$. Our analysis will not work if this occurs.

The threshold $r = \Theta(n^{1/3})$ presents a different problem. When the first degree three vertex emerges, it may not be unique. For example, if $r = cn^{1/3}$, then there is a probability $p_2 = p_2(c) > 0$ that $\{e_1, e_2, e_3, e_4\}$ form a simple hypergraph but e_5 contains both $e_1 \cap e_2$ and $e_3 \cap e_4$. What makes this case even more difficult is that t_4 need not be $t_3 + 1$. Recall that we proved that when $\omega(n^{1/3}) = r = o(n^{2/5})$ then $t_4 = t_3 + 1$. At $r = \Theta(n^{1/3})$ this is not the case. For example, if $r = cn^{1/3}$, there exists a probability $p_3 = p_3(c) > 0$ such that all of the following occurs: The edges $\{e_1, \ldots, e_6\}$ form a simple hypergraph. The edge e_7 contains intersection points $e_1 \cap e_2$ and $e_3 \cap e_4$ and no others. The edge e_8 contains intersection points $e_1 \cap e_2 \cap e_7$ and $e_5 \cap e_6$ but no others. Then, e_9 contains intersection points $e_3 \cap e_4 \cap e_7$ and $e_5 \cap e_6 \cap e_8$ but no others. So, $t_4 \geq t_3 + 2$ for this example.

In fact, there are numerous outcomes that can occur with nonzero probability after only $O(1)$ edges when $r = \Theta(n^{1/3})$. The simplicity of Theorem 2 is, therefore, all the more remarkable.

A subset of the authors of this paper intend to work further to describe the hypergraph that results when r is a threshold value as well as proceed to the case where $r = \omega(n^{5/12})$. We believe that precise structural results such as Theorems 1 and 2 are impossible, but the size of the hypergraph may be able to be described.

5 Thanks

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