On packing Hamilton Cycles in ϵ-regular Graphs

Alan Frieze[*] Michael Krivelevich[†]

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Abstract

A graph $G = (V, E)$ on n vertices is (α, ϵ)-regular if its minimal degree is at least αn, and for every pair of disjoint subsets $S, T \subset V$ of cardinalities at least ϵn, the number of edges $e(S, T)$ between S and T satisfies: $\left|\frac{e(S,T)}{|S|\,|T|} - \alpha\right| \leq \epsilon$. We prove that if $\alpha \gg \epsilon > 0$ are not too small, then every (α, ϵ)-regular graph on n vertices contains a family of $(\alpha/2 - O(\epsilon))n$ edge-disjoint Hamilton cycles. As a consequence we derive that for every constant $0 < p < 1$, with high probability in the random graph $G(n, p)$, almost all edges can be packed into edge-disjoint Hamilton cycles. A similar result is proven for the directed case.

Key-words ϵ-Regular Graphs, Hamilton Cycles.

1 Introduction

Hamiltonicity (see a recent survey of Gould [8]) is undoubtedly one of the most important topics in modern Graph Theory. There are great many papers devoted to finding sufficient conditions for a graph to be Hamilton.

In this paper we address a closely related question: how many edge disjoint Hamilton cycles can be found in a graph? Here, too, there have been quite a few results. For example,

[*]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA15213, U.S.A. Supported in part by NSF grant CCR-0200945.

[†]Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel. E-mail: krivelev@post.tau.ac.il. Research supported in part by USA-Israel BSF Grants 99-0013 and 2002-133, by grant 64/01 from the Israel Science Foundation, and by a Bergmann Memorial Grant.

Nash-Williams proved already in 1971 [15] that the Dirac condition for a graph G on n vertices (i.e., the assumption that all vertex degrees in G are at least $n/2$) guarantees the existence of a family of at least $\lfloor 5n/224 \rfloor$ edge-disjoint Hamilton cycles.

Obviously, if $\delta = \delta(G)$ is the minimum degree of a graph G, then G contains at most $\lfloor \delta/2 \rfloor$ edge-disjoint Hamilton cycles. Motivated by this observation, we denote by $\mathcal{H}_\delta$ the property of having $\lfloor \delta/2 \rfloor$ edge disjoint Hamilton cycles plus an edge disjoint matching of size $\lfloor n/2 \rfloor$ if δ is odd. As it turns out, in some probability spaces of random graphs one can prove that property $\mathcal{H}_\delta$ holds with high probability, or **whp**. For example, Bollobás and Frieze [2] proved this for the probability space $G(n,m)$ of labeled graphs on n vertices with m edges and with uniform probability:

Theorem 1 *Let $m = \frac{n}{2}(\ln n + k \ln\ln n + \omega)$ where k is constant and $\omega \to \infty$ with n. Then* **whp** *$G(n,m)$ contains $\lfloor k/2 \rfloor$ edge disjoint Hamilton cycles plus an edge disjoint matching of size $\lfloor n/2 \rfloor$ if k is odd.*

This result is best possible in the sense that if $\omega = o(\ln\ln n)$ then **whp** $G(n,m)$ has minimum degree k. We conjecture though that the above result can be extended to *all* values of $m = m(n)$:

Conjecture 1 **Whp** *$G(n,m)$ has property $\mathcal{H}_\delta$ for any $1 \le m \le \binom{n}{2}$.*

It is likely that the following slightly stronger conjecture is also true.

Conjecture 2 *Consider the graph process where $e_1, e_2, \ldots, e_N, N = \binom{n}{2}$ is a random permutation of the edges of K_n. Let $G_m = ([n], \{e_1, e_2, \ldots, e_m\})$. Then* **whp** *every graph in the sequence G_m, $1 \le m \le N$ has property $\mathcal{H}_\delta$.*

Note that Bollobás and Frieze proved that **whp** $G(n,m)$ has property $\mathcal{H}_\delta$ as long as $1 \le m \le \frac{n}{2}(\ln n + O(\ln\ln n))$.

As another example consider the probability space $G_{n,r}$ of all r-regular graphs on n vertices (nr is assumed to be even). There Kim and Wormald proved recently [10] that for a constant $r \ge 3$ property $\mathcal{H}_\delta$ holds **whp** in $G_{n,r}$.

Conjecture 1 appears to be quite hard for the case $m \gg n \log n$, where **whp** in $G(n,m)$ all degrees are almost equal. One can thus ask a weaker question of packing *almost all* edges of a graph into edge-disjoint Hamilton cycles. In this paper we resolve this question for a class of dense graphs. Let $0 < \alpha < 1$ be constant and suppose that

$$10\left(\frac{\ln n}{n}\right)^{1/6} \le \epsilon \ll \alpha \tag{1}$$

Let $\mathcal{G}_{n,\alpha,\epsilon}$ denote the set of graphs G on vertex set $[n]$ which have the following properties:

P1 $\delta(G) \geq \alpha n$.

P2 If S, T are disjoint subsets of $[n]$ and $|S|, |T| \geq \epsilon n$ then $\left|\frac{e_G(S,T)}{|S|\,|T|} - \alpha\right| \leq \epsilon$, where $e_G(S,T)$ is the number of $S - T$ edges in G.

Graphs of this sort are sometimes also called *pseudo-random* as their edge distribution approaches closely that of the random graph $G(n,m)$ with $m = \alpha n^2/2$ edges. (See [11] for a survey on pseudo-random graphs.) Hamiltonian properties of pseudo-random graphs have been considered in [17], [6], [7].

Our main result is the following theorem.

Theorem 2 *Suppose that α is constant and* (1) *holds. If $G \in \mathcal{G}_{n,\alpha,\epsilon}$ then G contains $(\frac{\alpha}{2} - 3\epsilon)n$ edge disjoint Hamilton cycles.*

This result improves an estimate of Thomason [17], who proved that (α, ϵ)-regular graphs on n vertices with $\alpha \gg \epsilon > 0$ contain a linear in n number of edge-disjoint cycles, but his constant is substantially less than $\alpha/2$ even for very small ϵ.

If $0 < p < 1$ is constant then **whp** the random graph $G_{n,p}$ is in $\mathcal{G}_{n,p',\epsilon}$ where $p' = p - O(n^{-1/2}\ln^{1/2} n)$ and $\epsilon = O(n^{-1/3}\ln^{1/3} n)$. We therefore have

Corollary 1 *Assume that $0 < p < 1$ is constant. Then* **whp** *$G_{n,p}$ contains $np/2 - O(n^{5/6}\ln^{1/6} n)$ edge disjoint Hamilton cycles.*

The constraint (1) prevents us from claiming $np/2 - O(n^{2/3}\ln^{1/3} n)$ Hamilton cycles, as one might think at first glance.

We can also prove a bipartite version of Theorem 2. Let $\mathcal{B}_{n,\alpha,\epsilon}$ denote the set of bipartite graphs G with vertex partition $V_1 = [n]$, $V_2 = [n]$ which have the following properties:

P1 $\delta(G) \geq \alpha n$.

P2 If $S \subseteq V_1, T \subseteq V_2$ and $|S|, |T| \geq \epsilon n$ then $\left|\frac{e_G(S,T)}{|S|\,|T|} - \alpha\right| \leq \epsilon$.

Theorem 3 *Suppose that α is constant and* (1) *holds. If $G \in \mathcal{B}_{n,\alpha,\epsilon}$ then G contains $(\frac{\alpha}{2} - 3\epsilon)n$ edge disjoint Hamilton cycles.*

The above results can be extended to digraphs, although our bound on the error term is weaker. Note the $10\epsilon^{1/2}$ in place of 3ϵ. Let $\mathcal{D}_{n,\alpha,\epsilon}$ denote the set of digraphs D on vertex set $[n]$ which have the following properties.:

R1 $\min\{\delta^+(D), \delta^-(D)\} \geq \alpha n$.

R2 If S, T are subsets of $[n]$ and $|S|, |T| \geq \epsilon n$ then $\left|\frac{e_D(S,T)}{|S|\,|T|} - \alpha\right| \leq \epsilon$, where $e_D(S,T)$ is the number of $S \to T$ arcs in D.

Theorem 4 *Suppose that α is constant and (1) holds. If $D \in \mathcal{D}_{n,\alpha,\epsilon}$ then G contains $(\alpha - 4\epsilon^{1/2})n$ edge disjoint directed Hamilton cycles.*

Remark 1 *As pointed out by the referee, the directed version (Theorem 4) can be used to prove that any $G \in \mathcal{G}_{n,2\alpha,\epsilon}$ contains $(\alpha/2 - o(1))n$ edge-disjoint Hamilton cycles, whenever $0 < \alpha < 0.5$ is a constant and $\epsilon = o(1)$. Indeed, orienting the edges of such G randomly one gets* **whp** *a random digraph $D \in \mathcal{D}_{n,\alpha',\epsilon'}$ with $\alpha' = \alpha - o(1)$ and $\epsilon' = o(1)$. However, this approach would result in less accurate estimates in the error term of Theorem 2 and Corollary 1.*

Lu [12, 13, 14] considered the following Maker-Breaker game. Maker and Breaker take turns choosing edges from the complete graph K_n. Maker aims to construct as many edge disjoint Hamilton cycles as possible. Lu conjectured that Maker could construct $\sim n/4$ cycles. Using the results of this paper we confirm this and related conjectures in [9].

2 Proof of Theorem 2

Outline of proof

We first choose a random subgraph Γ of G with edge density $5\epsilon/2$. Let $G_1 = G - \Gamma$. We then show that G_1 has an r-factor F, where $r = 2s = \lfloor(\alpha - 4\epsilon)n\rfloor$ is assumed to be even. We then extract $\tau = s - \lfloor\epsilon n\rfloor$ edge-disjoint 2-factors $F_1, F_2, \ldots, F_\tau$, where each F_i has $O(\epsilon^{-1}(n \ln n)^{1/2})$ cycles. Then, for $i = 1, 2, \ldots, \tau$ we convert F_i into a Hamilton cycle H_i, using the edges of $G \setminus (H_1 \cup \ldots \cup H_{i-1} \cup F_i \cup \ldots \cup F_\tau)$.

Assume from now on that $G \in \mathcal{G}_{n,\alpha,\epsilon}$. Let Γ be obtained from G by independently including each edge with probability $\frac{5\epsilon}{2\alpha}$.

Lemma 1 Whp

- $\delta(\Gamma) \geq 2\epsilon n$ *and for disjoint S, T with $|S|, |T| \geq \epsilon n$ we have $e_\Gamma(S,T) \geq \epsilon|S|\,|T|$.*
- $\delta(G_1) \geq (\alpha - 3\epsilon)n$ *and for disjoint S, T with $|S|, |T| \geq \epsilon n$ we have $e_{G_1}(S,T) \leq (\alpha - \epsilon)|S|\,|T|$.*

Proof Vertex degree dominates $Bin(\alpha n, 5\epsilon/2\alpha)$ in Γ and $Bin(\alpha n, 1-5\epsilon/2\alpha)$ in G_1 and so a Chernoff bound implies $\delta(\Gamma) \geq 2\epsilon n$ and $\delta(G_1) \geq (\alpha - 3\epsilon)n$ **whp**.

If $|S|, |T| \geq \epsilon n$ then in G_1 we have $e_{G_1}(S,T)$ dominated by $Bin((\alpha+\epsilon)|S|\,|T|, (1-5\epsilon/2\alpha))$ and a Chernoff bound gives the answer since there are less than 4^n choices for S, T. A similar argument works for Γ. □

So assume from now on that the conditions of Lemma 1 hold.

2.1 G_1 has an r-factor

This is a fairly simple task using a theorem of Tutte [18]: Let S, T, U be a partition of $[n]$. Then let

$$R(S,T) = \sum_{v \in T} d(v) - e_{G_1}(S,T) + r(|S| - |T|)\,,$$

where $d(v)$ is the degree of v in G_1.

Let $Q(S,T)$ be the number of *odd* components of the graph G_U induced by U. A component C of G_U is odd if $r|C| + e_{G_1}(C,T)$ is odd.

Theorem 5 *G_1 contains an r-factor iff for every partition of $[n]$ into S, T, U we have $R(S,T) \geq Q(S,T)$.*

Let us apply this theorem to G_1 and let S, T, U be a partition of $[n]$. Then

$$R(S,T) \geq (\alpha - 4\epsilon)n|S| + \epsilon n|T| - e_{G_1}(S,T) - ||S| - |T||\,, \tag{2}$$

where $||S| - |T||$ accounts for rounding.

Case 1: $|S|, |T| \geq \epsilon n$.

Then from (2) and from the second condition of Lemma 1 we see that

$$R(S,T) \geq |S|((\alpha - 4\epsilon)n - (\alpha - \epsilon)|T|) + \epsilon n|T| - ||S| - |T||\,.$$

If $|T| \leq (1 - \frac{3\epsilon}{\alpha - \epsilon})n$ then $R(S,T) \geq \epsilon n|T| - n \geq \epsilon^2 n^2 - n \gg n$ and $Q(S,T) \leq |U| < n$.

If $|T| > (1 - \frac{3\epsilon}{\alpha-\epsilon})n$ then $|S| < \frac{3\epsilon}{\alpha-\epsilon}n$ and $R(S,T) \geq \epsilon n|T| - n - 3\epsilon n|S| \geq \epsilon n \left(1 - \frac{3\epsilon}{\alpha-\epsilon}\right) n - n - \frac{9\epsilon^2}{\alpha-\epsilon}n^2 \gg n > |U|$.

Case 2: $\frac{2}{\alpha - 4\epsilon} \leq |S| < \epsilon n$.

Now $e_{G_1}(S,T) < \epsilon n|T|$ and from (2) we see that $R(S,T) > (\alpha - 4\epsilon)n|S| - n \geq 2n - n = n \geq |U|$.

Case 3: $|S| < \frac{2}{\alpha - 4\epsilon}$ and $|T| \geq (\alpha - 4\epsilon)n$.

Then (2) implies $R(S,T) > \epsilon n|T| - |S||T| - n \gg n$.

Case 4: $|S| < \frac{2}{\alpha - 4\epsilon}$ and $\frac{2}{\epsilon} \leq |T| < (\alpha - 4\epsilon)n$.

The $e_{G_1}(S,T) \leq (\alpha - 4\epsilon)n|S|$ and so (2) implies $R(S,T) \geq \epsilon n|T| - n \geq n \geq |U|$.

Case 5: $|S| < \frac{2}{\alpha - 4\epsilon}$ and $|T| < \frac{2}{\epsilon}$.

Now every component of U has size at least $\delta(G_1) - |S| - |T| \geq (\alpha - 3\epsilon)n - |S| - |T|$ and so there are at most $\frac{n}{(\alpha-3\epsilon)n - |S| - |T|} \leq \frac{2}{\alpha}$ components. Now either $T = \emptyset$ and $R(S,T) \geq 0$ while $Q(S,T) = 0$ because r is even, or $R(S,T) \geq \delta(G_1) - |S| \cdot |T| - \max\{|S|, |T|\} \geq (\alpha - 3\epsilon)n - O(1) > 2/\alpha$.

This completes the proof that G_1 contains an r-factor which we denote by F.

2.2 Extracting 2-factors

Petersen [16] showed that every $2s$-regular graph contains a 2-factor and so F can be decomposed into the disjoint union of 2-factors. We need however to bound the number of cycles in our 2-factors. To this end, we use well-known bounds on the permanent in a way similar to that of [1] by Alon.

Lemma 2 *Let H be a $2d$-regular graph on vertex set $[n]$, where $d \geq \epsilon n$. Then H contains a 2-factor with at most $10\epsilon^{-1}(n \ln n)^{1/2}$ cycles.*

Proof Suppose that H is a $2d$-regular graph on vertex set $[n]$. Orient the edges of H so that every vertex of $\vec{H}$ has in-degree=out-degree d. Now consider the d-regular bipartite graph B on vertex set $[n] + [n]$ where (x,y) is an edge of B iff (x,y) is an arc of $\vec{H}$. Every perfect matching M of B yields a collection C_M of vertex disjoint oriented cycles in $\vec{H}$ which cover all the vertices $[n]$. Each cycle is of length at least 3 since B does not contain edges (x,x) and at most one of $(x,y), (y,x)$ can be an edge of B. Thus ignoring orientation gives a 2-factor of H and distinct matchings give distinct 2-factors. (Note that this does not necessarily account for all 2-factors of H.)

Now let X denote the number of perfect matchings of B. It equals the permanent of the adjacency matrix A_B of B. Then

$$X \geq \left(\frac{d}{n}\right)^n n!. \tag{3}$$

This follows from the proof of Van der Waerden's conjecture [5], [4]: The Van der Waerden conjecture being that the permanent of a non-negative matrix with all row and column sums equal to 1 is at least $n!/n^n$. We apply this theorem to $d^{-1}A_B$.

Next let $X_{k,\ell}$ be the number of perfect matchings M of B such that C_M contains at least k cycles of length ℓ. Then

$$X_{k,\ell} \leq \binom{n}{k} d^{k(\ell-1)} \ell^{-k} \left(\frac{(n-2k\ell)d + k^2\ell^2}{n-k\ell} \right)^{n-k\ell} e^{-(n-k\ell)} (3n)^{10n/d} \tag{4}$$

Explanation of (4): We choose one vertex for each of k cycles $C_1, C_2, \ldots, C_k$ in $\binom{n}{k}$ ways. Then starting with one of these vertices, we can choose a sequence of $\ell - 1$ vertices to make a cycle in at most $d^{\ell-1}$ ways. Each collection of cycles is produced ℓ^k times by this construction, which expalins the factor ℓ^{-k}. If we remove the vertices of $C_1, C_2, \ldots, C_k$ from H then we remove $2k\ell$ vertices from B, $k\ell$ vertices from each side. The remaining bipartite sub-graph B' has $n - k\ell$ vertices on each side and at most $(n-2k\ell)d + k^2\ell^2$ edges. We will use Bregman's solution of the Minc conjecture [3] to show that

$$B' \text{ has at most } \left(\frac{(n-2k\ell)d + k^2\ell^2}{n-k\ell} \right)^{n-k\ell} e^{-(n-k\ell)} (3n)^{10n/d} \text{ perfect matchings,} \tag{5}$$

completing the explanation of (4).

Assume the truth of (5) for the moment. Estimating

$$\left(\frac{(n-2k\ell)d + k^2\ell^2}{n-k\ell} \right)^{n-k\ell} \leq d^{n-k\ell} \exp\left\{ -k\ell + \frac{k^2\ell^2}{d} \right\}$$

we get

$$\begin{aligned} X^{-1} X_{k,\ell} &\leq \left(\frac{ne}{k\ell d} \exp\left\{ \frac{k\ell^2}{d} \right\} \right)^k (3n)^{10n/d} \\ &\leq \left(\frac{e}{k\ell\epsilon} \exp\left\{ \frac{k\ell^2}{\epsilon n} \right\} \right)^k (3n)^{10/\epsilon} \end{aligned}$$

Now put $k = 20\epsilon^{-1} \ln n$ and assume $\ell \leq \ell_0 = \frac{\epsilon n^{1/2}}{(20 \ln n)^{1/2}}$. Then

$$X^{-1} X_{k,\ell} \leq (\ln n)^{-10\epsilon^{-1} \ln n}$$

and

$$X^{-1} \sum_{\ell=3}^{\ell_0} X_{k,\ell} < 1.$$

Consequently, H contains at least one 2-factor with at most $k\ell_0 + \frac{n}{\ell_0}$ cycles, giving the lemma.

We complete the proof of the lemma by verifying (5). Set $\nu = n - k\ell$ and let the degrees on one side of B' be $d_1, d_2, \ldots, d_\nu$. It follows from [3] that the number of perfect matchings $\mu(B')$ in B' satisfies

$$\mu(B') \leq \prod_{i=1}^{\nu} (d_i!)^{1/d_i}. \tag{6}$$

Indeed the RHS of (6) is Minc's conjectured upper bound on the permanent of an $n \times n$ 0-1 matrix with row sums $d_1, d_2, \ldots, d_\nu$.

We will argue later that we can restrict our attention to the case where

$$d_i \geq \frac{d}{10} \qquad i = 1, 2, \ldots, \nu. \tag{7}$$

Using Stirling's formula we then obtain

$$\mu(B') \leq A \prod_{i=1}^{\nu} e^{-1} d_i \leq A e^{-\nu} \left(\nu^{-1} \sum_{i=1}^{\nu} d_i \right)^{\nu}$$

where

$$A \leq \prod_{i=1}^{\nu} (3d_i)^{1/2d_i} \leq (3n)^{10n/d},$$

completing the proof of (5).

We show now that (7) is justified. We can assume that B' has $(n - 2k\ell)d + k^2\ell^2$ edges. Since $k\ell \leq (20n \ln n)^{1/2}$ we see that the average degree in B' is at least $d/2$. Suppose that for example, $d_1 = a < d/10$. Then we can assume that $d_2 = b \geq d/2$. Then

$$\left(\frac{a!^{1/a} b!^{1/b}}{(a+1)!^{1/(a+1)} (b-1)!^{1/(b-1)}} \right)^{ab(a+1)(b-1)} = \frac{(a+1)!^{b(b-1)}}{(a+1)^{(a+1)b(b-1)}} \cdot \frac{b^{a(a+1)b}}{b!^{a(a+1)}}. \tag{8}$$

Using Stirling's formula, the logarithm of the RHS of (8) is at most

$$a(a+1)b - b(b-1)(a + 1 - \ln 3 - \frac{1}{2} \ln(a+1)) < 0.$$

So, given our lower bound on the number of edges in B', the RHS of (6) is maximised by a degree sequence satisfying (7). □

Remark 2 *If all one wants is an upper bound of $o(n)$ cycles then one need not work as hard as we did in the above lemma. This will suffice if we only wish to assume that ϵ is a positive constant independent of n. But then we could not make the statement of Corollary 1.*

Thus starting with F we can pull out edge-disjoint 2-factors $F_1, F_2, \ldots, F_\tau$ each containing at most

$$s_0 = 10\epsilon^{-1}(n \ln n)^{1/2}$$

cycles.

2.3 Transforming 2-factors to Hamilton cycles

Assume inductively that for some $i \geq 0$ we have created edge-disjoint Hamilton cycles $H_1, H_2, \ldots, H_i$ which are edge-disjoint from $F_{i+1}, \ldots, F_\tau$. Assume further that $|H_j \setminus F_j| \leq 3s_0$ for $1 \leq j \leq i$.

Next let $\Gamma_1 = G \setminus (H_1 \cup \ldots \cup H_i \cup F_{i+1} \ldots \cup F_\tau)$. Then

Q1 $\delta(\Gamma_1) \geq \delta(\Gamma) \geq 2\epsilon n$.

Q2 If S, T are disjoint subsets of $[n]$ and $|S|, |T| \geq \epsilon n$ then $e_{\Gamma_1}(S,T) \geq e_\Gamma(S,T) - 3s_0 n \geq \frac{\epsilon^3}{2} n^2$.

It follows immediately that Γ_1 is connected. Let $\Gamma_2 = \Gamma_1 \cup F_{i+1}$.

Remark 3 *It is* **Q2** *that forces the lower bound of* $n^{-1/6+o(1)}$ *on* ϵ.

Next suppose that F_{i+1} comprises cycles $C_1, C_2, \ldots, C_t$ where $t \leq s_0$. We systematically merge cycles.

General Step: Given the current 2-factor (initially F_{i+1}) choose an edge $e = (x,y)$ of Γ_2 which joins two distinct cycles C, C'. This is always possible because Γ_2 is connected. Let f be an edge of C incident with x and f' be an edge of C' incident with y. Let P be the path $C \cup C' \cup \{e\} \setminus \{f, f'\}$. There are now several possibilities.

(a): There is an endpoint u say, of P which has a neighbour v in a cycle C'' disjoint from P. We *extend* P by replacing P, C'' by $P \cup C'' \cup \{(u,v)\} \setminus f''$ where f'' is an edge of C'' incident with v. We repeat this operation as long as we can. We then carry out (b) or (c).

(b) The endpoints u, v of P are connected by an edge in Γ_2. Adding (u,v) to P creates a 2-factor with at least one less cycle than at the start of the General Step and completes it.

(c) Let $P = (u_1, u_2 \ldots, u_k)$. Let X be the set of neighbors of u_1 in $P \setminus \{u_2\}$, and let Y be the set of neighbors of u_k in $P \setminus \{u_{k-1}\}$. Then due to **Q1** both sets X and Y contain at least $2\epsilon n$ elements. We denote by X_1, resp. Y_1, the set of the first ϵn vertices of X, resp. Y along P, and by X_2, resp. Y_2, the set of the last ϵn vertices of X, resp. Y, along P.

Consider first the case in where all of the vertices in X_1 precede all of the vertices in Y_2. Denote by X_1' the set of vertices which are the predecessors of X_1 along P, and by Y_2' the set of vertices which are the successors of Y_2 along P. It follows from **Q2** that $e(X_1', Y_2') > 0$. Then for some $2 \leq i < j \leq k-1$ the graph Γ_2 contains edges (u_1, u_i), (u_{i-1}, u_{j+1}), (u_j, u_k). In this case we get a cycle $u_1 u_2 \ldots u_{i-1} u_{j+1} u_{j+2} \ldots u_k u_j u_{j-1} \ldots u_i u_1$ through the vertices of P.

Given that the above case fails, we find that all of the vertices in X_2 precede all of the vertices in Y_1. Let X_2' be the set of vertices which are successors of X_2 along P, and let Y_1'

be the set of vertices which are predecessors of Y_1 along P. Again, $e(X_2', Y_1') > 0$ due to **Q2**, and therefore for some $1 \leq j < i < k$ the graph Γ_2 contains edges (u_1, u_i), (u_{i+1}, u_{j+1}), (u_j, u_k). We can form a cycle $u_1 u_2 \dots u_j u_k u_{k-1} \dots u_{i+1} u_{j+1} u_{j+2} \dots u_i u_1$ through the vertices of P.

End of description of General Step.

Each general step reduces the number of cycles by at least one and we require at most three edges of Γ_1 per step to do this. Thus, after all general steps have been executed we obtain the next Hamilton cycle H_{i+1} for which $|H_{i+1} \setminus F_{i+1}| \leq 3s_0$. This completes the induction and the proof of Theorem 2.

3 Bipartite Case

This is very similar to the previous case. We will therefore try to be brief. First let Γ be obtained from G by independently including each edge with probability $\frac{5\epsilon}{2\alpha}$. The following lemma is proved the same way as Lemma 1.

Lemma 3 **Whp**

- $\delta(\Gamma) \geq 2\epsilon n$ *and for* $S \subseteq V_1, T \subseteq V_2$ *with* $|S|, |T| \geq \epsilon n$ *we have* $e_\Gamma(S, T) \geq \epsilon |S|\,|T|$.
- $\delta(G_1) \geq (\alpha - 3\epsilon)n$ *and for* $S \subseteq V_1, T \subseteq V_2$ *with* $|S|, |T| \geq \epsilon n$ *we have* $e_{G_1}(S, T) \leq (\alpha - \epsilon)|S|\,|T|$.

□

So assume from now on that the conditions of Lemma 3 hold. Let $r = 2s = \lfloor (\alpha - 4\epsilon)n \rfloor$.

3.1 G_1 has an r-factor

This is a fairly simple task using the max-flow min-cut theorem. We construct a network $\mathcal{N}$ by adding vertices s, t. We add an arc (s, v) of capacity r for each $v \in V_1$ and an arc (w, t) of capacity r for each $w \in V_2$. The edges of G_1 are given capacity 1. We only have to show that $\mathcal{N}$ admits an $s - t$ flow of value rn. So consider an $s - t$ cut $S : \bar{S}$ where $S = \{s\} \cup S_1 \cup S_2$ and $S_i \subseteq V_i$ for $i = 1, 2$. The capacity of this cut is

$$r(n - |S_1|) + e(S_1, \bar{S}_2) + r|S_2|$$

and we need therefore to show that

$$e(S_1, \bar{S}_2) \geq r(|S_1| - |S_2|). \tag{9}$$

Assume therefore that

$$|S_1| \geq |S_2|.$$

Case 1: $|S_1|, |\bar{S}_2| \geq \epsilon n$.

$$\begin{aligned} e(S_1, \bar{S}_2) &\geq (\alpha - \epsilon)|S_1|(n - |S_2|) \\ &\geq (\alpha - \epsilon)n(|S_1| - |S_2|) \end{aligned}$$

which implies (9).

Case 2: $|\bar{S}_2| < \epsilon n$.

$$\begin{aligned} e(S_1, \bar{S}_2) &\geq (\alpha - 3\epsilon)n|\bar{S}_2| - |\bar{S}_1||\bar{S}_2| \\ &\geq (\alpha - 3\epsilon)n|\bar{S}_2| - \epsilon n|\bar{S}_2| \\ &= (\alpha - 4\epsilon)n(n - |S_2|) \end{aligned}$$

which implies (9).

3.2 Extracting 2-factors

Lemma 2 is applicable (with n replaced by $2n$) and so we can extract $\tau = s - \lfloor \epsilon n \rfloor$ edge-disjoint 2-factors $F_1, F_2, \ldots, F_\tau$, where each F_i has $O(\epsilon^{-1}(n \ln n)^{1/2})$ cycles.

3.3 Transforming 2-factors to Hamilton cycles

$F_1, F_2, \ldots, F_\tau$ can be transformed into Hamilton cycles in much the same way as before. The only point to note is that the paths formed in the general steps are always of odd length and so can be completed to cycles with a single edge.

4 Proof of Theorem 4

Outline of proof

Let $\gamma = \epsilon^{1/2}/2$. We first choose a random subdigraph Γ of D with edge density $9\gamma/2$. Let $D_1 = D - \Gamma$. We then show that D_1 has an r-difactor F, where $r = \lfloor (\alpha - 6\gamma)n \rfloor$ is assumed to be even. (F is a regular subgraph of indegree=outdegree $=r$). We then extract $\tau = r - \lfloor \epsilon n \rfloor$ edge-disjoint 1-difactors $F_1, F_2, \ldots, F_\tau$, where each F_i has $O(\epsilon^{-1}(n \ln n)^{1/2})$ cycles. Then, for $i = 1, 2, \ldots, \tau$ we convert F_i into a directed Hamilton cycle H_i, using the arcs of $D \setminus (H_1 \cup \ldots \cup H_{i-1} \cup F_i \cup \ldots \cup F_\tau)$.

Assume from now on that $D \in \mathcal{D}_{n,\alpha,\epsilon}$. Let Γ be obtained from D by independently including each edge with probability $\frac{9\gamma}{2\alpha}$.

Lemma 4 **Whp**

- $\delta^+(\Gamma), \delta^-(\Gamma) > 4\gamma n$ *and for disjoint* S, T *with* $|S|, |T| \geq \epsilon n$ *we have* $e_\Gamma(S, T) \geq 4\gamma |S|\,|T|$.
- $\delta^+(D_1), \delta^-(D_1) \geq (\alpha - 5\gamma)n$ *and for disjoint* S, T *with* $|S|, |T| \geq \epsilon n$ *we have* $e_{D_1}(S, T) \geq (\alpha - 5\gamma)|S|\,|T|$.

Proof Similar to the proof of Lemma 1. □

Assume from now on that the conditions of Lemma 4 hold.

4.1 D_1 has an r-difactor

We show next that D_1 has an r-difactor. Let B be the bipartite graph associated with D_1. An r-difactor in D_1 corresponds to an r-regular subgraph of B. Let the vertex bipartition of B be V, W. We will use the max-flow min-cut theorem. We add vertices s, t and join s to every vertex of V by an edge of capacity r and also join every vertex of W to t by an edge of capacity r. Every edge of B has capacity 1 and we need to prove that this network $\mathcal{N}$ has a flow of capacity rn from s to t.

A cut of $\mathcal{N}$ can be defined by $S_1 \subseteq V$ and $S_2 \subseteq W$. The capacity $c(S_1, S_2)$ of this cut is given by

$$c(S_1, S_2) = r(n - |S_1|) + e(S_1 : \bar{S}_2) + r|S_2|$$

where $\bar{S}_2 = W \setminus S_2$.

We need to show that $c(S_1, S_2) \geq rn$ for all S_1, S_2. This is trivially true if $|S_1| \leq |S_2|$ and so assume $|S_1| > |S_2|$ from here on.

Case 1: $|S_1|, |\bar{S}_2| \geq \epsilon n$.

Then

$$\begin{aligned} c(S_1, S_2) &\geq r(n - |S_1|) + (\alpha - 5\gamma)|S_1|(n - |S_2|) + r|S_2| \\ &\geq r(n - |S_1|) + \frac{r}{n}|S_1|(n - |S_2|) + r|S_2| = rn - \frac{r}{n}|S_1||S_2| + r|S_2| \\ &\geq rn. \end{aligned}$$

Case 2: $|S_2| < |S_1| \leq \epsilon n$.

In this case we have $e(S_1, \bar{S}_2) \geq |S_1|(\alpha - 5\gamma)n - |S_1|S_2| \geq |S_1|(\alpha - (5\gamma + \epsilon))n$ and so

$$c(S_1, S_2) \geq r(n - |S_1|) + |S_1|(\alpha - (5\gamma + \epsilon))n + r|S_2| \geq rn.$$

This completes the proof that D_1 has an r-difactor.

4.2 Extracting 1-difactors

We can use the same argument as in Section 2.2 to show we can find τ edge-disjoint 1-difactors $F_1, F_2, \ldots, F_\tau$, where each F_i has at most s_0 cycles, with $s_0 = 10\epsilon^{-1}(n \ln n)^{1/2}$ as before.

4.3 Transforming 1-difactors to directed Hamilton cycles

Assume inductively that for some $i \geq 0$ we have created arc disjoint directed Hamilton cycles $H_1, H_2, \ldots, H_i$ which are arc-disjoint from $F_{i+1}, \ldots, F_\tau$. Assume further that $|H_j \setminus F_j| \leq 5s_0$ for $1 \leq j \leq i$.

Next let $\Gamma_1 = D \setminus (H_1 \cup \ldots H_i \cup F_{i+1} \cup \ldots \cup F_\tau)$. Then

Q1 $\min\{\delta^+(\Gamma_1), \delta^-(\Gamma_1)\} > \min\{\delta^+(\Gamma), \delta^-(\Gamma)\} > 4\gamma n$.

Q2 If S, T are disjoint subsets of $[n]$ and $|S|, |T| \geq \gamma n - 2$ then $e_{\Gamma_1}(S, T) \geq e_\Gamma(S, T) - 5ns_0 \geq 4\gamma|S|\,|T|$.

Q3 If S, T are disjoint subsets of $[n]$ and $|S|, |T| \geq \epsilon n$ then $e_{\Gamma_1}(S, T) \geq e_\Gamma(S, T) - 5ns_0 \geq 1$.

It follows immediately that Γ_1 is strongly connected. Let $\Gamma_2 = \Gamma_1 \cup F_{i+1}$.

Next suppose that F_{i+1} comprises cycles $C_1, C_2, \ldots, C_t$ where $t \leq s_0$. We systematically merge cycles.

General Step: Given the current 1-difactor (initially F_{i+1}) choose an arc $e = (x, y)$ of Γ_2 which joins 2 distinct cycles C, C'. This is always possible because Γ_2 is strongly connected. Let f be the arc of C directed from x and f' be the arc of C' directed to y. Let P be the directed path $C \cup C' \cup \{e\} \setminus \{f, f'\}$ and suppose that it is directed from vertex a to vertex b. There are now several possibilities.

(a): The endpoint b of P has an out-neighbour v in a cycle C'' disjoint from P. We *extend* P by replacing P, C'' by $P \cup C'' \cup \{(u, v)\} \setminus f''$ where f'' is the arc of C'' directed into v. We make a similar extension if endpoint a has an in-neighbour outside P. We repeat these operations as long as we can. We then carry out (b) or (c). At this point, P has length at least $2\gamma n$.

(b) The endpoints a, b of P are connected by an arc (b, a) in Γ_2. Adding (b, a) to P creates a 1-difactor with at least one less cycle than at the start of the General Step and completes it.

(c) Let $P = (a = u_1, u_2 \ldots, u_k = b)$. Let X be the set of in-neighbours of a in P and let Y be the set of out-neighbours b on P. It follows from **Q1** that $|X|, |Y| \geq 4\gamma n$.

Let X_1 be the first $2\gamma n$ vertices in X along P and let X_2 be the last $2\gamma n$ vertices in X along P and define Y_1, Y_2 similarly. There are 2 cases to consider:

(i) Each vertex of X_1 precedes each vertex of Y_2 along P.

Let $X_1' = \{u_j : j \geq \gamma n \text{ and } u_{j-1} \in X_1\}$ and $Y_2' = \{u_j : j \leq k - \gamma n \text{ and } u_{j+1} \in Y_2\}$ and note that $|X_1'|, |Y_2'| \geq \gamma n$. Next let $X_1'' = \{u_j : j < \gamma n \text{ and } \exists \text{arc } X_1' \to u_{j-1}\}$ and $Y_2'' = \{u_j : j > k - \gamma n \text{ and } \exists \text{arc } Y_2' \to u_{j+1}\}$. It follows from **Q2** that $|X_1''| \geq \frac{(4\gamma)(\gamma n)|X_1'|}{|X_1'|} \geq \epsilon n$ and similarly $|Y_2''| \geq \epsilon n$.

X_1'', Y_2'' are clearly disjoint and it now follows from **Q3** that there exist $x = u_r \in X_1''$, $y = u_s \in Y_2''$ such that (y, x) is an arc of Γ_1. We may then replace P by the cycle C: Here $u_\rho \in X_1'$ witnesses $u_r \in X_1''$ and $u_\sigma \in Y_2'$ witnesses $u_s \in Y_2''$.

$C = (y, x = u_r, u_{r+1}, \ldots, u_{\rho-1}, u_1, \ldots, u_{r-1}, u_\rho, u_{\rho+1}, \ldots, u_\sigma, u_{s+1}, \ldots, u_k, u_{\sigma+1}, \ldots, u_s = y)$.

(see Figure 1). This creates a 1-difactor with at least one less cycle than at the start of the General Step and completes it.

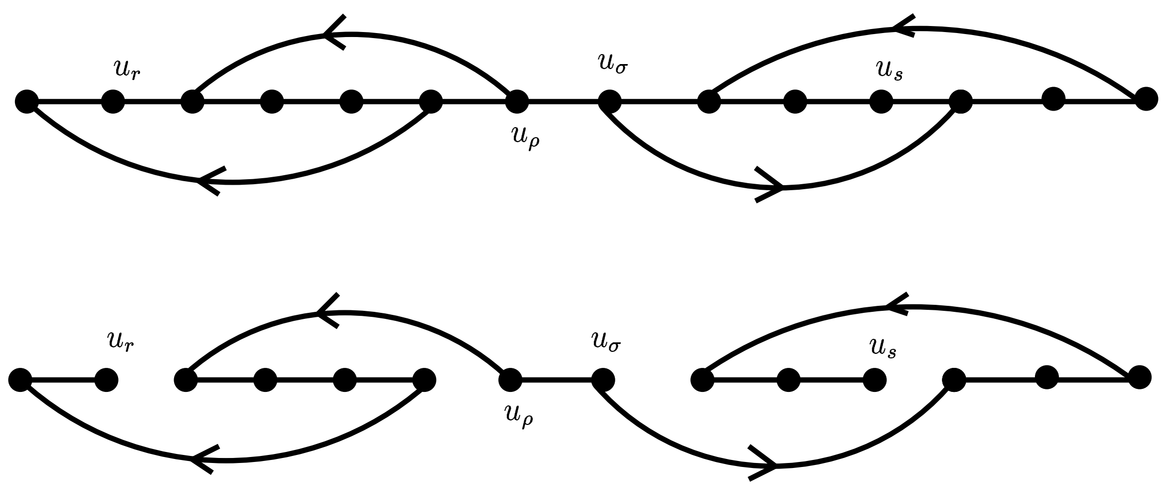

Figure 1:

(ii) Each vertex of Y_1 precedes each vertex of X_2 along P.

Let Y_1' denote the first γn members of Y_1 along P and let $j_0 = \max\{j : j \in Y_1\}$. Let $X_2' = \{u_j : j > j_0 + \gamma n \text{ and } u_{j-1} \in X_2\}$ and note that $|X_2'|, |Y_1'| \geq \gamma n$. Next let $X_2'' = \{u_j : j_0 < j < j_0 + \gamma n \text{ and } \exists \text{arc } u_{j-1} \to X_2'\}$ and $Y_1'' = \{u_j : j_0 - \gamma n < j < j_0 \text{ and } \exists \text{arc } Y_1' \to u_{j+1}\}$. It follows from **Q2** that $|X_2''| \geq \frac{(4\gamma)(\gamma n)|X_2'|}{|X_2'|} \geq \epsilon n$ and similarly $|Y_1''| \geq \epsilon n$.

It now follows from **Q3** that there exist $x = u_r \in X_2''$, $y = u_s \in Y_1''$ such that (y, x) is an arc of Γ_1. We may then replace P by the cycle C: Here $u_\rho \in X_2'$ witnesses $u_r \in X_2''$ and $u_\sigma \in Y_1'$ witnesses $u_s \in Y_1''$.

$$C = (y, x = u_r, u_{r+1}, \ldots, u_\rho, u_1, \ldots, u_{\sigma-1}, u_{s+1}, \ldots, u_{r-1}, u_{\rho+1}, \ldots, u_k, u_\sigma, \ldots, u_s = y).$$

(see Figure 2).

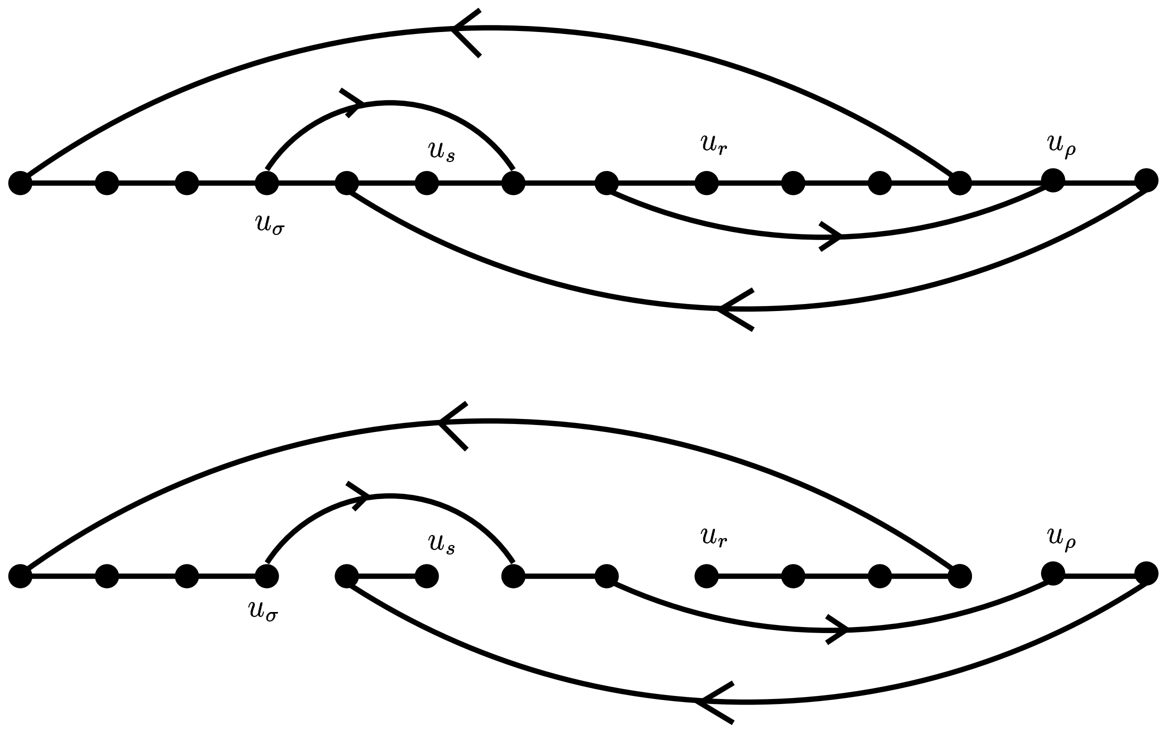

Figure 2:

End of description of General Step. Each general step reduces the number of cycles by at least one and we require at most five edges of Γ_1 per step to do this. Thus after all general steps have been completed we create a Hamilton cycle H_{i+1} for which $|H_{i+1} \setminus F_{i+1}| \leq 5s_0$. This completes the induction and the proof of Theorem 4.

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