Perfect matchings in random graphs with prescribed minimal degree

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Abstract

We consider the existence of perfect matchings in random graphs with n vertices (or $n+n$ vertices in the bipartite case) and m random edges, *subject to a lower bound on minimum vertex degree.* We establish tight thresholds in terms of the number of random edges. For example we show that a random bipartite graph with $n+n$ vertices and cn random edges, and minimum degree at least 2, has a perfect **whp**.

1 Introduction

To quote from Lovász [16], "the problem of the existence of 1- factors (perfect matchings), the solution of which (the Kőnig-Hall theorem for bipartite graphs and Tutte's theorem for the general case) is an outstanding result making this probably the most developed field of graph theory". Erdős and Rényi ([9],[10]) found a way to use these results for a surprisingly sharp study of existence of perfect matchings in random graphs. For $B_{n,m}$, a random bipartite graph with $n+n$ vertices and $m = n(\ln n + c_n)$ random edges, they proved [9] that

$$\begin{aligned}\lim_{n\to\infty} \mathbf{Pr}(B_{n,m} \text{ has a perfect matching}) &= \lim_{n\to\infty} \mathbf{Pr}(\delta(B_{n,m}) \geq 1) \\ &= \begin{cases} 0 & c_n \to -\infty, \\ e^{-2e^{-c}}, & c_n \to c, \\ 1 & c_n \to \infty, \end{cases}\end{aligned}$$

where δ denotes minimum degree. Of course minimum degree at least one is a trivial necessary condition for the existence of a perfect matching. The Hall theorem turned out to be perfectly tailored for use in combination with probabilistic techniques, pioneered by the authors several years earlier, [9]. Even though Tutte's theorem for the non-bipartite case is considerably more involved, in [10] they managed to extend the analysis to the random graph $G_{n,m}$, a random general graph with n vertices and $m = \frac{n}{2}(\ln n + c_n)$ edges, showing that

$$\begin{aligned}\lim_{\substack{n\to\infty \\ n \text{ even}}} \mathbf{Pr}(G_{n,m} \text{ has a perfect matching}) &= \lim_{n\to\infty} \mathbf{Pr}(\delta(G_{n,m}) \geq 1) \\ &= \begin{cases} 0 & c_n \to -\infty, \\ e^{-e^{-c}} & c_n \to c, \\ 1 & c_n \to \infty. \end{cases}\end{aligned}$$

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In both cases a perfect matching becomes likely as soon as one has sufficiently many random edges for the minimum degree to be at least one with high probability (**whp**). This has led researchers to consider the existence of perfect matchings in models of a random graph in which the minimum degree requirement is always satisfied. Perhaps the first result along these lines is due to Walkup [21]. He considered a κ-out model $B_{\kappa-\text{out}}$ of a random bipartite graph, again with $n+n$ vertices V_1+V_2. Here each vertex $v \in V_i$ "chooses" κ random neighbours in its complementary class V_{3-i}. Walkup showed that

$$\lim_{n\to\infty} \mathbf{Pr}(B_{\kappa-out} \text{ has a perfect matching}) = \begin{cases} 0 & \kappa = 1 \\ 1 & \kappa \geq 2 \end{cases}$$

Frieze [11] proved a non-bipartite version of this result, the argument being based on Tutte's theorem and considerably harder. Very recently Karoński and Pittel [13] have proven **whp** existence of a perfect matching in what they called the $B_{(1+e^{-1})-out}$ graph, a subgraph of B_{2-out}, obtained from B_{1-out} by letting each of its degree 1 vertices select another random neighbor in the complementary class. Observe that in all of these results [21], [11] and [13] the number of random edges depends linearly on the number of vertices, and the minimum degree has been raised to 2, in a sharp contrast with the case m being of order $n \log n$. Here is why. When there are order $n \ln n$ random edges, there are few vertices of degree 1 and they are far apart. In sparser models, with minimum degree 1, **whp** there will be a linear (in n) number of vertices of degree 1, and some two vertices of degree 1 will have a common neighbor, which rules out a perfect matching. In the case of random regular graphs it turns out that minimum degree 3 is required, Bollobás [3]: Let G_r denote a random r-regular graph on vertex set $[n]$, n even. Then

$$\lim_{n\to\infty} \mathbf{Pr}(G_r \text{ has a perfect matching}) = \begin{cases} 0 & r = 2, \\ 1 & r = 1 \text{ or } r \geq 3. \end{cases}$$

The case $r = 1$ is trivial since then G_r is itself a perfect matching of $[n]$. G_2 is **whp** a collection of $O(\ln n)$ disjoint cycles and they will all have to be even for G_2 to have a perfect matching. The meat of the result is therefore in the case $r \geq 3$ and this follows from r-connectivity and Tutte's theorem.

Another approach was considered by Bollobás and Frieze [6]. Let $\mathcal{G}_{n,m}^{\delta\geq\kappa}$ denote the set of graphs with vertex set $[n]$, m edges and minimum degree at least κ. Let $G_{n,m}^{\delta\geq\kappa}$ be sampled uniformly from $\mathcal{G}_{n,m}^{\delta\geq\kappa}$. By conditioning on minimum degree 1, say, we will need fewer random edges to get a perfect matching **whp**: Let $m = \frac{n}{4}(\ln n + 2\ln\ln n + c_n)$.

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \mathbf{Pr}(G_{n,m}^{\delta\geq 1} \text{ has a perfect matching}) = \begin{cases} 0 & c_n \to \infty \textit{ sufficiently slowly}, \\ e^{-\frac{1}{8}e^{-c}} & c_n \to c, \\ 1 & c_n \to \infty. \end{cases} \tag{1}$$

The restriction "sufficiently slowly" may seem out of place, but bear in mind that if $m = n/2$ then the probability of a perfect matching is 1. The precise threshold between $n/2$ and $\frac{1}{4}n\ln n$ for the non-existence of a perfect matching was not determined. (Using the approach developed in the present paper for the bipartite case, we have found that "sufficiently slowly" in (1) can be replaced simply by $m > n/2$.) This work was extended in Bollobás, Fenner and Frieze [4] who considered the probability that $G_{n,m}^{\delta\geq k}$ has $\lfloor k/2 \rfloor$ disjoint Hamilton cycles plus a further disjoint perfect matching if k is odd.

In this paper we continue this line of research. We first consider the bipartite version of (1). Let $\mathcal{B}_{n,m}^{\delta\geq\kappa}$ denote the set of bipartite graphs with vertex set $[n], [n]$, m edges and minimum degree at least κ. Let $B_{n,m}^{\delta\geq\kappa}$ be sampled uniformly from $\mathcal{B}_{n,m}^{\delta\geq\kappa}$.

Theorem 1. *Let $m = \frac{n}{2}(\ln n + 2\ln\ln n + c_n)$. then*

$$\lim_{n\to\infty} \mathbf{Pr}(B^{\delta\geq 1}_{n,m} \text{ has a perfect matching}) = \begin{cases} 0 & c_n \to -\infty,\ m > n, \\ e^{-\frac{1}{4}e^{-c}} & c_n \to c, \\ 1 & c_n \to \infty. \end{cases} \tag{2}$$

The probability on the RHS of (2) is the limiting probability that a pair of vertices of degree 1 have a common neighbor. Thus, the probability that a perfect matching exists is (close to) 1 when either $m = n/2$ or c_n is large, and the probability is very small for m everywhere in between, except c_n not far to the left from 0.

The next natural question in this line of research is: How many random edges are needed if we constrain the minimum degree to be at least 2, so ruling out the possibility of two vertices of degree 1 having a common neighbour. These are the two principal results of the paper.

Theorem 2. *Let $c \geq 2$ be an absolute constant. Then*

$$\lim_{n\to\infty} \mathbf{Pr}(B^{\delta\geq 2}_{n,cn} \text{ has a perfect matching}) = 1.$$

If $c = 2$ then we are dealing with 2-regular bipartite graphs and all such graphs have a perfect matching. Thus the content of the theorem lies in the case $c > 2$.

If we consider $G^{\delta\geq 2}_{n,cn}$ then we have to allow for the existence of small components which are isolated *odd* cycles i.e. we will not have a "probability one" result. Also, for a change, we will allow n to assume odd values as well. For a graph $G = (V, E)$, let $\mu^*(G)$ denote the maximum matching number. Slightly stretching, we say that G has a perfect matching if $\mu^*(G) = [|V|/2]$. Let $X(G)$ stand for the total number of odd isolated cycles in G. Clearly

$$\mu^*(G) \leq \nu(G) := \left[\frac{|V| - X(G)}{2}\right].$$

Let μ^*_n, X_n, ν_n stand for μ^*, X, ν computed at $G = G^{\delta\geq 2}_{n,cn}$.

Theorem 3. *Let $c > 1$ be an absolute constant. Then*

$$\lim_{n\to\infty} \mathbf{Pr}(\mu^*_n = \nu_n) = 1,$$

and X_n is, in the limit, Poisson (λ),

$$\lambda = \frac{1}{4}\log\frac{1+\sigma}{1-\sigma} - \frac{\sigma}{2}, \quad \sigma := \frac{\rho}{e^\rho - 1},$$

and ρ satisfies

$$\frac{\rho(e^\rho - 1)}{e^\rho - 1 - \rho} = 2c.$$

In particular, as $n \to \infty$,

$$\mathbf{Pr}(G^{\delta\geq 2}_{n,cn} \text{ has a perfect matching}) = o(1) + \begin{cases} e^{-\lambda}, & \text{if } n \text{ even}, \\ e^{-\lambda} + \lambda e^{-\lambda}, & \text{if } n \text{ odd}. \end{cases} \tag{3}$$

Notice that $c = 1$ corresponds to the random 2-regular (nonbipartite) graph, which typically has $\Theta(\log n)$ odd (isolated) cycles. Sure enough, the explicit term in the RHS of (3) approaches zero as $c \downarrow 1$.

It was shown in Aronson, Frieze and Pittel [2] that **whp** a simple greedy algorithm of Karp and Sipser [14] found a matching that was within $\tilde{O}(n^{1/5})$ of optimal. Theorem 3 shows that the Karp-Sipser algorithm is **whp** also $\Omega(n^{1/5})$ from optimal.

For integer $k \geq 2$ let graph G have property $\mathcal{A}_k$ if G contains $\lfloor (k-1)/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is even, a further edge disjoint matching of size $\lfloor n/2 \rfloor$. Bollobás, Cooper, Fenner and Frieze [5] show that for $k \geq 3$, there exists a constant $c_k \leq 2(k+1)^3$ such that if $c \geq c_k$ $G_{n,cn}^{\delta \geq k+1}$ has property $\mathcal{A}_k$. Thus the current paper deals with the property $\mathcal{A}_2$ and proves a sharp result. It is reasonable to conjecture that the true value for c_k is $(k+1)/2$. Note that if $c = (k+1)/2$ and cn is integer then $G_{n,cn}^{\delta \geq k+1}$ is a random $(k+1)$-regular graph and this is known to have $\mathcal{A}_k$ **whp**, Robinson and Wormald [20] .

2 Enumerational Preliminaries

2.1 Enumerating bipartite graphs meeting vertex degree bounds.

In our probabilistic model, the sample space $\mathcal{B}_d(n,m)$ is the set of all bipartite graphs on the bipartition $[n] \cup [n]$ with m edges, and the minimum degree d at least ($d = 1$ or $d = 2$). The probability measure is uniform, i.e. each sample graph $B_d(n,m)$ is assigned the same probability, $N_d(n,m)^{-1}$, where $N_d(n,m) = |\mathcal{B}_d(n,m)|$. We will obtain a sharp asymptotic formula for $N_d(n,m)$, as a special case for the number of bipartite graphs meeting more general conditions on vertex degrees.

Consider the bipartite graphs with vertex bipartition $R \cup C$, (Rows and Columns), $R = [\nu_1]$ and $C = [\nu_2]$. Given μ, the ν_1-tuple $\mathbf{a}$, and the ν_2-tuple $\mathbf{b}$ of nonnegative integers a_i, $i \in [\nu_1]$, b_j, $j \in [\nu_2]$, let $N(\mathbf{a},\mathbf{b})$ denote the total number of the bipartite graphs with the row degree sequence $\mathbf{a}$ and the column degree sequence $\mathbf{b}$. Using the bipartite version of the pairing model, we see that

$$N(\mathbf{a},\mathbf{b}) \leq N^*(\mathbf{a},\mathbf{b}); \quad N^*(\mathbf{a},\mathbf{b}) := \frac{\mu!}{\prod\limits_{i\in[\nu_1]} a_i! \cdot \prod\limits_{j\in[\nu_2]} b_j!}. \tag{4}$$

The fudge factor, i.e. the ratio

$$F(\mathbf{a},\mathbf{b}) = \frac{N(\mathbf{a},\mathbf{b})}{N^*(\mathbf{a},\mathbf{b})}, \tag{5}$$

is the probability that the uniformly random pairing is graph induced. A sharp asymptotic formula for $F(\mathbf{a},\mathbf{b})$ has been a subject of many papers. A culmination point is [?] by McKay who proved that if $D^3/\mu \to 0$, D being the maximum degree, then

$$N(\mathbf{a},\mathbf{b}) = N^*(\mathbf{a},\mathbf{b}) \exp\left(-\frac{1}{2}\lambda(\mathbf{a})\lambda(\mathbf{b}) + O(D^3/\mu)\right), \tag{6}$$

$$\lambda(\mathbf{a}) := \frac{1}{\mu} \sum_{i\in[\nu_1]} a_i(a_i - 1), \tag{7}$$

$$\lambda(\mathbf{b}) := \frac{1}{\mu} \sum_{j\in[\nu_2]} b_j(b_j - 1), \tag{8}$$

The formulas (4) and (6) are instrumental in asymptotic evaluation (estimation) of the total number of bipartite graphs with a given number of edges and certain restrictions on the degree sequence.

Let the ν_1-tuple $\mathbf{c} = (c_1, \ldots, c_{\nu_1})$ and the ν_2- tuple $\mathbf{d} = (d_1, \ldots, d_{\nu_2})$ of nonnegative integers be given. Introduce $N_{\mathbf{cd}}(\boldsymbol{\nu}, \mu)$, $\boldsymbol{\nu} = (\nu_1, \nu_2)$, the total number of bipartite graphs with μ edges, such that $a_i \ge c_i$, $(i \in [\nu_1])$, and $b_j \ge d_j$, $(j \in [\nu_2])$. Of course, $N_{\mathbf{cd}}(\boldsymbol{\nu}, \mu) = 0$ if $\mu < \sum_i c_i$, or $\mu < \sum_j d_j$. So we assume that $\mu \ge \max\{\sum_i c_i, \sum_j d_j\}$. Neglecting for now the fudge factor in (6),

$$N_{\mathbf{cd}}(\boldsymbol{\nu}, \mu) \le \sum_{a_i \ge c_i, b_j \ge d_j \sum_i a_i = \sum_j b_j = \mu} N^*(\mathbf{a}, \mathbf{b}), \tag{9}$$

In order to rewrite (4) in a more manageable way, we observe that

$$\sum_{\mu_1, \mu_2 \ge 0} x^{\mu_1} y^{\mu_2} \sum_{a_i \ge c_i, b_j \ge d_j \sum_i a_i = \mu_1, \sum_j b_j = \mu_2} \frac{1}{\prod_{i \in [\nu_1]} a_i! \cdot \prod_{j \in [\nu_2]} b_j!} = G_{\mathbf{c}}(x) H_{\mathbf{d}}(y);$$

$$\begin{aligned} G_{\mathbf{c}}(x) &= \prod_{i \in [\nu_1]} f_{c_i}(x); && (10) \\ H_{\mathbf{d}}(y) &= \prod_{j \in [\nu_2]} f_{d_j}(y), && (11) \end{aligned}$$

where

$$f_t(z) = \sum_{\ell \ge t} \frac{z^\ell}{\ell!} = e^z - \sum_{\ell < t} \frac{z^\ell}{\ell!}. \tag{12}$$

Therefore (4) becomes

$$\begin{aligned} N_{\mathbf{cd}}(\boldsymbol{\nu}, \mu) &\le \mu! [x^\mu y^\mu] G_{\mathbf{c}}(x) H_{\mathbf{d}}(y) && (13) \\ &= \mu! (2\pi i)^{-1} \oint_{|x| = r_1} x^{-\mu-1} G_{\mathbf{c}}(x)\, dx \cdot (2\pi i)^{-1} \oint_{|y| = r_2} y^{-\mu-1} H_{\mathbf{d}}(y)\, dy, && (14) \end{aligned}$$

for all $r_1, r_2 > 0$. Using an inequality (Pittel [18])

$$|f_t(z)| \le f_t(|z|) \exp\left(-\frac{|z| - \text{Re } z}{t+1}\right), \tag{15}$$

(10), (11), and the fact that

$$|z| - \text{Re } z = r(1 - \cos\theta) \ge cr\theta^2, \quad \text{when } z = re^{i\theta},\, \theta \in (-\pi, \pi],$$

we see from (14), after a straightforward estimation, that

$$N_{\mathbf{cd}}(\boldsymbol{\nu}, \mu) \le_b \mu! \frac{1}{\sqrt{r_1 \sum_i (c_i + 1)^{-1}}} \cdot \frac{G_{\mathbf{c}}(r_1)}{r_1^\mu} \times \frac{1}{\sqrt{r_2 \sum_j (d_j + 1)^{-1}}} \cdot \frac{H_{\mathbf{d}}(r_2)}{r_2^\mu}. \tag{16}$$

Here and elsewhere $A <_b B$ means that $A = O(B)$, uniformly for all feasible parameters that determine the values of A and B. In the sequel we consider only $\max_i c_i = O(1)$, $\max_j d_j = O(1)$, in which case the bound (16) simplifies to

$$N_{\mathbf{cd}}(\boldsymbol{\nu}, \mu) \le_b \mu! \cdot (\nu_1 r_1)^{-1/2} \frac{G_{\mathbf{c}}(r_1)}{r_1^\mu} \cdot (\nu_2 r_2)^{-1/2} \frac{H_{\mathbf{d}}(r_2)}{r_2^\mu}. \tag{17}$$

The task of determining the "best" values of r_1 and r_2 and incorporating the left-out fudge factor will be made easier by looking at the above through probabilistic lenses.

Fix $r_1, r_2 > 0$ and introduce the independent random variables Y_i, Z_j, with the distributions

$$\mathbf{Pr}(Y_i = \ell) = \frac{r_1^\ell/\ell!}{f_{c_i}(r_1)}, \ (\ell \geq c_i), \tag{18}$$

$$\mathbf{Pr}(Z_j = \ell) = \frac{r_2^\ell/\ell!}{f_{d_j}(r_2)}, \ (\ell \geq d_j), \tag{19}$$

so that, in distribution, Y_i is Poisson (r_1) conditioned on $\{\text{Poisson}(r_1) \geq c_i\}$, and Z_j is Poisson(r_2) conditioned on $\{\text{Poisson}(r_2) \geq d_j\}$. In short, $Y_i = \text{Po}(r_1; \geq c_i)$ and $Z_j = \text{Po}(r_2; \geq d_j)$. Now (13) can be rewritten as

$$\begin{aligned} N_{\mathbf{cd}}(\boldsymbol{\nu}, \mu) &\leq \mu! \times \frac{G_{\mathbf{c}}(r_1)}{r_1^\mu}[x^\mu]\frac{G_{\mathbf{c}}(xr_1)}{G_{\mathbf{c}}(r_1)} \cdot \frac{H_{\mathbf{d}}(r_1)}{r_1^\mu}[y^\mu]\frac{H_{\mathbf{d}}(yr_1)}{H_{\mathbf{d}}(r_1)} \\ &= \mu! \times \frac{G_{\mathbf{c}}(r_1)}{r_1^\mu}\mathbf{Pr}\left(\sum_i Y_i = \mu\right) \cdot \frac{H_{\mathbf{d}}(r_2)}{r_2^\mu}\mathbf{Pr}\left(\sum_j Z_j = \mu\right). \end{aligned} \tag{20}$$

Now the RHS expressions in (13), (14) and (20) are equal to each other and the RHS of inequality (17) bounds them all. Therefore,

$$\sup_\mu \mathbf{Pr}\left(\sum_i Y_i = \mu\right) \leq_b \frac{1}{\sqrt{\nu_1 r_1}}, \quad \sup_\mu \mathbf{Pr}\left(\sum_j Z_j = \mu\right) \leq_b \frac{1}{\sqrt{\nu_2 r_2}}. \tag{21}$$

Furthermore, (9) becomes equality when $N^*(\mathbf{a}, \mathbf{b})$ is replaced by $N(\mathbf{a}, \mathbf{b})$. So, analogously to (20),

$$\begin{aligned} N_{\mathbf{cd}}(\boldsymbol{\nu}, \mu) &= \frac{\mu!}{(r_1 r_2)^\mu} \sum_{a_i \geq c_i, b_j \geq d_j \sum_i a_i = \mu_1, \sum_j b_j = \mu_2} \prod_{i \in [\nu_1]} \frac{r_1^{a_i}}{a_i!} \prod_{j \in [\nu_2]} \frac{r_2^{b_j}}{b_j!} \\ &= \mu! \frac{G_{\mathbf{c}}(r_1) H_{\mathbf{d}}(r_2)}{(r_1 r_2)^\mu} \sum_{a_i \geq c_i, b_j \geq d_j \sum_i a_i = \mu_1, \sum_j b_j = \mu_2} \mathbf{Pr}(\mathbf{Y} = \mathbf{a}, \mathbf{Z} = \mathbf{b}) F(\mathbf{a}, \mathbf{b}) \\ &= \mu! \frac{G_{\mathbf{c}}(r_1) H_{\mathbf{d}}(r_2)}{(r_1 r_2)^\mu} \cdot \mathbf{E}\left(F(\mathbf{Y}, \mathbf{Z}) \cdot 1_{\{\sum_i Y_i = \mu\}} 1_{\{\sum_j Z_j = \mu\}}\right), \end{aligned} \tag{22}$$

where $F(\cdot, \cdot)$ is defined in (5). To make this formula useful, we need to show that, for a proper choice of r_1, r_2, asymptotically we can replace $\lambda(\mathbf{Y})\lambda(\mathbf{Z})$ in the formula (6) by $\mathbf{E}(\lambda(\mathbf{Y}))\mathbf{E}(\lambda(\mathbf{Z})))$.

From now on let us assume that and

$$\mu^{-1} \leq_b r_1, r_2 \leq_b \log \mu \text{ and that } \nu_1, \nu_2 = O(\mu). \tag{23}$$

Since $\max_i c_i$, $\max_j d_j$ are both $O(1)$, using the definition of Y_i, Z_j and the conditions on ν_1, ν_2 and r_1, r_2, we have: for $0 < \alpha' < \alpha$,

$$\begin{aligned} \mathbf{Pr}(\max\{\max_i Y_i, \max_j Z_j\} \geq \mu^\alpha) &\leq \sum_i \mathbf{Pr}(Y_i \geq \mu^\alpha) + \sum_j \mathbf{Pr}(Z_j \geq \mu^\alpha) \\ &\leq e^{-\mu^{\alpha'}}. \end{aligned}$$

Therefore, for $\alpha < 1/3$, $\mathbf{E}_{\boldsymbol{\nu}\mu}$, the expected value in (22), is given by

$$\mathbf{E}_{\boldsymbol{\nu}\mu} = (1 + O(\mu^{-1+3\alpha}))\mathbf{E}^*_{\boldsymbol{\nu},\mu} + O(e^{-\mu^{\alpha'}}); \tag{24}$$

$$\mathbf{E}^*_{\boldsymbol{\nu}\mu} = \mathbf{E}\left(F^*(\mathbf{Y}, \mathbf{Z}) \cdot 1_{\{\sum_i Y_i = \mu\}} 1_{\{\sum_j Z_j = \mu\}}\right); \tag{25}$$

$$F^*(\mathbf{a}, \mathbf{b}) := \exp\left(-\frac{1}{2}\lambda(\mathbf{a})\lambda(\mathbf{b})\right), \tag{26}$$

see (6). In particular, see (21),

$$\mathbf{E}^*_{\nu\mu} \leq_b (\nu_1\nu_2 r_1 r_2)^{-1/2}.$$

Let us estimate the effect of replacing $\lambda(\mathbf{Y}), \lambda(\mathbf{Z})$ in (26) by their expected values. To this end, let us introduce

$$U_i = (Y_i)_2 - \mathbf{E}((Y_i)_2), \quad V_j = (Z_j)_2 - \mathbf{E}((Z_j)_2).$$

Simple computation shows that $\mathbf{E}((Y_i)_2)$ is of order $O(1+r_i^2)$, whence of order $O(\log^2 \mu)$, and likewise $\mathbf{E}(Z_j(Z_j-1)) = O(\log^2 \mu)$. ¿From $\lambda(\mathbf{Y}) = \mu^{-1}\sum_i (Y_i)_2$, $\lambda(\mathbf{Z}) = \mu^{-1}\sum_j (Z_j)_2$, it follows then that $\mathbf{E}(\lambda(\mathbf{Y})), \mathbf{E}(\lambda(\mathbf{Z})) = O(\log^2 \mu)$ and that after using the expansion

$$ab - \bar{a}\bar{b} = (a-\bar{a})(b-\bar{b}) + \bar{a}(b-\bar{b}) + \bar{b}(a-\bar{a})$$

we have

$$\begin{aligned} |\lambda(\mathbf{Y})\lambda(\mathbf{Z}) - \mathbf{E}(\lambda(\mathbf{Y}))\mathbf{E}(\lambda(\mathbf{Z}))| &\leq_b (\log^2 \mu)\Delta(\mathbf{Y},\mathbf{Z}) + \Delta^2(\mathbf{Y},\mathbf{Z}), \\ \Delta(\mathbf{Y},\mathbf{Z}) &= |\lambda(\mathbf{Y}) - \mathbf{E}(\lambda(\mathbf{Y}))| + |\lambda(\mathbf{Z}) - \mathbf{E}(\lambda(\mathbf{Z}))|. \end{aligned} \tag{27}$$

Therefore, if we replace $F^*(\mathbf{Y},\mathbf{Z})$ in (25) by $\exp(-\frac{1}{2}\mathbf{E}(\lambda(\mathbf{Y}))\mathbf{E}(\lambda(\mathbf{Z})))$, then the compensating factor is $\exp(O(\log^2 \mu\Delta(\mathbf{Y},\mathbf{Z}) + \Delta^2(\mathbf{Y},\mathbf{Z})))$. Furthermore, setting $u = \log^{10}\mu$, we estimate

$$\begin{aligned} \mathbf{Pr}(|U_i| \geq u) &\leq \sum_{\ell^2 \geq u} \frac{r_1^\ell/\ell!}{f_{c_i}(r_1)} \\ &\leq_b r_1^{-c_i}\left(\frac{er_1}{\log^5 \mu}\right)^{\log^5 \mu} \\ &\leq \exp(-\Omega(\log^5 \mu)). \end{aligned}$$

Likewise

$$\begin{aligned} \mathbf{E}(U_i; |U_i| \geq u) &= \sum_{\ell : (\ell)_2 - \mathbf{E}((Y_i)_2) \geq u} [(\ell)_2 - \mathbf{E}((Y_i)_2)] \frac{r_1^\ell/\ell!}{f_{c_i}(r_1)} \\ &\leq_b r_1^{2-c_i}\left(\frac{2er_1}{\log^5 \mu}\right)^{\frac{1}{2}\log^5 \mu} \\ &\leq \exp(-\Omega(\log^5 \mu)). \end{aligned} \tag{28}$$

Let $\mathcal{U}_i = U_i \mathbf{I}_{\{|U_i|<u\}}$, so that $|\mathcal{U}_i| \leq u$. Then (Azuma-Hoeffding inequality), for every $t > 0$,

$$\mathbf{Pr}\left(\left|\sum_i (\mathcal{U}_i - \mathbf{E}\mathcal{U}_i)\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2u^2\nu_1}\right).$$

Since $\mathbf{E}U_i = 0$, from (28) we have

$$\left|\sum_i \mathbf{E}\mathcal{U}_i\right| = \left|\sum_i \mathbf{E}(U_i; |U_i| \geq u)\right| \leq \exp(-\Omega(\log^5 \mu)).$$

Therefore, for $t \geq 1$,

$$\begin{aligned}
\mathbf{Pr}\left(\left|\sum_i U_i\right| \geq t\right) &\leq \sum_i \mathbf{Pr}(|U_i| \geq u) + \mathbf{Pr}\left(\left|\sum_i \mathcal{U}_i\right| \geq t\right) \\
&\leq \sum_i \mathbf{Pr}(|U_i| \geq u) + \mathbf{Pr}\left(\left|\sum_i (\mathcal{U}_i - \mathbf{E}\mathcal{U}_i)\right| \geq t - \left|\sum_i \mathbf{E}\mathcal{U}_i\right|\right) \\
&\leq_b \exp(-\Omega(\log^5 \mu)) + \exp\left(-\frac{\left(t - \exp(-\Omega(\log^5 \mu))\right)^2}{2u^2\nu_1}\right) \\
&= \exp(-\Omega(\log^5 \mu)), \qquad (29)
\end{aligned}$$

the latter inequality holding if $t = \mu^{1/2}\log^5 \mu$. An analogous inequality holds for $\sum_j V_j$. Equivalently

$$\begin{aligned}
\mathbf{Pr}\left(|\lambda(\mathbf{Y}) - \mathbf{E}(\lambda(\mathbf{Y}))| \geq \mu^{-1/2}\log^{10}\mu\right) &\leq \exp(-\Omega(\log^5 \mu)), \\
\mathbf{Pr}\left(|\lambda(\mathbf{Z}) - \mathbf{E}(\lambda(\mathbf{Z}))| \geq \mu^{-1/2}\log^{10}\mu\right) &\leq \exp(-\Omega(\log^5 \mu)).
\end{aligned}$$

Combining these bounds with (27) and (25), and denoting $R = \sum_i Y_i$, $S = \sum_j Z_j$, we get

$$\begin{aligned}
\mathbf{E}^*_{\boldsymbol{\nu}\mu} &= (1 + O(\mu^{-1/2}\log^{12}\mu))e^{-\frac{1}{2}\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})}\mathbf{Pr}\,(R = \mu)\,\mathbf{Pr}\,(S = \mu) + D_{\boldsymbol{\nu}\mu}; \qquad (30) \\
|D_{\boldsymbol{\nu}\mu}| &\leq_b \exp(-\Omega(\log^5 \mu)). \qquad (31)
\end{aligned}$$

In (30), the exponential factor is $\exp(-O(\log^4 \mu))$, and, by (21) and the conditions on μ, ν_i, r_i, the product of the probabilities is of order $(\nu_1 r_1 \nu_2 r_2)^{1/2}$, the latter being $\Omega((\mu\log\mu)^{-1})$. The resulting bound makes the remainder $D_{\boldsymbol{\nu}\mu}$ relatively negligible, so that

$$\mathbf{E}^*_{\boldsymbol{\nu}\mu} \leq_b \frac{e^{-\frac{1}{2}\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})}}{(\nu_1 r_1 \nu_2 r_2)^{1/2}}.$$

The power of this bound is due to wide range of the parameters r_i for which it holds. However, we will also need an asymptotic formula for $\mathbf{E}^*_{\boldsymbol{\nu}\mu}$, and this requires asymptotic formulas for the local probabilities, rather than their upper bounds.

Intuitively, we stand a better chance of achieving this goal when the parameters r_1, r_2 are such that the events $\{\sum_i Y_i = \mu\}$ and $\{\sum_j Z_j = \mu\}$ have "sizeable" probabilities. What better candidates than $r_1 = \rho_1$ and $r_2 = \rho_2$ for which

$$\sum_{i=1}^{\nu_1} \mathbf{E}Y_i = \mu, \quad \sum_{j=1}^{\nu_2} \mathbf{E}Z_j = \mu.$$

Explicitly, using (18) and (19), ρ_1 and ρ_2 are the roots of

$$\sum_{i=1}^{\nu_1} \frac{x f_{c_i-1}(x)}{f_{c_i}(x)} = \mu, \qquad (32)$$

and

$$\sum_{j=1}^{\nu_2} \frac{x f_{d_j-1}(x)}{f_{d_j}(x)} = \mu, \qquad (33)$$

respectively; ($f_t(z) := e^z$, for $t \le 0$). For $x \to 0$, the LHS of (32) and (33) approach $\sum_i c_i \le \mu$ and $\sum_j d_j \le \mu$, respectively. Each LHS is strictly increasing, asymptotic to $\nu_1 x$ and $\nu_2 x$ respectively, as $x \to \infty$. Assuming that $\sum_i c_i < \mu$ and $\sum_j d_j < \mu$, we see that the positive roots ρ_1 and ρ_2 exist uniquely, and that $\rho_i < \mu/\nu_i$. Assuming from now that $\mu = O(\nu_i \log \nu_i)$, $i = 1, 2$, we obtain that $\rho_i = O(\log \mu)$ which puts ρ_1 and ρ_2 into the set of feasible (meeting (23)) r_1 and r_2, respectively.

How do the probabilities in (30) behave if $r_1 = \rho_1, r_2 = \rho_2$? For $Y_i = \mathrm{Po}(\rho; \ge 2)$ and $\mu = \nu \mathbf{E}(Y)$, it was proved in [2] that

$$\mathbf{Pr}\left(\sum_{i=1}^{\nu} Y_i = \mu + a\right) = \frac{1 + O(a^2(\rho\nu)^{-1})}{(2\pi\nu\mathbf{Var}(Y))^{1/2}},$$

provided that $\rho\nu \to \infty$, and $a^2/(\rho\nu) \to 0$. (The condition $\rho\nu \to \infty$ is equivalent to $\nu\mathbf{Var}(Y) \to \infty$ since $\mathbf{Var}(Y) = \Theta(\rho)$). Suppose that in the present context $\nu_1\rho_1 \to \infty$, i.e. $\sum_i \mathbf{Var}(Y_i) \to \infty$, which is equivalent to $\mu - \sum_i c_i \to \infty$. Only simple modifications of the proof in [2] are needed to prove

Lemma 1. *If $\sum_i \mathbf{Var}(Y_i) \to \infty$ and the c_i are uniformly bounded, then*

$$\mathbf{Pr}\left(\sum_{i=1}^{\nu_1} Y_i = \mu + a\right) = \frac{1 + O(a^2(\nu_1\rho_1)^{-1})}{(2\pi\sum_i \mathbf{Var}(Y_i))^{1/2}},$$

if $a^2(\nu_1\rho_1)^{-1} \to 0$. An analogous formula holds for $\mathbf{Pr}\left(\sum_j Z_j = \mu + a\right)$.

Proof See Appendix A.

Suppose, say, that $\nu_1\rho_1$ is bounded, or equivalently that $\sigma_1 := \mu_1 - \sum_i c_i = O(1)$. Extending an argument in [19] (which covers the case of identically distributed Y_i), we can show that

$$\mathbf{Pr}\,(R = \mu) = (1 + O(\nu_1^{-1}))e^{-\sigma_1}\frac{\sigma_1^{\sigma_1}}{\sigma_1!}. \tag{34}$$

An analogous relation holds for $\mathbf{Pr}(S = \mu)$ if $\sigma_2 := \mu - \sum_j d_j = O(1)$. Clearly then, regardless of the behavior of σ_1, σ_2, in (30) the remainder term $D_{\nu\mu}$ is negligible compared to the explicit term.

Thus we have proved the following statement.

Lemma 2. *Suppose that $\nu_1, \nu_2, \mu \to \infty$ in such a way that $\nu_1, \nu_2 = O(\mu)$ and $\mu = O(\nu_i \log \nu_i)$, $i = 1, 2$. Let $G_{\mathbf{c}}(x)$ and $H_{\mathbf{d}}(y)$ be defined by (10) and (11).*

(i) *Suppose that $\mu^{-1} \le_b r_1, r_2 = O(\log \mu)$. Then*

$$N_{\mathbf{cd}}(\boldsymbol{\nu}, \mu) \sim \mu!\frac{G_{\mathbf{c}}(r_1)H_{\mathbf{d}}(r_2)}{(r_1r_2)^{\mu}}\left[e^{-\frac{1}{2}\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})}\mathbf{Pr}(R = \mu)\mathbf{Pr}(S = \mu) + O(e^{-\Omega(\log^5 \mu)})\right], \tag{35}$$

$$N_{\mathbf{cd}}(\boldsymbol{\nu}, \mu) \le_b \mu!\frac{G_{\mathbf{c}}(r_1)H_{\mathbf{d}}(r_2)}{(r_1r_2)^{\mu}\sqrt{\nu_1\nu_2r_1r_2}}e^{-\frac{1}{2}\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})}, \tag{36}$$

where $Y_i = Po(r_1; \ge c_i)$, $Z_j = Po(r_2; \ge d_j)$ are all independent, and $R = \sum_i Y_i$, $S = \sum_j Z_j$.

(ii) *Suppose also that* $\max_i c_i = O(1)$, $\max_j d_j = O(1)$, *and* $\mu > \max\{\sum_i c_i, \sum_j d_j\}$. *Then there exist (unique) positive roots* ρ_1, ρ_2 *of (32) and (33), and*

$$N_{\mathbf{cd}}(\boldsymbol{\nu}, \mu) \sim \mu! \frac{G_{\mathbf{c}}(\rho_1) H_{\mathbf{d}}(\rho_2)}{(\rho_1\rho_2)^{\mu}} e^{-\frac{1}{2}\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})} \cdot \mathbf{Pr}(R = \mu)\mathbf{Pr}(S = \mu), \tag{37}$$

where $Y_i = Po(\rho_1; \geq c_i)$, $Z_j = Po(\rho_2; \geq c_j)$.

Furthermore

$$\mathbf{Pr}(R = \mu) \sim \frac{1}{(2\pi \sum_i \mathbf{Var}(Y_i))^{1/2}}, \quad or \quad \mathbf{Pr}(R = \mu) \sim e^{-\sigma_1} \frac{\sigma_1^{\sigma_1}}{\sigma_1!},$$

dependent upon whether $\sigma_1 := \mu - \sum_i c_i$ *approaches infinity or stays bounded, with the analogous formula for* $\mathbf{Pr}(S = \mu)$.

Corollary 4. *Suppose* $n = O(m)$, $dn < m = O(n \log n)$. *Then*

$$B_d(n, m) \sim m! \left(\frac{f_d(\rho)^n \mathbf{Pr}\left(\sum_i Y_i = m\right)}{\rho^m} \right)^2 \exp\left(-\frac{n^2}{2m^2} \mathbf{E}^2[(Y)_2] \right), \tag{38}$$

where Y *is Poisson* $(\rho; \geq d)$ *such that* $\mathbf{E}Y = r = m/n$. *Note that*

$$\mathbf{E}[(Y)_2] = \begin{cases} r^2 & d = 0 \\ \rho r & d = 1 \\ \rho r/(1 - e^{-\rho}) & d = 2 \end{cases}. \tag{39}$$

Further

$$\mathbf{Pr}\left(\sum_i Y_i = m\right) \sim (2\pi n \mathbf{Var} Y)^{-1/2}, \qquad \text{if } m - dn \to \infty, \tag{40}$$

and

$$\mathbf{Pr}\left(\sum_i Y_i = m\right) \sim e^{-\sigma} \frac{\sigma^{\sigma}}{\sigma!} \qquad \text{if } m - dn > 0 \text{ is fixed.} \tag{41}$$

As we will see, these results are all we need to evaluate (bound) the probabilities arising in the proofs of Theorem 1 and Theorem 2. We will also need a crude upper bound for the fraction of bipartite graphs in question, with the maximum degree exceeding m^{α}. This bound is already implicit in the preceding analysis! Indeed, from (22), (24), (25), and the observation that the factor

$$\mathbf{Pr}^2\left(\sum_i Y_i = m\right) \exp\left(-\frac{n^2}{2m^2} \mathbf{E}^2[(Y)_2] \right) \tag{42}$$

in (38) is $\exp(-\Theta(\log^2 n))$, it follows that, for $\alpha' < \alpha < 1/3$, this fraction is $e^{-m^{\alpha'}}$ at most.

One is tempted to call this "overpowering both the conditioning and the fudge factor". Needless to say, this trick would work for the counts (fractions) of other graph classes, as long as the degrees restrictions are so severe that the probability that Y_i, Z_j meet them is negligible compared to the factor in (42).

3 Proof of Theorem 1

We will use Hall's necessary and sufficient condition for the existence of a perfect matching in a bipartite graph to prove (1).

The random graph $B_{n,m}^{\delta\geq 1}$ has no perfect matching iff for some $k \geq 2$ there exists a *k-witness*. A k-witness is a pair of sets $K \subseteq R, L \subseteq C$, or $K \subseteq C, L \subseteq R$, such that $|K| = k, |L| = k-1$ and $N(K) \subseteq L$. Here $N(K)$ denotes the set of neighbours of vertices in K. A k-witness is *minimal* if there does not exist $K' \subset K, L' \subset L$ such that (K', L') is a k'-witness, where $k' < k$. It is straightforward that if (K, L) is a minimal k-witness then every member of L has degree at least two in $B_n(K \cup L)$, the subgraph of B_n induced by $K \cup L$. Therefore the subgraph has at least $2(k-1)$ edges. We can restrict our attention to $k \leq n/2$ since for $k > n/2$ we can consider $C \setminus L, R \setminus K$. For $2 \leq k \leq n/2$, let $W_{nk\mu}$ denote the random number of minimal k-witnesses, such that $B_n(K \cup L)$ has μ edges, $\mu \geq 2(k-1)$. Actually, since $k \leq n/2$, we also have $\mu \leq m - n$.

(i) Suppose $m = O(n \log n)$ and $m \geq (1/3 + \epsilon) n \log n$, $\epsilon > 0$. Let us prove that whp B_n has no k-witnesses with $k \geq 3$, i.e.

$$\mathbf{Pr}\left(\sum_{k\geq 3, \mu \geq 2(k-1)} W_{nk\mu} = 0 \right) \to 1, \quad n \to \infty.$$

It siffices to show that

$$\sum_{k\geq 3, \mu\geq 2(k-1)} E_{nk\mu} \to 0, \quad E_{nk\mu} := \mathbf{E} W_{nk\mu}.$$

Let us bound $E_{nk\mu}$. Suppose for certainty that $K \subset R$, $L \subset C$. We can choose a pair (K, L) in $\binom{n}{k}\binom{n}{k-1}$ ways. (K, L) being a witness imposes the above listed conditions on degrees of the subgraph induced by $K \cup L$. If we delete the row set K, we get a remainder graph, which is a bipartite graph with bipartition (R', C), $R' = R \setminus K$; it has $m - \mu$ edges and every vertex in $R' \cup (C \setminus L)$ has degree 1 at least. We bound N_1, the total number of those subgraphs, and N_2 the total number of the remainder graphs using Lemma 2 (i), emphasizing the possibility to choose the corresponding parameters r_1, r_2 anyway we want. The product of these two bounds divided by the asymptotic expression for $B_1(n, m)$ in Corollary 4 provides an upper bound for the probability that (K, L) is a k-witness with μ edges. Multiplying this bound by $2\binom{n}{k}\binom{n}{k-1}$, we get a bound for $E_{nk\mu}$. To implement this program, we consider separately $k \leq m^\beta$ and $k \geq m^\beta$, where $\beta \in (0, 1)$ will be specified in the course of the argument.

Let $k \leq m^\beta$. Pick $\alpha' < \alpha = (1-\beta)/2$. ¿From the note following Corollary 4, with probability $1 - e^{-m^{\alpha'}}$ at least, the maximum vertex degree in the uniformly random bipartite graph is m^α at most. So, backpedaling a bit, we will consider $\mu \leq m^\gamma$, $(\gamma := (1+\beta)/2)$, only. To bound N_1 we use (17) with $r_1 = \mu/k, r_2 = \mu/(k-1)$, and to bound N_2 we use (36) with $r_1 = r_2 = \rho$. Here ρ is the parameter of Y_i in Corollary 4, the root of $x f_0(x)/f_1(x) = r$, $r := m/n$, so that

$$\rho = r(1 - e^{-\rho}) < r, \quad \rho = r - \Theta(re^{-r}). \tag{43}$$

The r_i for N_1 seem natural, if one interprets them as parameters of Poissons approximating the vertex degrees that should add to μ on either side of the subgraph induced by $K \cup L$. Since k, μ are relatively small, $r_1 = r_2 = \rho$ should be expected to deliver a good enough bound for N_2. Most importantly, this choice does the job!

After cancellations and trivial tinkering, the resulting bound is

$$\begin{aligned}
E_{nk\mu} &\leq_b \frac{m\binom{n}{k}\binom{n}{k-1}}{\mu(m-\mu)\binom{m}{\mu}} \cdot \rho^{2\mu} \left(\frac{k}{\mu}\right)^{\mu} \left(\frac{k-1}{\mu}\right)^{\mu} \\
&\times \frac{f_1(\mu/k)^k f_1(\mu/(k-1))^{k-1} f_0(\rho)^{k-1}}{f_1(\rho)^{2k-1}} \\
&\times \frac{\exp\left(-\frac{1}{2}\frac{(n-k)\mathbf{E}(Y)_2}{m-\mu} \cdot \frac{(k-1)\rho^2+(n-k+1)\mathbf{E}(Y)_2}{m-\mu}\right)}{\exp\left(-\frac{1}{2}\frac{(n\mathbf{E}(Y)_2)^2}{m^2}\right)}.
\end{aligned} \tag{44}$$

Some explanation: $k-1$ vertices from L in the remaining graph have degrees not bounded away from zero, whence the factor $f_0(\rho)^{k-1} = e^{\rho(k-1)}$ in the second line, and $k-1$ usual Poissons (ρ), each with the second factorial moment equal ρ^2, contributing $(k-1)\rho^2$ in the last line. Also, we have used $f_1(\mu/(k-1))^{k-1}$ where we could have used the smaller $f_2(\mu/(k-1))^{k-1}$.

The last line fraction is of order $O(1)$, as $\mathbf{E}(Y)_2 = \Theta(\rho^2)$. Further, since $\log f_1(z) = \log(e^z - 1)$ is concave,

$$\begin{aligned}
&k \log f_1\left(\frac{\mu}{k}\right) + (k-1)\log f_1\left(\frac{\mu}{k-1}\right) - (2k-1)\log f_1(\rho) \leq \\
&\quad (2k-1)\left(\log f_1\left(\frac{2\mu}{2k-1}\right) - \log f_1(\rho)\right) \leq (2k-1)(\log f_1)'(\rho)\left(\frac{2\mu}{2k-1} - \rho\right) \\
&\qquad \leq 2\mu - (2k-1)\rho + 3\mu e^{-\rho}.
\end{aligned} \tag{45}$$

Using the last observations and $\mu! = \Theta(\mu^{1/2}(\mu/e)^{\mu})$, we see that $E_{nk\mu}$ is of order $E^*_{nk\mu}$ at most, where

$$E^*_{nk\mu} = \frac{n^{2k-1}e^{-k\rho}}{k!(k-1)!} \cdot \frac{(m-\mu)!}{m!\mu!} k^{2\mu}\rho^{2\mu} \cdot \exp(3\mu e^{-\rho}). \tag{46}$$

The rest is easy! First, since $2(k-1) \leq \mu \leq m^{\gamma}$,

$$\frac{E^*_{nk,\mu+1}}{E^*_{nk\mu}} = \frac{k^2\rho^2}{(m-\mu)(\mu+1)} = O(m^{-1/3}\log^2 n),$$

so that

$$\sum_{2(k-1)\leq\mu\leq m^{\gamma}} E^*_{nk\mu} \sim E^*_{nk,2(k-1)}.$$

Second

$$\begin{aligned}
\frac{E^*_{n,k+1,2k}}{E^*_{nk,2(k-1)}} &= \frac{n^2(k+1)^{4k}\rho^4 e^{-\rho}}{k(k+1)(2k-1)2k(m-2k+1)(m-2k+2)k^{4(k-1)}} \\
&\leq_b \frac{n^2}{m^2}\rho^4 e^{-\rho} = O(\rho^2 e^{-\rho}).
\end{aligned}$$

Therefore

$$\sum_{3\leq k\leq m^{\gamma}} E^*_{nk,2(k-1)} \sim E^*_{n34} \leq_b \frac{n^5}{m^4}\rho^4 e^{-3\rho} \sim ne^{-3m/n} = O(n^{-3\epsilon}),$$

as $m \geq (1/3+\epsilon)n\log n$. In summary,

$$\sum_{3\leq k\leq m^{\gamma}} \sum_{2(k-1)\leq\mu\leq m^{\gamma}} E_{nk\mu} = O(n^{-3\epsilon}). \tag{47}$$

Consider now $k \geq m^\beta$. This time we use (35) not only for N_1 but for N_2 as well, using for the latter $r_1 = (\rho n - \mu)/(n-k)$ and $r_2 = (\rho n - \mu)/n$. That the latter r_i are positive follows from $\mu \leq m - n/2$ and (43). (For μ, k not being relatively small anymore, the count N_2 of the remaining graphs would hardly be well bounded via the previous choice $r_1 = r_2 = \rho \sim m/n$. What we have chosen turns out to be a working compromise between that old choice and the "naive" $r_1 = (m-\mu)/(n-k)$, $r_2 = (m-\mu)/n$.) The resulting bound is

$$\begin{aligned}
E_{nk\mu} &\leq_p \exp\left(\frac{(n\mathbf{E}(Y)_2)^2}{2m^2}\right) E'_{nk\mu}, \\
E'_{nk\mu} &= \rho^{2m} \frac{\binom{n}{k}^2}{\binom{m}{\mu}} \cdot \left(\frac{k}{\mu}\right)^{2\mu} \left(\frac{n-k}{\rho n - \mu}\right)^{m-\mu} \left(\frac{n}{\rho n - \mu}\right)^{m-\mu} \\
&\times \frac{f_1(\frac{\mu}{k})^k f_1(\frac{\mu}{k-1})^{k-1} f_1(\frac{\rho n-\mu}{n-k})^{n-k} f_1(\frac{\rho n-\mu}{n})^{n-k+1} f_0(\frac{\rho n-\mu}{n})^{k-1}}{f_1(\rho)^{2n}}.
\end{aligned} \tag{48}$$

(We use the notation $a_n \leq_p b_n$ to indicate that a_n/b_n is polynomially large, at most.) Using again convexity of $\log f_1(z)$ and denoting $h = (k-1)(\rho n - \mu)/n$, we obtain that the logarithm of the last line fraction is less than

$$2n \log f_1\left(\rho - \frac{h}{2n}\right) - 2n \log f_1(\rho) + h \leq -h((\log f_1)'(\rho) - 1) = -\frac{h}{e^\rho - 1} \leq 0.$$

Thus the fraction is bounded, 1 at most, like its counterpart for $k \leq m^\beta$. (Our search for the proper r_1, r_2 was driven, in fact, by desire to make that fraction bounded again!)

Introduce $x = k/n$, $y = \mu/n$. Using the Stirling formula for factorials, we obtain easily then that

$$E'_{nk\mu} \leq_p \exp(nH(x,y)),$$

where

$$H(u,v) = 2r \log \rho + 2H(u) - rH(v/r) + 2v \log u/v + (r-v) \log \frac{1-u}{(\rho - v)^2},$$

($u \in (x_n, 1/2]$, $v \in (0, \rho)$), $x_n := \frac{m^\beta}{n}$, and

$$H(w) = w \log 1/w + (1-w) \log 1/(1-w).$$

It follows that

$$\begin{aligned}
H_v(u,v) &= \log \frac{u^2(\rho - v)^2}{v(r-v)(1-u)} - 2\frac{\rho - r}{\rho - v}, \\
H_{vv}(u,v) &= \frac{1}{r-v} - \frac{2}{\rho - v} - \frac{1}{v} - \frac{2(\rho - r)}{(\rho - v)^2} < 0.
\end{aligned}$$

So $H_v(u,v)$ decreases with v, and $H_v(u, 0+) = \infty$, $H_v(u, \rho-) = -\infty$. Hence, given u, $H(u,v)$ attains its maximum at a unique root $v(u)$ of the equation

$$\frac{(\rho - v)^2 u^2}{(r-v)(1-u)v} = \exp\left(\frac{2(\rho - r)}{\rho - v}\right). \tag{49}$$

By (43), $\rho < r$ and $\rho - r = O(re^{-r}) \to 0$; so we should expect $v(u)$ to be close to $v^*(u)$, the root of (49) with ρ replaced by r, i.e.

$$v^*(u) = \frac{u^2}{1 - u + u^2} r \leq \frac{r}{3},$$

$(u \leq 1/2)$. Careful computations reveal that

$$H^*(u, v^*(u)) = r \log(1 - u + u^2) + 2H(u),$$

where $H^*(u, v)$ is obtained from $H(u, v)$ by replacing ρ with r. Furthermore, as the RHS of (49) is $1 + O(e^{-r})$, it can be shown that

$$v(u) = v^*(u)(1 + O(e^{-r})).$$

In this setting, strictly speaking, $v(u)$ is also a function of ρ, and so is $H(u, v(u))$, both explicitly and implicitly, via $v(u)$. Since $H_v(u, v(u)) = 0$, the derivative of $H(u, v(u))$ with respect to ρ is just the partial derivative, which is

$$\frac{2r}{\rho} - \frac{2(r - v(u))}{\rho - v(u)} = \frac{2v(u)(\rho - r)}{\rho(\rho - v(u))} = O(e^{-r});$$

therefore

$$H(u, v(u)) = H^*(u, v^*(u)) + O(e^{-r}(r - \rho)) = H^*(u, v^*(u)) + O(re^{-2r}).$$

Using

$$\log(1 - u + u^2) \leq -u/2,\ (u \leq 1/2), \quad (1 - u)\log(1 - u)^{-1} \leq u,$$

we see that, for $u \in (m^\beta/n, 1/2]$ (and $m \geq (1/3 + \epsilon)n \log n$),

$$\begin{aligned} H^*(u, v^*(u)) &\leq -u(r/2 - 2 - 2\log(1/u)) \\ &\leq -u((1/6 + \epsilon/2)\log n - 2(1 - \beta)\log n + O(\log\log n)) \\ &\leq -cu \log n, \end{aligned}$$

where $c = c(\beta) > 0$ if $\beta > 11/12 - \epsilon/4$. So, for this choice of β,

$$\begin{aligned} H(u, v(u)) &= H^*(u, v^*(u)) + O(r^{-2r}) \\ &\leq -cm^\beta n^{-1} \log n + O(e^{-2m/n} \log n) \\ &\leq -m^\beta n^{-1}. \end{aligned}$$

This inequality shows that

$$E'_{nk\mu} \leq_p \exp(-m^\beta) \Rightarrow E_{nk\mu} \leq \exp(-0.5m^\beta),$$

as the fudge factor in (48) is only $\exp(O(\log^2 n))$. Consequently

$$\sum_{m^\beta \leq k \leq n/2\mu \leq m} E_{nk\mu} \leq \exp(-0.4m^\beta). \tag{50}$$

Combining (47) and (50) we obtain

$$\sum_{3 \leq k \leq m^\beta 2(k-1) \leq \mu \leq m^\gamma} E_{nk\mu} + \sum_{m^{1/3} < k \leq n/2\mu < m E_{nk\mu}} = O(n^{-3\epsilon}),$$

so that

$$\mathbf{Pr}\left(\sum_{k \geq 3,\, \mu \geq 2(k-1)} W_{nk\mu} = 0\right) = 1 - O(n^{-3\epsilon}). \tag{51}$$

(ii) Turn now to the 2-witnesses. From (46), it follows that

$$E_{n21} \le_b \frac{n^3 e^{-2\rho}\rho^4}{m^2} = O(n\log^2 n e^{-2m/n}) = O(e^{-c_n}), \tag{52}$$

with c_n defined by the notation

$$m = \frac{n}{2}(\log n + 2\log\log n + c_n).$$

Case 1 $c_n \to \infty$**, and** $m = O(n\log n)$.
Then (52) shows that, with probability more than $1 - e^{-c_n} \to 1$, there are no 2-witnesses. By (52), with probability $1 - O(n^{-1/2})$ at least, there are no 3-witnesses either. Thus, with probability approaching 1, there exists a perfect matching.

Case 2: $c_n \to c \in (-\infty, \infty)$.
We want to prove that W_{n21}, the number of 2-witnesses, is, in the limit, Poisson $(e^{-c}/4)$. We do so via the factorial moments method. To evaluate $\mathbf{E}(W_{n21})$ sharply, we notice that in this case $N_1 = 2\binom{n}{2}n$ exactly, and for N_2 we use the part (ii) of Lemma 1 with $r_1 = r_2 = \rho$. So (compare to (44)

$$E_{n21} \sim \frac{2n\binom{n}{2}}{m^2}\cdot\rho^4 e^{-2\rho} \sim \frac{1}{4}e^{-c_n} \to \lambda := \frac{1}{4}e^{-c}. \tag{53}$$

We need to show that, for each $t \ge 2$

$$\lim_{n\to\infty} \mathbf{E}(W_{n21})_t = \lambda^t, \quad \lambda = \frac{1}{4}e^{-c}.$$

To simplify our task, let us consider instead W^*_{n21}, the total number of vertex-disjoint 2-witnesses. The difference $W_{n21} - W^*_{n21}$ is **whp** (51) $\mathcal{W}_n$ at most, where $\mathcal{W}_n$ is the total number of subset pairs (K, L), $K \subset R$, $L \subset C$, or $K \subset C$, $L \subset R$, such that $|K| = 3$, $|L| = 1$, and $L = N(K)$. Analogously to (44),

$$\mathbf{E}\mathcal{W}_n \le_b \frac{n\binom{n}{3}\rho^6}{\binom{m}{3}}e^{-3\rho} = O(n^{-1/2}).$$

Therefore, $W_{n21} = W^*_{n21}$ with probability $1 - O(n^{-1/2})$ at least, and it suffices to show that

$$\lim_{n\to\infty} \mathbf{E}(W^*_{n21})_2 = \lambda^t, \quad t \ge 1. \tag{54}$$

This is obviously true for $t = 1$. Let $t \ge 2$. Combinatorially, $(W^*_{n21})_t$ is the total number of ordered t-tuples of (vertex-disjoint) 2-witnesses. Given $r + s = t$, let us compute E_{rs}, the expected number of t-tuples containing r "2 rows, 1 column" (first kind) witnesses, and s "2 columns, 1 row" (second kind) witness. The r vertex-disjoint first kind of witnesses can be chosen in $\binom{n}{2r}\binom{n}{r}(2r-1)!!r!$ ways. (Indeed, once $2r$ rows and r columns are selected, we pair the rows in $(2r-1)!!$ ways and assign the formed r pairs to r columns in $r!$ ways.) Given any such choice, the s 2-nd witnesses, disjoint among themselves *and* from the r first kind witnesses, can be chosen in $\binom{n-r}{2s}\binom{n-2r}{s}(2s-1)!!s!$ ways. There are $t! = (r+s)!$ ways to order all $r + s$ witnesses. Hence $N_1(r, s)$, the total number of the ordered t-tuples of the "alleged" witnesses, is given by

$$\begin{aligned} N_1(r,s) &= \binom{n}{2r}\binom{n}{r}(2r-1)!!r!\binom{n-r}{2s}\binom{n-2r}{s}(2s-1)!!s!(r+s)! \\ &\sim \binom{t}{r}\frac{n^{3t}}{2^t}. \end{aligned}$$

Deleting $2r$ rows and $2s$ columns involved in first kind and second kind witnesses respectively produces a bipartite graph with $m-2t$ edges that meets the following conditions. (a) Every row (column) vertex not involved in the s 2-nd (in the r first) kind witnesses has degree at least 1. (b) No edge can be added to one of (just deleted) $r+s$ 2-witnesses to form a pair (K, L), such that $|K| = 3, |L| = 1$, $K \subset R, L \subset C$, or $K \subset C, L \subset R$, and $N(K) = L$. (This condition is necessary and sufficient for the $(r+s)$ 2-witnesses to be disjoint from all other 2-witnesses.) Denote the total number of such graphs by $N_2(r,s)$. Clearly $N_2(r,s) \le \mathcal{N}_2(r,s)$, where $\mathcal{N}_2(r,s)$ is the total number of bipartite graphs with the condition (b) dropped. Using (35) with $r_1 = r_2 = \rho$, we have

$$
\begin{aligned}
\mathcal{N}_2(r,s) \quad &\sim \quad (m-2t)! \cdot \frac{(e^\rho - 1)^{2n-3t} e^{t\rho}}{\rho^{2(m-2t)}} \\
&\times \quad \left(e^{-\frac{\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})}{2}} \cdot \mathbf{Pr}(R = m-2t)\mathbf{Pr}(S = m-2t) + O(e^{-\log^5 m}) \right).
\end{aligned}
$$

Here $R = \sum_{i=1}^{n-2r} Y_i$, $S = \sum_{j=1}^{n-2s} Z_j$, $Y_i, Z_j = \mathrm{Po}(\rho; \ge 1)$ for $1 \le i \le n-2r-s$, $1 \le j \le n-r-2s$, and $Y_i, Z_j = \mathrm{Po}(\rho)$ for $n-2r-s < i \le n-2r$, $n-r-2s < j \le n-2s$. Using (1) for both local probabilities, we obtain that the second line in the above formula is asymptotic to

$$
\exp\left(-\frac{(n\mathbf{E}(Y_1)_2)^2}{2m^2} \right) \cdot \frac{1}{2\pi n \mathbf{Var}(Y_1)}.
$$

Thus

$$
\frac{\mathcal{N}_2(r,s)}{B_d(n,m)} \sim m^{-2t}\rho^{4t}(e^\rho - 1)^{-3t}e^{t\rho} \sim \left(\frac{e^{-c_n}}{4n^3} \right)^t. \tag{55}
$$

Now

$$
\mathcal{N}_2(r,s) - N_2(r,s) \le r(n-2r-s)N_2^{(1)}(r,s) + s(n-r-2s)N_2^{(2)}(r,s);
$$

here $N_2^{(1)}(r,s)$ ($N_2^{(2)}(r,s)$ resp.) is the total number of the remaining graphs, such that a particular row (column resp.) vertex is incident to a single column (row resp.) vertex, which happens to be one of the vertices from r first kind (s second kind resp.) witnesses. Consider $N_2^{(1)}(r,s)$. Deleting that row we get a graph with one less number of row vertices and one less number of edges. So, using (35) with $r_1 = r_2 = \rho$ and $e^\rho - 1 \sim e^\rho$, we obtain that

$$
\begin{aligned}
\frac{N_2^{(1)}(r,s)}{B_1(n,m)} \quad &\le_b \quad \frac{\mathcal{N}_2(r,s)}{B_1(n,m)} \frac{\rho^2}{me^\rho}, \\
\frac{N_2^{(2)}(r,s)}{B_1(n,m)} \quad &\le_b \quad \frac{\mathcal{N}_2(r,s)}{B_1(n,m)} \frac{\rho^2}{me^\rho}.
\end{aligned}
$$

Therefore

$$
\frac{\mathcal{N}_2(r,s) - N_2(r,s)}{B_1(n,m)} \le_b \frac{n\rho^2}{me^\rho} \le \rho e^{-\rho} \to 0.
$$

Collecting the pieces, we obtain that

$$
E_{rs} \sim \binom{t}{r} \frac{n^{3t}}{2^t} \cdot \left(\frac{\rho^4}{m^2 e^{2\rho}} \right)^t \to \binom{t}{r} \left(\frac{e^{-c}}{8} \right)^t,
$$

i.e.

$$
E(W_{n21}^*)_t \to \sum_{r=0}^{t} \binom{t}{r} \left(\frac{e^{-c}}{8} \right)^t = \lambda^t.
$$

Hence W^*_{n21} is in the limit Poisson (λ), and then so is W_{n21}. Consequently, a perfect matching exists with the limiting probability equal

$$\lim_{n\to\infty} \mathbf{Pr}(W_{n21}=0) = e^{-\lambda} = \exp\left(-\frac{e^{-c}}{4}\right).$$

Case 3: $c_n \to -\infty$, $m > n$.
3a: $m \ge (\frac{1}{3}-\epsilon)n\log n$.
In this case, after with trivial modifications in the above derivation,

$$\mathbf{E}(W^*_{n21})_2 \sim \left(\frac{e^{-c_n}}{4}\right)^2 \to \infty,$$

and, by Chebyshev's inequality,

$$\lim_{n\to\infty} \mathbf{Pr}(W^*_{n21} > 0) = 1.$$

So, whp, a perfect matching does not exist.

3b: $m \le (1/3-\epsilon)n\log n$, $m-n\to\infty$.
Note that $n\rho\to\infty$. Let X_n denote the total number of isolated trees with 2 row vertices and 1 column vertex. ($X_n > 0$ implies that there is no perfect matching.) If the X_n trees are deleted, the remaining graph has $n-2t$ row vertices, $n-t$ column vertices, and $m-2t$ edges, and every vertex has degree 1, at least. Evaluating the number of such graphs by (35), we easily obtain

$$\begin{aligned}\mathbf{E}(X_n)_t &\sim \binom{n}{2t}\binom{n}{t}(2t-1)!!(t!)^2\frac{(m-2t)!}{m!}\frac{\rho^{4t}}{(e^\rho-1)^{3t}}\\ &\sim \left(\frac{m\rho e^{-3\rho}}{2}\right)^t,\end{aligned}$$

using the definition of ρ for the second equality. Also from this definition, $\rho\sim 2(m-n)/n$ if $\rho\to 0$, and $\rho\le m/n$ always. So, if $\rho\to 0$,

$$m\rho e^{-3\rho} \sim 2m(m-n)/n \ge 2(m-n)\to\infty,$$

and, if $\lim\rho > 0$, then

$$m\rho e^{-3\rho} \ge \rho n e^{-3m/n} \ge \rho n^{3\epsilon}\to\infty.$$

Thus

$$\mathbf{E}(X_n)\to\infty, \quad \mathbf{E}(X_n)_2\sim \mathbf{E}^2(X_n),$$

so that (Chebyshev's inequality) $\mathbf{Pr}(X_n>0)\to 1$. That is, whp, there is no perfect matching.

3c: $\sigma := m-n > 0$ is fixed.
If we form $4n-3m$ isolated edges, the remaining $3(m-n)$ row vertices and $3(m-n)$ column vertices can be partitioned into $2(m-n)$ trees of size 3, half of the trees each containing 2 row vertices and 1 column vertex, and another half - 1 row vertex and 2 column vertices. The total number of such bipartite graphs is

$$\begin{aligned}N^*(n,m) &= \binom{n}{4n-3m}^2(4n-3m)!\cdot\left[\binom{3(m-n)}{2(m-n)}(2(m-n)-1)!!(m-n)!\right]^2\\ &\sim \frac{(n!)^2}{(n-3\sigma)!2^{2\sigma}(\sigma!)^2}.\end{aligned} \tag{56}$$

As for $B_1(n,m)$, the total number of all bipartite graphs, by Corollary 4 and (34), it is given by

$$B_1(n,m) \sim m! \left(\frac{f_1(\rho)^n e^{-(m-n)}(m-n)^{m-n}/(m-n)!}{\rho^m} \right)^2 \exp\left(-\frac{(n\mathbf{E}(Y)_2)^2}{2m^2} \right),$$

where, using the definition of ρ,

$$\rho = \frac{2\sigma}{n} \left(1 - \frac{2\sigma}{3n} + O(\sigma^2/n^2) \right).$$

So, after simple computations,

$$B_1(n,m) \sim \frac{m! n^{2\sigma}}{2^{2\sigma}(\sigma!)^2}. \tag{57}$$

Since, for fixed σ,

$$\frac{(n!)^2}{(n-3\sigma)!} \sim m! n^{2\sigma},$$

it follows from (56) and (57) that, with probability approaching 1, the random graph has $2\sigma > 0$ isolated trees of size 3, thus no perfect matching exists.

Theorem 1 is proved completely. $\square$

4 Proof of Theorem 2

Let now $m = cn$ where $c > 2$ is a fixed constant and let $B_n = B_{n.m}^{\delta \geq 2}$. A direct application of Hall's theorem has resisted our efforts. Along these lines we can only manage

Lemma 3. *There exists an $\epsilon = \epsilon(c)$ such that,*

$$\mathbf{Pr}(\exists K \subseteq R: \ |K| \leq \epsilon n \ and \ |N(K)| < |K|) = O(n^{-1}).$$

However this lemma can be used in the proof of the following: Let $\mu^*(G)$ denote the size of a maximum matching in G.

Lemma 4. *For $t \geq 1$, $m \sim cn$, $\epsilon_1 > 0$ and $\omega = K \log n$ for some sufficiently large $K = K(c)$,*

If $\mathbf{Pr}(\mu^*(B_{n,m-\omega}^{\delta \geq 2} \geq n - t) \geq 1 - \epsilon_1$ *then*

$$\mathbf{Pr}(\mu^*(B_{n,m}^{\delta \geq 2}) \geq n - t + 1) \geq 1 - \epsilon_1 + \frac{(\log n)^3}{n^{1/2}}.$$

Lemma 5. *If $m \sim cn$ then*

$$\mathbf{Pr}(\mu^*(B_{n,m}^{\delta \geq 2}) \leq n - O(n^{.49})) \leq n^{-4}.$$

With these two preceding lemmas we can easily prove Theorem 2. Let $m \sim cn$ and let $m_r = m - r\omega$, $r = 0, 1, \ldots, n^{.49}$. Then

$$\mathbf{Pr}(\mu^*(B_{n,m_r}^{\delta \geq 2}) \leq n - r) \leq n^{-1} + (O(n^{.49}) - r)\frac{(\log n)^3}{n^{1/2}}. \tag{58}$$

This is proved by downwards induction on r with the base case $r = An^{.49}$ (for some constant $A > 0$) being verified by Lemma 5 and Lemma 4 providing the inductive step. Theorem 2 is the case $r = 0$ of (58).

4.1 Proof of Lemma 3

In the notations of the proof of Theorem 1, we need to show that

$$\sum_{\substack{2\le k\le \epsilon m\\ 2k\le\mu<m}} E_{nk\mu}\to 0, \tag{59}$$

for some $\epsilon>0$. Here $E_{nk\mu}$ is the expected number of the minimal k-witnesses (K,L), $|K|=k,|L|=k-1$, $N(K)=L$, with μ edges. We know that every vertex from L has at least two neighbors from K. Picking $z\in(0,\rho)$, we use (36) with $r_1=r_2=z$ to bound N_1, the number of such bipartite graphs. We use (35) with $r_1=r_2=\rho$ to bound N_2, the total number of feasible bipartite graphs that remain after deletion of all vertices belonging to K. Here ρ is the parameter of truncated Poissons Y_i in Corollary 4, i.e. $\rho\in(0,\infty)$ satisfies

$$\frac{\rho f_1(\rho)}{f_2(\rho)}=\frac{\rho(e^\rho-1)}{e^\rho-1-\rho}=\frac{m}{n}=c. \tag{60}$$

Using Corollary 4 and (40), we obtain that $E_{nk\mu}$ is of order $E^*_{nk\mu}$ at most, where

$$E^*_{nk\mu}=\frac{\binom{n}{k}\binom{n}{k-1}}{kz\binom{m}{\mu}}\cdot\frac{f_2(z)^{2k-1}}{z^{2\mu}}\cdot\frac{\rho^{2\mu}e^{(k-1)\rho}}{f_2(\rho)^{2k-1}}. \tag{61}$$

Consequently

$$\frac{E^*_{nk,\mu+1}}{E^*_{nk\mu}}=\frac{\mu+1}{m-(\mu+1)}\cdot\frac{\rho^2}{z^2}\le\begin{cases}\frac{1}{2} & \mu<\overline{\mu}:=\lfloor\frac{z^2}{3\rho^2}m\rfloor\\ \frac{2}{3} & \mu\le\lfloor 1.1\overline{\mu}\rfloor\end{cases} \tag{62}$$

if z is small. Therefore, if $k\le 0.5\overline{\mu}$, then

$$\sum_{\mu=2k}^{\max\{\overline{\mu},2.2k\}} E^*_{nk\mu}\le_b E^*_{nk,2k}. \tag{63}$$

Furthermore, using (61) for k in question and $\mu=2k$, $rf_2(\rho)=\rho(e^\rho-1)$ and the equation (60),

$$\begin{aligned}\frac{E^*_{n,k+1,2(k+1)}}{E^*_{nk,2k}} &\le 4\frac{(n-k)_2}{(m-2k)_2}\cdot\frac{f_2(z)^2}{z^4}\cdot\frac{\rho^4e^\rho}{f_2(\rho)^2}\\ &= (1+O(z^2))\left(\frac{f_2(z)}{z^2/2}\right)^2\left(\frac{\rho/2}{\sinh\rho/2}\right)^2\\ &\le \frac{1+O(z)}{1+\rho^2/12}\\ &< 1,\end{aligned}$$

provided that z is chosen sufficiently small. So

$$\sum_{k=2}^{\overline{\mu}/2}E^*_{nk,2k}\le_b E^*_{n24}=O(n^3m^{-4})=O(n^{-1}),$$

and, invoking (63),

$$\sum_{k=2}^{\overline{\mu}/2}\sum_{\mu=2k}^{\max\{\overline{\mu},2.2k\}} E_{nk\mu}=O(n^{-1}). \tag{64}$$

Let us bound $E_{nk\mu}$ for the same k's, but $\mu > \max\{\overline{\mu}, 2.2k\}$. To bound N_1 this time, we choose $r_1 = r_2 = \mu/k$. In particular, $r_i \geq 2.2$, thus bounded away from zero, just like the optimal r, the root of $rf_1(r)/f_2(r) = \mu/k$. (For $\mu = 2k$, the root would be zero!) Using $f_2(\mu/k) \leq e^{\mu/k}$, we get

$$E_{nk\mu} \leq_p \exp(nJ(x,y)), \quad x = k/n,\ y = \mu/n, \tag{65}$$

where

$$J(x,y) = 2H(x) - cH(y/c) + 2y\log\frac{x\rho e}{y} + x\log\frac{e^\rho}{f_2(\rho)^2},$$

and $(x,y) \in D$,

$$\begin{aligned} D &:= \{(x,y)|\, x \leq \overline{x},\ y \geq \max\{\overline{y}, 2.2x\}\}, \\ \overline{x} &:= \frac{cz^2}{6\rho^2}, \quad \overline{y} := \frac{cz^2}{3\rho^2}. \end{aligned}$$

Notice at once that, for $(x,y) \in D$,

$$\begin{aligned} J_x(x,y) &= 2(-\log x + \log(1-x)) + \frac{2y}{x} + \log\frac{e^\rho}{f_2(\rho)^2} \\ &\geq 4\log z^{-1} + O(1) > 0, \end{aligned}$$

if z is small enough. For such a z, $J(x,y)$ increases with x for every y, as long as $(x,y) \in D$. In addition, the equation $J_y(x,y) = 0$ has two roots $y_\pm(x) > 0$,

$$y_\pm = \frac{1}{2}(c \pm \sqrt{c^2 - 4\rho^2 x^2}),$$

as $x \leq 1/2$ and $\rho < c$, and $y_- = O(x^2) = O(z^4)$, $y_+ = c - O(z^4)$. In particular, $y^*(x) := \max\{\overline{y}, 2.2x\} \in (y_-, y_+)$, if z is sufficiently small. Furthermore, as a function of y, $J(x,y)$ decreases on $[y_-, y_+]$, and increases on $[y_+, c]$. Therefore,

$$\max\{J(x,y) :\ (x,y) \in D\} = \max_{x \leq \overline{x}}\{J(x, y^*(x)), J(x,c)\}.$$

If $y^*(x) = 2.2x$, then

$$\begin{aligned} J(x,y) &= 2x\log(1/x) - 2.2x\log(1/x) + O(x) \\ &= -0.2x\log(1/x) + O(x) \\ &\leq -0.05\frac{cz^2}{\rho^2}\log(1/z). \end{aligned}$$

If $y^*(x) = \overline{y}$, so that $x \leq \overline{y}/2.2$, then

$$\begin{aligned} J(x,y) &= 2x\log(1/x) - \overline{y}\log(1/\overline{y}) + O(\overline{y}) \\ &\leq \left(\frac{2}{2.2} - 1\right)\overline{y}\log(1/\overline{y}) + O(\overline{y}) \\ &\leq -0.05\frac{cz^2}{\rho^2}\log(1/z). \end{aligned}$$

Finally

$$J(x,c) = -2(c-2)\log(1/x) + O(1).$$

Therefore, for z sufficiently small,

$$\max\{J(x,y) :\ (x,y) \in D\} \leq -0.05\frac{cz^2}{\rho^2}\log\frac{1}{z} < 0.$$

¿From (65) we get then

$$E_{nk\mu} \leq e^{-\alpha n}, \quad \alpha = -0.04 \frac{cz^2}{\rho^2},$$

so

$$\sum_{k=2}^{\overline{\mu}/2} \sum_{\mu \geq \max\{\overline{\mu}.2.2k\}} E_{nk\mu} \leq e^{-\alpha^* n}, \quad \alpha^* = 0.03 \frac{cz^2}{\rho^2} \log \frac{1}{z}.$$

Combining this with (64) and the definition of $\overline{\mu}$ in (62), we prove (59) with $\epsilon = \frac{cz^2}{6\rho^2}$. □

4.2 Proof of Lemma 4

Let $\omega = \lceil K \log n \rceil$ for some large constant $K > 0$. Consider the bipartite graph Γ with vertex set $\mathcal{B}_{n,m-\omega}^{\delta \geq 2} + \mathcal{B}_{n,m}^{\delta \geq 2}$ and an edge (G, H) iff

$$E(G) \subseteq E(H) \text{ and } E(G) \setminus E(H) \text{ is a matching.}$$

Consider the following experiment SAMPLE:

- Choose G randomly from $\mathcal{B}_{n,m-\omega}^{\delta \geq 2}$
- Add a random matching M, disjoint from $E(G)$ of size ω to obtain $H \in \mathcal{B}_{n,m}^{\delta \geq 2}$.

This induces a probability measure $\mathbf{Q}$ on $\mathcal{B}_{n,m}^{\delta \geq 2}$. Let d_Γ denote degree in Γ.

Lemma 6.

$G \in \mathcal{B}_{n,m-\omega}^{\delta \geq 2}$ *implies*

$$\frac{(n^2 - m - 2\omega n)^\omega}{\omega!} \leq d_\Gamma(G) \leq \binom{n^2}{\omega}.$$

Proof The RHS is obvious. For the LHS let us bound from below the number of *ordered* sequences $e_1, e_2, \dots, e_\omega$ of ω edges which are disjoint from $E(G)$ and form a matching. Observe that after choosing $e_1, e_2, \dots, e_i$ we rule out at most $m - \omega + 2in$ choices for e_{i+1}. (The $m - \omega$ edges of G plus the further $\leq 2in$ edges incident with $e_1, e_2, \dots, e_i$). Thus there are always at least $n^2 - m - 2\omega n$ choices for e_{i+1}. Dividing by $\omega!$ accounts for removing the ordering. □

Thus for n large and $G, G' \in \mathcal{B}_{n,m-\omega}^{\delta \geq 2}$,

$$\left| \frac{d_\Gamma(G)}{d_\Gamma(G')} - 1 \right| \leq \frac{4\omega^2}{n}. \tag{66}$$

We now consider the degrees $d_\Gamma(H)$ for $H \in \mathcal{B}_{n,m}^{\delta \geq 2}$.

For $H \in \mathcal{B}_{n,m}^{\delta \geq 2}$ let $E_>(H)$ be the edges of H joining vertices of degree at least 3. How large is $E_>(H)$? If $e \in E(H) \setminus E(G)$ then other edges of H incident to e must already be in $E(G)$. So, if (G, H) is an edge of Γ then $E(H) \setminus E(G) \subseteq E_>(H)$.

Lemma 7. *Let*

$$\theta = c^{-1} y^2, \text{ where } \frac{y(e^y - 1)}{e^y - 1 - y} = c.$$

If H is chosen uniformly at random from $\mathcal{B}_{n,m}^{\delta \geq 2}$ then **qs**[1]

[1]A sequence of events $\mathcal{E}_n$ is said to occur *quite surely* if $\mathbf{Pr}(\mathcal{E}_n) = 1 - O(n^{-K})$ for any $K > 0$.

(a)

$$\Delta(H) \leq \log n.$$

(b)

$$|E_>(H) - \theta n| = O(n^{1/2} \log n).$$

Proof Let $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$ be independent copies of $\text{Po}(y; \geq 2)$. Next let $m_{\mathbf{X}} = X_1 + \cdots + X_n$ and $m_{\mathbf{Y}} = Y_1 + \cdots + Y_n$. Then, given $\mathbf{X}, \mathbf{Y}$, let $(x_1, x_2, \ldots, x_{m_{\mathbf{X}}})$ be a (uniformly) random permutation σ_X of the multi-set $\{1^{X_1}, \ldots, n^{X_n}\}$, i.e. the set which contains X_i copies of the row vertex i, and let $(y_1, y_2, \ldots, y_{m_{\mathbf{Y}}})$ be a random permutation σ_Y of the multi-set $\{1^{Y_1}, \ldots, n^{Y_n}\}$, containing Y_i copies of the column vertex i, $1 \leq i \leq n$. Conditional on $\mathcal{E}_0 = \{m_{\mathbf{X}} = m_{\mathbf{Y}} = m \text{ and (simplicity)}\}$ the pairing $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ defines a uniformly random graph $H \in \mathcal{B}_{n,m}^{\delta \geq 2}$. In the notation of Section 2.1 we have that

$$\mathbf{Pr}(\mathcal{E}_0) = \mathbf{E}\left(F(\mathbf{Y}, \mathbf{Z}) \cdot 1_{\{\sum_i Y_i = m\}} 1_{\{\sum_j Z_j = m\}}\right).$$

Then comparing (22) and (37), and using (39) and (40) we see that

$$\mathbf{Pr}(\mathcal{E}_0) = (1 + o(1))\frac{\beta}{n}, \quad \beta = \frac{\exp\left(-\frac{y^2 e^{2y}}{2(e^y - 1)^2}\right)}{2\pi Var(P_0(y))}, \tag{67}$$

i.e. $\mathbf{Pr}(\mathcal{E}_0)$ is only polynomially small. This implies that if $\{(\mathbf{X}, \mathbf{Y}) \in A\}$, $(A \supset \{2, 3, \ldots\}^{2n})$, is a **qs**event, then so is the event $\{\deg(H) \in A\}$, $\deg(H)$ denoting the degree sequence of $H \in \mathcal{B}_{n,m}^{\delta \geq 2}$. The part (a) follows then immediately since, for $L = \log n$,

$$\mathbf{Pr}(\max(\max_i X_i, \max_j Y_j) \geq L) \leq 2n\mathbf{Pr}(X_1 \geq L = \log n) = O(ny^L / L!),$$

which is $O(n^{-K})$ for any $K > 0$. Turn to (b). Let W be the number of pairs (x_i, y_i), $i \leq \min\{m_{\mathbf{X}}, m_{\mathbf{Y}}\}$ such that X_{x_i}, $Y_{y_i} \geq 3$. We know that, conditioned $\mathcal{E}_0$, $W = E_>(H)$. Assuming for certainty that $m_{\mathbf{X}} \leq m_{\mathbf{Y}}$, we see that

$$\mathbf{E}(W \mid \mathbf{X}, \mathbf{Y}) = \frac{m_{\mathbf{X},3} m_{\mathbf{Y},3}}{m_{\mathbf{X}}}, \tag{68}$$

where

$$m_{\mathbf{X},3} = \sum_i X_i \mathbf{I}_{\{X_i > 2\}}, \quad m_{\mathbf{Y},3} = \sum_j Y_j \mathbf{I}\{Y_j > 2\}.$$

Now

$$\begin{aligned} \mathbf{E}(m_{\mathbf{X}}) &= n\mathbf{E}X_1 = n\frac{y(e^y - 1)}{e^y - 1 - y} = nc, & (69) \\ \mathbf{E}(m_{\mathbf{X},3}) &= n(\mathbf{E}X_1 - 2\mathbf{Pr}(X_1 = 2)) = n\left(\frac{y(e^y - 1)}{e^y - 1 - y} - \frac{y^2}{e^y - 1 - y}\right) & (70) \\ &= ny, & (71) \end{aligned}$$

with the same results for $\mathbf{E}(m_{\mathbf{Y}})$, $\mathbf{E}(m_{\mathbf{Y},3})$. Now $m_{\mathbf{X}}$, $m_{\mathbf{Y}}$ are the sums of independent copies of $\text{Po}(y; \geq 2)$, whose pgf is $(e^{xy} - 1 - xy)/(e^y - 1 - y)$. Using this function and (69), in a standard (Chernoff-type) way, we obtain that **qs**

$$|m_{\mathbf{X}} - nc| \leq n^{1/2} \log n, \quad |m_{\mathbf{Y}} - nc| \leq n^{1/2} \log n. \tag{72}$$

Analogously, or using the Chernoff tail bounds for the binomial distribution, we obtain also that **qs**

$$|m_{\mathbf{X},3} - ny| \leq n^{1/2} \log n, \quad |m_{\mathbf{Y},3} - ny| \leq n^{1/2} \log n. \tag{73}$$

Suppose now the conditions (72)) and (73) hold simultaneously, which we call the event $\mathcal{E}_1$. Then $\mathcal{E}_1$ holds **qs**. It follows from (68) that

$$\mathbf{E}(W \mid \mathcal{E}_1) = \theta n + O(n^{1/2} \log n), \quad \theta := c^{-1} y^2. \tag{74}$$

We can now use the Azuma-Hoeffding inequality to show concentration of W on the event $\mathcal{E}_1$. Transposing any two elements of σ_X and any two elements of σ_Y may change W by at most 2. So, for every $u > 0$,

$$\mathbf{Pr}(|W - \mathbf{E}(W \mid \mathcal{E}_1)| \geq u \mid \mathcal{E}_1) \leq 2e^{-u^2/(9cn)}.$$

Removing the conditioning on $\mathcal{E}_1$ we obtain

$$\mathbf{Pr}(|W - \mathbf{E}(W \mid \mathcal{E}_1)| \geq u) \leq \mathbf{Pr}(\mathcal{E}_1^c) + 2e^{-u^2/(8cn)}.$$

So, substituting $u = n^{1/2} \log n$ and using (74), we see that **qs**

$$|W - \theta n| \leq A n^{1/2} \log n,$$

if a constant A is sufficiently large. Recalling that $W = E_>(H)$ on the event $\mathcal{E}_0$, and that $\mathbf{Pr}(\mathcal{E}_0)$ is of order n^{-1}, we have proved the part (b). □

Now let $\tilde{\mathcal{B}}$ be the set of $H \in \mathcal{B}_{n,m}^{\delta \geq 2}$ satisfying the conditions of the above lemma i.e.

- The number of edges joining two vertices of degree ≥ 3 is in the range $\theta n \pm A n^{1/2} \log n$ for some constant $A > 0$.
- The maximum degree $\Delta(H) \leq \log n$.

According to the lemma

$$|\mathcal{B}_{n,m}^{\delta \geq 2} \setminus \tilde{\mathcal{B}}| \leq |\tilde{\mathcal{B}}| n^{-K}, \quad \forall K > 0. \tag{75}$$

Note next that

Lemma 8.

$H \in \tilde{\mathcal{B}}$ implies

$$\frac{(\theta n - A n^{1/2} \log n - 2\omega \log n)^\omega}{\omega!} \leq d_\Gamma(H) \leq \binom{\theta n + A n^{1/2} \log n}{\omega}.$$

Proof The upper bound is obvious. As in Lemma 6, for the LHS let us bound from below the number of *ordered* sequences $e_1, e_2, \ldots, e_\omega$ of ω edges which are contained in $E_>(H)$ and form a matching. Observe that after choosing $e_1, e_2, \ldots, e_i$ we rule out at most $2i\Delta$ choices for e_{i+1}. Thus there are always at least $\theta n - A n^{1/2} \log n - 2\omega\Delta$ choices for e_{i+1}. Dividing by $\omega!$ accounts for removing the the ordering. □

So for $H, H' \in \tilde{\mathcal{B}}$,

$$\left| \frac{d_\Gamma(H)}{d_\Gamma(H')} - 1 \right| \leq \frac{2A\omega \log n}{\theta n^{1/2}}. \tag{76}$$

Finally, for $H \in \mathcal{B}_{n,m}^{\delta \geq 2} \setminus \tilde{\mathcal{B}}$, $H' \in \tilde{\mathcal{B}}$,

$$\frac{d_\Gamma(H)}{d_\Gamma(H')} \leq \frac{\binom{m}{\omega}}{\frac{(\theta n - A n^{1/2} \log n - 2\omega \log n)^\omega}{\omega!}} \leq \left(\frac{2c}{\theta} \right)^\omega, \tag{77}$$

as the total number of ways to delete a matching of size ω from $H \in \mathcal{B}_{n,m}^{\delta\geq 2}$ is $\binom{m}{\omega}$ at most.

Let $G_0 \in \mathcal{B}_{n,m-\omega}^{\delta\geq 2}$ be fixed. By (66), if $H \in \mathcal{B}_{n,m}^{\delta\geq 2}$ then

$$\begin{aligned}
\mathbf{Q}(H) &= \mathbf{Pr}(\textsc{sample} \text{ chooses } H) \\
&= \frac{1}{|\mathcal{B}_{n,m-\omega}^{\delta\geq 2}|} \times \sum_{(G,H)\in E(\Gamma)} \frac{1}{d_\Gamma(G)} \\
&= \frac{1+O(\omega^2/n)}{|\mathcal{B}_{n,m-\omega}^{\delta\geq 2}|} \cdot \frac{d_\Gamma(H)}{d_\Gamma(G_0)}. \qquad (78)
\end{aligned}$$

From this relation, (76), and (77), it follows that

$$H, H' \in \tilde{\mathcal{B}} \quad \text{implies} \quad \left|\frac{\mathbf{Q}(H)}{\mathbf{Q}(H')} - 1\right| \leq \frac{3A\omega \log n}{\theta n^{1/2}}, \qquad (79)$$

$$H \in \mathcal{B}_{n,m}^{\delta\geq 2} \setminus \tilde{\mathcal{B}},\ H' \in \tilde{\mathcal{B}} \quad \text{implies} \quad \frac{\mathbf{Q}(H)}{\mathbf{Q}(H')} \leq \left(\frac{3c}{\theta}\right)^{\omega}. \qquad (80)$$

Furthermore, invoking also

$$\sum_{G\in\mathcal{B}_{n,m-\omega}^{\delta\geq 2}} d_\Gamma(G) = \sum_{H\in\mathcal{B}_{n,m}^{\delta\geq 2}} d_\Gamma(H),$$

and picking $H' \in \tilde{\mathcal{B}}$, we obtain (see (66), (76)):

$$\frac{d_\Gamma(H')}{d_\Gamma(G_0)} \leq \left(1 + \frac{3A\omega \log n}{\theta n^{1/2}}\right) \frac{|\mathcal{B}_{n,m-\omega}^{\delta\geq 2}|}{|\tilde{\mathcal{B}}|}. \qquad (81)$$

Combining (75), (78), (80), and (81), we get: for every $K > 0$,

$$\begin{aligned}
\mathbf{Q}(\mathcal{B}_{n,m}^{\delta\geq 2} \setminus \tilde{\mathcal{B}}) &\leq \mathbf{Q}(H')\left(\frac{3c}{\theta}\right)^{\omega} \cdot n^{-2K}|\tilde{\mathcal{B}}| \\
&= \frac{1+O(\omega^2/n)}{|\mathcal{B}_{n,m-\omega}^{\delta\geq 2}|} \cdot \frac{d_\Gamma(H')}{d_\Gamma(G_0)} \left(\frac{3c}{\theta}\right)^{\omega} n^{-2K}|\tilde{\mathcal{B}}| \\
&= O\big((3c/\theta)^{\omega} n^{-2K}\big) \\
&\leq n^{-K}. \qquad (82)
\end{aligned}$$

Since $\mathbf{Q}(\mathcal{B}_{n,m}^{\delta\geq 2}) = 1$, from (82) and (79) we deduce that, for $H \in \tilde{\mathcal{B}}$,

$$\left|\mathbf{Q}(H) - \frac{1}{|\mathcal{B}_{n,m}^{\delta\geq 2}|}\right| \leq \frac{1}{|\mathcal{B}_{n,m}^{\delta\geq 2}|} \times \frac{4A\omega \log n}{\theta n^{1/2}} \times \frac{1}{1-n^{-L}} \leq \frac{1}{|\mathcal{B}_{n,m}^{\delta\geq 2}|} \times \frac{5A\omega \log n}{\theta n^{1/2}}. \qquad (83)$$

This means that on the graph set $\tilde{\mathcal{B}}$ the probability measure $\mathbf{Q}$ is almost uniform. It is worth repeating that $\mathbf{Q}$ is induced by picking a random $G \in \mathcal{B}_{n,m-\omega}^{\delta\geq 2}$, and adding to G a random matching M of cardinality ω which is disjoint from $E(G)$.

Now let $\mathbf{Pr}_M$ denote probability w.r.t. a graph chosen uniformly from $\mathcal{B}_{n,M}^{\delta\geq 2}$ and let $\mu^*(G)$ denote the size of the largest matching in G. We want to prove, using the near uniformity of $\mathbf{Q}$, that $\mathbf{Pr}_m(\mu^*(G) = n) \to 1$, if $m = cn$ and $c > 2$.

From the previous part we know that there exists $\alpha = \alpha(c)$ such that

$$\mathbf{Pr}_{m-\omega}(|N(S)| \geq |S| : \forall\, S \subset [n],\ 0 < |S| \leq \alpha n) \geq 1 - \frac{\gamma}{n} \qquad (84)$$

for some $\gamma = \gamma(c)$. (Here S is a set rows, or a set of columns.)

Now, given $G \in \mathcal{B}^{\delta\geq 2}_{n,m-\omega}$ such that $\mu^*(G) \in [n-t, n)$ and the event in (84), fix some matching M of size $n-t$ and let x be a row vertex and y be a column vertex not covered by M.

Suppose G does not contain a matching of size $n-t+1$, i.e. $\mu^*(G) = n-t$. Let A be the set of row vertices reachable from x by an alternating path w.r.t. M, and let B, the set of column vertices, be defined analogously for y. (Of course, the sets A and B depend on the choice of a maximum matching M. To achieve uniqueness, we assume that M is the lexicographically first among all maximum matchings.) Each such path is of even length, and we include x into A, and y into B, as corresponding to the paths of zero length. There does not exist an edge connecting A and B, since otherwise we could use the resulting path between x and y to get, in a standard way, a larger matching. (Therefore if any of the ω edges added to G in SAMPLE join A to B, $\mu^*(G') > \mu^*(G)$ for the new graph G'.) Furthermore, for every row vertex in A, all its column neighbors must be covered by M, since otherwise there would exist an alternating path connecting x and an uncovered column vertex, and there would exist a larger matching M'. This implies that $N(A)$ consists of all column vertices on the paths from x, so that $|N(A)| = |A| - 1$, as x is the only vertex in A not covered by M. Similarly, $|N(B)| = |B| - 1$. Then necessarily $|A| \geq \alpha n$, $|B| \geq \alpha n$. So if G is such that the event in (84) holds, then—conditioned on G—the probability that none of the ω added edges of SAMPLE join A to B is at most

$$\left(1 - \frac{(\alpha n - \omega)^2}{n^2}\right)^{\omega} \leq \left(1 - \frac{\alpha^2}{2}\right)^{\omega} \leq \frac{\gamma}{n},$$

if we pick K in $\omega = \lceil K \log n \rceil$ sufficiently large. Therefore, if $H \in \mathcal{B}^{\delta\geq 2}_{n,m}$ then

$$\begin{aligned}
\mathbf{Q}(\mu^*(H) < n-t+1) &\leq \mathbf{Pr}_{m-\omega}(\mu^*(G) < n-t) + \frac{\gamma}{n} + (1-\alpha^2/2)^{\omega} \leq \epsilon(m-\omega, t) + \frac{2\gamma}{n}; \\
\epsilon(m-\omega, t) : &= \mathbf{Pr}_{m-\omega}(\mu^*(G) < n-t) = 1 - \mathbf{Pr}_{m-\omega}(\mu^*(G) \geq n-t).
\end{aligned}$$

So

$$\mathbf{Q}(\{\mu^*(H) < n-t+1\} \wedge \{H \in \tilde{\mathcal{B}}\}) \leq \epsilon(m-\omega, t) + \frac{2\gamma}{n}$$

and then, using (83),

$$\mathbf{Pr}_m(\{\mu^*(H) < n-t+1\} \wedge \{H \in \tilde{\mathcal{B}}\}) \leq \left(\epsilon(m-\omega, t) + \frac{2\gamma}{n}\right)\left(1 + \frac{5A\omega \log n}{\theta n^{1/2}}\right)$$

and

$$\begin{aligned}
\mathbf{Pr}_m(\mu^*(H) < n-t+1) &\leq \left(\epsilon(m-\omega, t) + \frac{2\gamma}{n}\right)\left(1 + \frac{5A\omega \log n}{\theta n^{1/2}}\right) + n^{-2L} \\
&\leq \epsilon(m-\omega, t) + \frac{6A\omega \log n}{\theta n^{1/2}},
\end{aligned}$$

(where n^{-2L} bounds $\mathbf{Pr}_m(H \notin \tilde{\mathcal{B}})$).

To summarise, using the notation $\mathbf{Pr}_M$ for the probability distribution of the graph chosen randomly from $\mathcal{B}^{\delta\geq 2}_{n,M}$ and

$$\mathbf{Pr}_M(\mu^*(G) < n-\tau) = \epsilon(M, \tau),$$

we have

$$\epsilon(m, t-1) \leq \epsilon(m-\omega, t) + \frac{6A\omega \log n}{\theta n^{1/2}},$$

provided that $m = cn$ and $c > 2$. So, shifting m and iterating this t times,

$$\epsilon(m, 0) \leq \epsilon(m - t\omega, t) + \frac{7At\omega \log n}{n^{1/2}}.$$

Thus it only remains to show that **whp** $G_{m-O(n^{1/2})}$ has a matching of size at least $n - n^{1/4}$.

4.3 Proof of Lemma 5

We first describe the graph model we use for our analysis. It is a bipartite version of the "random sequence model" considered in Section 2 of [14]. Let $\mu \geq 1$ and the disjoint sets R, C be given. R and C have meaning of a row set and a column set respectively, and μ a number of edges. For $\mathbf{x} \in R^\mu$ and $\mathbf{y} \in C^\mu$, we define a multi-bipartite graph $G_{\mathbf{x},\mathbf{y}}$ as having a vertex set $R + C$, and the edge set $E(G_{\mathbf{x},\mathbf{y}}) = \{(x_\ell, y_\ell);\ 1 \leq \ell \leq \mu\}$. Then the *degree* of $i \in R$ ($j \in C$ resp.) in $G\mathbf{x},\mathbf{y}$ equals $d_{\mathbf{x}}(i) = |\{\ell \in [\mu] :\ x_\ell = i\}|$ ($d_{\mathbf{x}}(j) = |\{\ell \in [\mu] :\ y_\ell = j\}|$ resp.). Define

$$R^\mu_{\geq 1} = \{\mathbf{x} \in R^\mu :\ d_{\mathbf{x}}(i) \geq 1,\ i \in R\}, \quad C^\mu_{\geq 1} = \{\mathbf{y} \in C^\mu :\ d_{\mathbf{y}}(j) \geq 1,\ j \in C\}.$$

That is $R^\mu_{\geq 1}$, say, is a set of all $\mathbf{x}$ such that every $i \in R$ has positive degree in $\mathbf{x}$. For $\boldsymbol{\nu} = (v_{1,R}, v_{1,C}, v_R, v_C, \mu, \zeta)$, let

$$\begin{aligned}\mathcal{B}_{R,C}(\boldsymbol{\nu}) = \{(\mathbf{x},\mathbf{y}) \in R^\mu_{\geq 1} \times C^\mu_{\geq 1} :&\\ &\text{there are } v_{1,R} \text{ indices of degree 1 in } R\\ &\text{there are } v_R \text{ indices of degree} \geq 2 \text{ in } R\\ &\text{there are } v_{1,C} \text{ indices of degree 1 in } C\\ &\text{there are } v_C \text{ indices of degree} \geq 2 \text{ in } C\}\end{aligned}$$

Thus $\mathcal{B}_{R,C}(\boldsymbol{\nu})$ is the set of all multi-bipartite graphs $G_{\mathbf{x},\mathbf{y}}$ without isolated vertices, and with the specified numbers of (*light*) vertices of degree 1, and of (*heavy*) vertices of degree 2 at least, separately among the row vertices and the column vertices. Clearly the total degree of v_R (v_C resp.) heavy vertices is $\mu - v_{1,R}$ ($\mu - v_{1,C}$ resp.). We will also use

$$v_1 = v_{1,R} + v_{1,C} \text{ and } v = v_R + v_C.$$

Clearly as well, the total degree of all v heavy vertices is $2\mu - v_1$. We first discuss the degree sequence of $G_{\mathbf{x},\mathbf{y}}$. Fix $\mathbf{x}, \mathbf{y}$ and let $R_1 = \{i \in R :\ d_{\mathbf{x}}(i) = 1\}$, $C_1 = \{j \in C :\ d_{\mathbf{y}}(j) = 1\}$.

Lemma 9. *Suppose that $(\mathbf{x}, \mathbf{y})$ is chosen uniformly at random from $\mathcal{B}_{R,C}(\boldsymbol{\nu})$. Then $\{d_{\mathbf{x}}(i) :\ i \in R \setminus R_1\}, \{d_{\mathbf{y}}(j) :\ j \in C\}$ is distributed as*

$$(\mathbf{Z}, \mathbf{Z}') = (\{Z_i :\ i \in R \setminus R_1\}, \{Z'_j :\ j \in C \setminus C_1\}).$$

Here Z_i are independent copies of $Po(\lambda; \geq 2)$ conditioned on $\sum_{i \in R\setminus R_1} Z_i = \mu - v_{1,R}$, Z'_j are independent copies of $Po(\lambda'; \geq 2)$ conditioned on $\sum_{j\in C\setminus C_1} Z'_j = \mu - v_{1,C}$, and $\mathbf{Z}$, $\mathbf{Z}'$ are mutually independent. The parameters $\lambda > 0$, λ' are arbitrary.

Proof Since $\mathbf{x}$, $\mathbf{y}$ are mutually independent, it suffices to consider $\mathbf{x}$ only. Assume without loss of generality that $R \setminus R_1 = [v_R]$. Let $s = \mu - v_{1,R}$ and

$$S = \left\{\vec{x} \in [s]^{v_R} :\ \sum_{1 \leq i \leq v_R} x_i = s \text{ and } \forall i,\ x_i \geq 2\right\}.$$

Fix $\vec{\xi} \in S$. Then, by the definition of $\mathbf{x}$ and $\vec{Z} = \{Z_i\}_{i \in [v_R]}$,

$$\mathbf{Pr}(\vec{\mathbf{x}} = \vec{\xi}) = \left(\frac{s!}{\xi_1!\xi_2!\ldots\xi_v!}\right) \Big/ \left(\sum_{\vec{x}\in S} \frac{s!}{x_1!x_2!\ldots x_v!}\right).$$

On the other hand,

$$\begin{aligned}
\mathbf{Pr}\left(\vec{Z}=\vec{\xi}\,\middle|\,\sum_{1\le i\le v_R} Z_i=s\right) &= \left(\prod_{1\le i\le v_R}\frac{z^{\xi_i}}{(e^z-1-z)\xi_i!}\right)\Bigg/\left(\sum_{\vec{x}\in S}\prod_{1\le j\le v_R}\frac{z^{x_i}}{(e^z-1-z)x_i!}\right)\\
&= \left(\frac{(e^z-1-z)^{-v_R}z^s}{\xi_1!\xi_2!\dots\xi_{v_R}!}\right)\Bigg/\left(\sum_{\vec{x}\in S}\frac{(e^z-1-z)^{-v}z^s}{x_1!x_2!\dots x_{v_R}!}\right)\\
&= \mathbf{Pr}(\mathbf{x}=\vec{\xi}).
\end{aligned}$$

□

To make the most out of this underlying independence, we set—like on other similar occasions in this paper—$\lambda = z_R$, $\lambda' = z_C$, where

$$\frac{\mu - v_{1,R}}{v_R} = \frac{z_R(e^{z_R}-1)}{f_R},\quad \frac{\mu - v_{1,C}}{v_C} = \frac{z_C(e^{z_C}-1)}{f_C},$$

where $f(x) = f_2(x) = e^x - 1 - x$ and we have abbreviated $f_R = f(z_R)$, $f_C = f_2(z_C)$. We will also use z, the root of

$$\frac{\mu - v_1}{v} = \frac{z(e^z-1)}{f(z)}.$$

In our analysis below we will only need to consider graphs for which

$$v_1 \le_b n^{.32} < n^{.64} \le_b v_R, v_C \le n. \tag{85}$$

$$z_R, z_C \le 3c. \tag{86}$$

Lemma 10. *Suppose that $v_Rz_R \to \infty$ and a is such that $a^2(v_Rz_R)^{-1} \to 0$. Then*

$$\mathbf{Pr}\left(\sum_{i\in R\setminus R_1} Z_i = \mu - v_{1,R} - a\right) = \frac{1 + O((1+a^2)(v_Rz_R)^{-1})}{(2\pi v_R \mathbf{Var}(Z))^{1/2}},$$

where $Z = Po(z_R; \ge 2)$. Analogous estimate holds for the column set C.

Proof This follows immediately from Lemma 1. □

Notice that $\mathbf{Var}(Z) = \Theta(z_R)$. Using Lemma 10 with $a = 0$, we see that the probability of the conditioning event is of order $(v_Rz_R)^{-1/2} \ge \Theta(n^{-1/2})$. So, a **qs** event expressed in terms of $\vec{Z}$ remains a **qs** event when $\vec{Z}$ is replaced by $\vec{x}$. The same relation holds between $\vec{Z}'$ and $\vec{y}$. In particular, since

$$\mathbf{Pr}(Po(z_R; \ge 2) \ge \log n) = O(n^{-\Omega(\log\log n)})$$

we have that for $(\mathbf{x}, \mathbf{y})$ chosen uniformly from $\mathcal{B}_{R,C}(\boldsymbol{\nu})$, with probability $1 - O(n^{-\Omega(\log\log n)})$,

$$\max\{\Delta_{\mathbf{x}}, \Delta_{\mathbf{y}}\} \le \log n, \tag{87}$$

where $\Delta_{\mathbf{x}} = \max_{j\in R}\{d_{\mathbf{x}}(j)\}$ and $\Delta_{\mathbf{y}} = \max_{j\in C}\{d_{\mathbf{y}}(j)\}$.

We next look at the expected number of vertices of a given degree in $G_{\mathbf{x},\mathbf{y}}$. We use the notation $V_{k,X}$, $X = R, C$ to denote the set of vertices of degree of degree k in X and $v_{k,X} = |V_{k,X}|$.

Lemma 11. *For vertices $i \in R \setminus R_1, j \in C \setminus C_1$, and $2 \le k, l \le \log n$,*

$$\mathbf{Pr}(d_{\mathbf{x}}(i) = k) = \frac{z_R^k}{k!f_R}\left(1 + O\left(\frac{(\log v)^2}{v_Rz_R}\right)\right) \tag{88}$$

$$\mathbf{Pr}(d_{\mathbf{x}}(i) = k, d_{\mathbf{y}}(j) = \ell) = \frac{z_R^k}{k!f_R}\frac{z_C^\ell}{\ell!f_C}\left(1 + O\left(\frac{(\log v)^2}{v_Rz_R} + \frac{(\log v)^2}{v_Cz_C}\right)\right). \tag{89}$$

Proof Since $\mathbf{x}$,$\mathbf{y}$ are independent, it is enough to prove (88). Using Lemma 10,

$$\begin{aligned}
\mathbf{Pr}(d_{\mathbf{x}}(i)=k) &= \frac{\mathbf{Pr}\left(Y_i=k \text{ and } \sum_{j\neq i} Y_j = v_R - v_{1,R}-k\right)}{\mathbf{Pr}\left(Y_i=k \text{ and } \sum_{j} Y_j = v_R - v_{1,R}\right)} \\
&= \frac{\frac{z_R^k}{f_R k!}\frac{1+O(k^2(v_Rz_R)^{-1})}{(2\pi(v_R-1)\mathbf{Var}(Y))^{1/2}}}{\frac{1+O((v_Rz_R)^{-1})}{(2\pi v_R \mathbf{Var}(Y))^{1/2}}} \\
&= \frac{z_R^k}{k!f_R}\left(1+O\left(\frac{(\log v)^2}{v_Rz_R}\right)\right)
\end{aligned}$$

□

Thus we can write that for $2\leq k\leq \log n$,

$$\mathbf{E}(v_{k,X}) = \frac{v_X z_X^k}{k!f_X}+O\left(\frac{(\log v)^2}{z_X}\right) \tag{90}$$

$$\mathbf{E}(v_{k,R}v_{\ell,C}) = \frac{v_Rv_Cz_R^kz_C^\ell}{k!\ell!f_Rf_C}+O\left(\frac{v(\log v)^2}{z_R}+\frac{v(\log v)^2}{z_C}\right) \tag{91}$$

Lemma 12. *Suppose* $(\mathbf{x},\mathbf{y})$ *is chosen randomly from* $\mathcal{B}_{R,C}(\boldsymbol{\nu})$ *where* $R=C=[n], v_1=0, v=2n, m=cn$.

(a) *Conditional on being simple,* $G_{\mathbf{x},\mathbf{y}}$ *is distributed as* $B_{n,cn}^{\delta\geq 2}$.

(b) $\lim_{n\to\infty}\mathbf{Pr}(G_{\mathbf{x},\mathbf{y}}$ *is simple*$)=\exp\left(-\frac{\rho}{1-e^{-\rho}}\right)$ *where* $\frac{\rho(e^\rho-1)}{e^\rho-1-\rho}=c$.

Proof
(a) If $G_{\mathbf{x},\mathbf{y}}$ is simple then it has vertex set $[n]+[n]$ and m edges. Also, there are $(m!)^2$ distinct equally likely values of $(\mathbf{x},\mathbf{y})$ which yield the same graph.

(b) If we condition on the degree sequence $d_{\mathbf{x}}, d_{\mathbf{y}}$ then the probability that $G_{\mathbf{x},\mathbf{y}}$ is simple is $F(\mathbf{x},\mathbf{y})$ of (5). Using (6) and (87) we see that

$$\mathbf{Pr}(G_{\mathbf{x},\mathbf{y}} \text{ is simple})\sim e^{-\lambda(d_{\mathbf{x}})\lambda(d_{\mathbf{y}})/2}$$

where

$$\mathbf{E}(n\lambda(d_{\mathbf{x}}))=\mathbf{E}(n\lambda(d_{\mathbf{y}}))\sim n\frac{\rho}{1-e^{-\rho}}.$$

Now the random variable $n\lambda(d_{\mathbf{x}})$ is the sum of independent (truncated) Poissons and so is concentrated around its mean. Thus $\mathbf{Pr}(\left|nl(d_{\mathbf{x}})-n\frac{\rho}{1-e^{-\rho}}\right|\geq n^{2/3})\leq n^{-a}$ for any constant $a>0$ and the lemma follows. □

We prove Lemma 5 by considering the following algorithm for finding a matching in a graph G. It is a technical modification of one described first by Karp and Sipser [14]. We apply the algorithm to the bipartite multigraph $G=G_{\mathbf{x},\mathbf{y}}$ where $(\mathbf{x},\mathbf{y})$ is chosen randomly from $\mathcal{B}_{R,C}(\boldsymbol{\nu})$ where $R=C=[n], v_1=0, v=2n, m=cn$. In the light of Lemma 12, we need only show that the following algorithm KSGREEDY finds a matching of size $n-O(n^{.49})$ with sufficiently high probability.

KSGREEDY

begin

$M \leftarrow \emptyset$;
while $E(G) \neq \emptyset$ **do**
begin
A1: If G has vertices of degree one in R and C, choose one, x say, randomly from R if $v_R \leq v_C$ and randomly from C otherwise.
If $v_{1,R} = 0$ or $v_{1,C} = 0$ and $v_1 > 0$ choose x randomly from the set with vertices of degree 1.
Let $e = \{x, y\}$ be the unique edge of G incident with x;
A2: Otherwise, (no vertices of degree one) choose
$e = \{x, y\} \in E$ randomly
$G \leftarrow G \setminus \{x, y\}$;
$M \leftarrow M \cup \{e\}$
end;
Output M
end

The reason for choosing a vertex of R when $v_R \leq v_C$ in Step A1 is that we must try to ensure that $|v_R - v_C|$ does not grow too large. This is because $|v_R - v_C|$ is a lower bound on the number of isolated vertices that will be created from now on. The choice of R in this case reduces $|v_R - v_C|$, in expectation.

KSGREEDY is defined on graphs. Formally, we need to define its action on pairs of sequences $\mathbf{x}, \mathbf{y}$. As in [2] we use $\star$'s to denote deleted edges i.e. if $x_i = a, y_i = b$ and the algorithm requires the removal of edge (a, b) then we make the assignments $x_i = y_i = \star$. Thus at a general step of the algorithm we are left with a pair of sequences $\mathbf{x}, \mathbf{y}$ from $([n] \cup \{\star\})^m$ which satisfy $x_i = \star$ iff $y_i = \star$ for $i \in [m]$. The sets R, C are defined by $R = \{j \in [n] : \exists i \text{ such that } x_i = j\}$ and $C = \{j \in [n] : \exists i \text{ such that } y_i = j\}$. The edges of this extended definition of $G_{\mathbf{x},\mathbf{y}}$ are simply $\{(x_i, y_i) : x_i \neq \star\}$. The next step in analogy to the argument from [2] is relate the evolution of $G_{\mathbf{x},\mathbf{y}}$ to a Markov chain on $\boldsymbol{\nu}$. So let $\boldsymbol{\nu}(0) = (0, 0, n, n, cn)$ and let $\boldsymbol{\nu}(t), t \geq 0$ be the sequence of states seen during KSGREEDY. The following lemma can be justified by arguments similar to those used for Lemma 3 of [2].

Lemma 13. *The random sequence $\boldsymbol{\nu}(t)$, $t = 0, 1, 2, \ldots$, is a Markov chain.*

Proof See Appendix B. □

We shall for convenenience introduce a stopping time $\mathcal{S}$ where

$$\mathcal{T}_0 = \begin{cases} \min\{t > 0 : \ |\boldsymbol{\nu}(t) - \boldsymbol{\nu}(t-1)| \geq \log n) & \text{if such } t \text{ exist} \\ n & \text{otherwise} \end{cases}$$

Note that

$$\mathbf{Pr}(\exists t : \ |\boldsymbol{\nu}(t) - \boldsymbol{\nu}(t-1)| \geq \log n) = O(n^{-K}) \tag{92}$$

for any constant $K > 0$. This follows from (87).

Note that $t < \mathcal{S}$ implies that

$$|z(t+1) - z(t)| = O\left(\frac{\log n}{v}\right). \tag{93}$$

$\mathcal{S}$ is our generic stopping time for the first occurrence of one of some unlikely events. As we proceed, we will find other unlikely events and we will update cS accordingly, with just a remark.

4.3.1 One step parameter changes

We now consider the the expected change in $\boldsymbol{\nu}$ due to one step of KSGREEDY.

Notice that

$$z \leq \frac{2\mu - v_1}{v},$$

the fraction being the average degree of a heavy vertex. Now a simple calculation shows that with probabilty $1 - O(n^{-4})$

$$\text{no vertex subset of } B_{n,m}^{\delta\geq 2} \text{ has average degree more than } 3m/n. \tag{94}$$

Given d, the property $P = \{G : \text{no vertex subset of } G \text{ has average degree more than } d\}$ is monotone increasing. Let two states $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ be such that the transition probability $p(\boldsymbol{\nu}'|\boldsymbol{\nu})$ is positive. Let G be chosen uniformly among all $G_{\mathbf{x},\mathbf{y}}$ such that $\boldsymbol{\nu}(\mathbf{x},\mathbf{y}) = \boldsymbol{\nu}$. One step of KSGREEDY applied to G produces a subgraph $G' = G_{\mathbf{x}',\mathbf{y}'}$. We know that $\mathbf{Pr}(\boldsymbol{\nu}(G') = \boldsymbol{\nu}'|G) = p(\boldsymbol{\nu}'|\boldsymbol{\nu}) > 0$, and that, conditioned on $\boldsymbol{\nu}(G') = \boldsymbol{\nu}'$, the graph G' is distributed unformly. Thus, for $p(\boldsymbol{\nu}'|\boldsymbol{\nu}) > 0$, we can couple two random graphs G and G', distributed uniformly on the set of all graphs with $\boldsymbol{\nu}(\mathbf{x},\mathbf{y}) = \boldsymbol{\nu}$ and $\boldsymbol{\nu}(\mathbf{x},\mathbf{y}) = \boldsymbol{\nu}'$ respectively, and such that $G' \subset G$. This means that $\mathbf{Pr}(G(t) \in P) \geq \mathbf{Pr}(G(t+1) \in P)$, $t \geq 0$. Using (94)), we get: for every $t \geq 0$, with probability $1 - O(n^{-4})$

$$\text{no vertex subset of } G(t) \text{ has average degree more than } 3m/n.$$

Hence, with probability $1 - O(n^{-3})$ the last event holds for all t simultaneously. So we will proceed assuming that

$$z \leq 3c. \tag{95}$$

Next let

$$\gamma = \gamma(\boldsymbol{\nu}) = |v_R - v_C| + v_1 \text{ and } \theta_v = \frac{\gamma}{v}.$$

A simple estimation, under the assumption that $\theta_v = o(1)$, yields

$$\left| \frac{2m - v_1}{v} - \frac{m - v_{1,R}}{v_R} \right| = \left| \frac{m(v_R - v_C)}{vv_R} - \frac{v_1}{v} + \frac{v_{1,R}}{v_R} \right| \leq 8c\theta_v.$$

Let $g(x) = \frac{x(e^x-1)}{e^x-1-x}$. We know that g assumes values $(\mu - v_{1,X})/v_X$ at z_X, $X = R, C$, and $(2\mu - v_1)/v$ at z. Then calculations yield that $g'(x) = \frac{x^2e^x+2e^x-1-e^{2x}}{(e^x-1-x)^2} \in [1,3]$. It follows immediately that there exists a constant c_1 such that

$$|z_R - z|, \quad |z_C - z| \leq c_1\theta_v.$$

Thus we can replace (90) and (91) by

$$\mathbf{E}(v_{k,X}) = \frac{vz^k}{2k!f} + O\left(v\theta_v \log v + \frac{(\log v)^2}{z}\right) \tag{96}$$

$$\mathbf{E}(v_{k,R}v_{\ell,C}) = \frac{v^2z^{k+\ell}}{4k!\ell!f^2} + O\left(v^2\theta_v \log v + \frac{v(\log v)^2}{z}\right) \tag{97}$$

In the following we will abbreviate the error terms to

$$\Theta = O\left(\theta_v \log v + \frac{(\log v)^2}{vz}\right).$$

In the analysis we will be able to concentrate on cases where

$$z \geq n^{-.17},\ v = \Omega(nz^2) \text{ and } \gamma = O(n^{.28}(\log n)^8). \tag{98}$$

Thus

$$\Theta = o(z^2) \quad \textit{throughout}. \tag{99}$$

Now we go through the steps of the algorithm and compute the expected changes in the parameters.

Case 1: Deleting a vertex x of degree 1 and its neighbour y. Assume that $x \in X$ and $\overline{X} = \{R, C\} \setminus X$.

Let $\boldsymbol{\nu}'$ refer to the state after one step.

Lemma 14. *Assume that* $\log n = O((vz)^{1/2})$, $v_{1,R} > 0$, *and the conditions (85), (86) hold. Let* $\mathcal{I}$ *be the event that KSGREEDY removes an isolated edge in this step. Then* **qs**

$$\begin{aligned}
\mathbf{E}(v'_{1,X} \mid \boldsymbol{\nu}) &= v_{1,X} - 1 + \frac{z^2 e^z}{(e^z-1)^2} + \Theta \\
v'_{1,\overline{X}} &= v_{1,\overline{X}} - 1_{\mathcal{I}} && (100)\\
\mathbf{E}(v'_1 \mid \boldsymbol{\nu}) &= v_1 - 1 + \frac{z^2 e^z}{(e^z-1)^2} + \Theta && (101)\\
\mathbf{E}(v'_X \mid \boldsymbol{\nu}) &= v_X - \frac{z^2 e^z}{(e^z-1)^2} + \Theta && (102)\\
\mathbf{E}(v'_{\overline{X}} \mid \boldsymbol{\nu}) &= v_{\overline{X}} - 1 + \Theta && (103)\\
\mathbf{E}(v' \mid \boldsymbol{\nu}) &= v - 1 - \frac{z^2 e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_X - v'_{\overline{X}} \mid \boldsymbol{\nu}) &= v_X - v_{\overline{X}} + 1 - \frac{z^2 e^z}{(e^z-1)^2} + \Theta && (104)\\
\mathbf{E}(\mu' \mid \boldsymbol{\nu}) &= \mu - 1 - \frac{z e^z}{e^z - 1} + \Theta \\
\mathbf{E}(v_{0,X} \mid \boldsymbol{\nu}) &= O\left(\frac{v_1 + (\log n)^2}{m}\right) \\
\mathbf{E}(v_{0,\overline{X}} \mid \boldsymbol{\nu}) &= 0.
\end{aligned}$$

Here $v_{0,X}$ *is the number of isolated vertices in* X *that are created by the step and we will let* $v_0 = v_{0,R} + v_{0,C}$.

Furthermore, if $|v_X - v_{\overline{X}}| \geq \log n$ *then*

$$v_X > v_{\overline{X}} \textit{ implies } \gamma' = \gamma - v_0. \tag{105}$$

$$v_X \leq v_{\overline{X}} \textit{ implies } \gamma' = \gamma + 2(v'_1 - v_1) + v_0 \tag{106}$$

$$v_X \leq v_{\overline{X}} \textit{ implies } \mathbf{E}(\gamma' \mid \boldsymbol{\nu}) = \gamma - 2\left(1 - \frac{z^2 e^z}{(e^z-1)^2}\right) + \Theta. \tag{107}$$

Proof Assume that $X = R$. We begin by conditioning on the degree sequence of $\mathbf{x}, \mathbf{y}$, assuming that it meets the conditions (85), (86). As we know, under these conditions, **qs** the maximum degree is at most $\log n$, (87). So we proceed assuming that this maximum degree condition holds. Now $\mathbf{x}$,$\mathbf{y}$ are just *mutually independent* random permutations of the multi-sets

$\prod_{i\in R} i^{d_{\mathbf{x}}(i)}$ and $\prod_{j\in C} j^{d_{\mathbf{y}}(j)}$, respectively. Suppose we delete $x \in R$ of degree 1 and its neighbour $y \in C$.

If $\mathcal{I}$ occurs, i.e. $y \in C_1$ then $v'_{1,X} = v_{1,X} - 1, v'_X = v_X$ $X = R, C$ and $m' = m - 1$. We note that

$$\mathbf{Pr}(y \in C_1 \mid \mathbf{x}, \mathbf{y}) = \frac{v_{1,C}}{m} = \Theta,$$

so this case only contributes to the error term. Note also that (106) holds in this case.

Now for $2 \le k \le \log n$ we have

$$\mathbf{Pr}(d_{\mathbf{y}}(y) = k \mid \mathbf{x}, \mathbf{y}) = \frac{kv_{k,C}}{m}. \tag{108}$$

Add the event $\{d_{\mathbf{y}}(y) = k\}$ to the conditioning on $\mathbf{x}, \mathbf{y}$, denoting the resulting conditioning by $\mathcal{H}$. Suppose y has $k_{i,j}$ neighbours (excluding x) of degree i that are joined to y by j edges. Then

$$\begin{aligned}
\mu' &= \mu - k && (109)\\
v'_{1,C} &= v_{1,C} && (110)\\
v'_{1,R} &= v_{1,R} - 1 - k_{1,1} + \sum_{i\ge 2} k_{i,i-1} && (111)\\
v'_C &= v_C - 1 && (112)\\
v'_R &= v_R - \sum_{i\ge 2} k_{i,i-1} - \sum_{i\ge 2} k_{i,i} && (113)\\
v'_{0,R} &= \sum_{j\ge 1} k_{j,j} && (114)\\
v'_{0,C} &= 0 && (115)
\end{aligned}$$

Then we have

$$v_0 = \sum_{i\ge 1} k_{i,i}$$

and

$$\begin{aligned}
v'_1 - v_1 &= -1 - k_{1,1} + \sum_{i\ge 2} k_{i,i-1}\\
\gamma' - \gamma &= v'_1 - v_1 - 1 + \sum_{i\ge 2} k_{i,i-1} + \sum_{i\ge 2} k_{i,i}
\end{aligned}$$

and (106) follows.

Note next that

$$\mathbf{E}(k_{1,1} \mid \mathcal{H}) = \frac{(k-1)(v_{1,R} - 1)}{\mu - 1}. \tag{116}$$

Further, for $\max(2, j) \le i \le \log n$,

$$\mathbf{E}(k_{i,j} \mid \mathcal{H}) = v_{i,R}\binom{k-1}{j}\frac{(i)_j(\mu - 1 - i)_{k-1-j}}{(\mu-1)_{k-1}} = \begin{cases} \frac{(k-1)iv_{i,R}}{\mu}\left(1 + O\left(\frac{(\log n)^2}{\mu}\right)\right) & j = 1,\\ O\left(\frac{(\log n)^{2j}}{\mu^{j-1}}\right) & j \ge 2. \end{cases} \tag{117}$$

Thus from (108), (111) and (117) we get

$$\begin{aligned}
\mathbf{E}(v'_{1,R} \mid \mathbf{x}, \mathbf{y}) &= v_{1,R} - 1 - \mathbf{E}(k_{1,1} \mid \mathbf{x}, \mathbf{y}) + \mathbf{E}\left(\sum_{i\geq 2} k_{i,i-1} \mid \mathbf{x}, \mathbf{y}\right) \\
&= v_{1,R} - 1 + \sum_{k\geq 2} \frac{kv_{k,C}}{\mu} \frac{(k-1)2v_{2,R}}{\mu} + \Theta.
\end{aligned}$$

Removing the conditioning on vertex degrees we get

$$\begin{aligned}
E(v'_{1,R} \mid \boldsymbol{\nu}) &= v_{1,R} - 1 + \sum_{k\geq 2} \frac{kz_C^k v_C}{k! f_C \mu} \cdot (k-1) \cdot \frac{z_R^2 v_R}{f_R \mu} + \Theta \\
&= v_{1,R} - 1 + \frac{z_R^2 z_C^2 v_R v_C e^{z_C}}{f_R f_C \mu^2} + \Theta \\
&= v_{1,R} - 1 + \frac{z^4 v^2 e^z}{4 f^2 \mu^2} + \Theta \\
&= v_{1,R} - 1 + \frac{z^2 e^z}{(e^z - 1)^2} + \Theta.
\end{aligned}$$

The remaining quantities can now be filled in the same way using (108) and (109) – (113).

$$\begin{aligned}
\mathbf{E}(v'_{1,C} \mid \boldsymbol{\nu}) &= v_{1,C} + \Theta \\
\mathbf{E}(v'_R \mid \boldsymbol{\nu}) &= v_R - (\mathbf{E}(v'_{1,R} \mid \boldsymbol{\nu}) - (v_{1,R} - 1)) + \Theta \\
&= v_R - \frac{z^2 e^z}{(e^z - 1)^2} + \Theta \\
\mathbf{E}(v'_C \mid \boldsymbol{\nu}) &= v_C - 1 + \Theta \\
\mathbf{E}(v'_{0,R} \mid \boldsymbol{\nu}) &= \sum_{k\geq 2} \frac{k\mathbf{E}(v_{k,C} \mid \boldsymbol{\nu})}{\mu} \sum_{j\geq 1} \mathbf{E}(k_{j,j} \mid \boldsymbol{\nu}) \\
&= \sum_{k\geq 2} \frac{k\mathbf{E}(v_{k,C} \mid \boldsymbol{\nu})}{\mu} \left(\frac{(k-1)(v_{1,R} - 1)}{\mu - 1} + O\left(\frac{(\log n)^2}{\mu}\right)\right) \\
&= O\left(\frac{v_1 + (\log n)^2}{\mu}\right) \\
v'_{0,C} &= 0 \\
\mathbf{E}(m' \mid \boldsymbol{\nu}) &= m - 1 - \sum_{k\geq 2} \frac{k\mathbf{E}(v_{k,C} \mid \boldsymbol{\nu})}{m - v_{1,C}} + \Theta \\
&= m - 1 - \sum_{k\geq 2} \frac{kz_C^k v_C}{k! f_C m} \cdot (k-1) + \Theta \\
&= m - 1 - \frac{z_C^2 v_C e^{z_C}}{m f_C} + \Theta \\
&= m - 1 - \frac{z^2 v e^z}{2mf} + \Theta \\
&= m - 1 - \frac{ze^z}{e^z - 1} + \Theta
\end{aligned}$$

Now note that since $|v' - v| = o(\log n)$ **qs**, we see that if $|v_C - v_R| \geq \log n$ then $v'_C - v'_R$ has the same sign as $v_C - v_R$ and we can use the equations (101) – (104) to get (107).

Finally (105) follows from

$$\gamma' - \gamma = -\sum_{i \geq 1} k_{i,i}.$$

□

Note that

$$\begin{aligned}
1 - \frac{z^2 e^z}{(e^z-1)^2} &= 1 - \left(\frac{z}{e^{z/2} - e^{-z/2}}\right)^2 \\
&= 1 - \left(\frac{1}{\sum_{j \geq 0} \frac{z^{2j}}{2^{2j}(2j+1)!}}\right)^2 \\
&\geq \min\left\{\frac{z^2}{48}, \frac{1}{2}\right\},
\end{aligned} \tag{118}$$

cf. Corollary 3 of [2].

So in Case 1, for $\boldsymbol{\nu}(t) \in W_1$, we have

$$\mathbf{E}(v_1' \mid \boldsymbol{\nu}) \leq v_1 - \min\left\{\frac{z^2}{48}, \frac{1}{2}\right\} + \Theta \leq v_1 - \min\left\{\frac{z^2}{50}, \frac{1}{2}\right\}, \tag{119}$$

on using (99).

Remark 1. *In what follows there is a proliferation of large related constants and the reader may find it difficult to check where these constants come from. We will adopt the convention that the subscript of such constants is defined by the equation number where they are first used. In this spirit* $C_{119} = 50$*.*

Furthermore, introduce $M = 2\mu - v_1$. Lemma 14 implies that

$$\frac{\mathbf{E}(M' - M \mid \boldsymbol{\nu})}{\mathbf{E}(v' - v \mid \boldsymbol{\nu})} = \frac{1 + \frac{2ze^z}{e^z-1} + \frac{z^2e^z}{(e^z-1)^2}}{1 + \frac{z^2e^z}{(e^z-1)^2}} + \Theta. \tag{120}$$

Case 2: Deleting a random edge when $v_1 = 0$.

Lemma 15. *Assume that* $\log n = O((vz)^{1/2})$ *and* $v_1 = 0$*. Then*

$$
\begin{aligned}
\mathbf{E}(v'_{1,R} \mid \boldsymbol{\nu}) &= \frac{z^2 e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}v'_{1,C} \mid \boldsymbol{\nu}) &= \frac{z^2 e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_1 \mid \boldsymbol{\nu}) &= 2\frac{z^2 e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_R \mid \boldsymbol{\nu}) &= v_R - 1 - \frac{z^2 e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_C \mid \boldsymbol{\nu}) &= v_C - 1 - \frac{z^2 e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v' \mid \boldsymbol{\nu}) &= v - 2 - 2\frac{z^2 e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_R - v'_C \mid \boldsymbol{\nu}) &= v_R - v_C + \Theta \qquad (121) \\
\mathbf{E}(\mu' \mid \boldsymbol{\nu}) &= \mu - 1 - 2\frac{ze^z}{e^z-1} + \Theta \\
\mathbf{E}(v_{0,R} \mid \boldsymbol{\nu}) &= O\left(\frac{\log^2 n}{\mu}\right) \qquad (122) \\
\mathbf{E}(v_{0,C} \mid \boldsymbol{\nu}) &= O\left(\frac{\log^2 n}{\mu}\right) \qquad (123)
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
\mathbf{Pr}(v'_{1,R} > 0 \mid \boldsymbol{\nu}) &= \frac{z}{e^z-1} + \Theta = 1 - \frac{z}{2} + O(z^2) + \Theta. \qquad (124) \\
\mathbf{Pr}(v'_{1,R} > 0 \text{ or } v'_{1,C} > 0 \mid \boldsymbol{\nu}) &= 1 - \left(1 - \frac{z}{e^z-1}\right)^2 + \Theta = 1 - \frac{z^2}{4} + O(z^3) + \Theta. \qquad (125) \\
\mathbf{Pr}(v'_{1,R} > 0 \text{ and } v'_{1,C} = 0 \mid \boldsymbol{\nu}) &= \frac{z}{e^z-1}\left(1 - \frac{z}{e^z-1}\right) + \Theta = \frac{z}{2} + O(z^2) + \Theta \qquad (126)
\end{aligned}
$$

Proof We again condition on the degree sequence $\mathbf{x}, \mathbf{y}$. Choosing a random edge means choosing a random x from $\mathbf{x}$, and then an x's random neighbour y from $\mathbf{y}$. Given $k\epsilon 2$, $\ell \geq 2$, let us add the event $\{d_{\mathbf{y}}(y) = k, d_{\mathbf{x}}(x) = \ell\}$ to the conditioning on $\mathbf{x}, \mathbf{y}$, denoting the resulting conditioning by $\mathcal{H}$. Besides the numbers $\{k_{i,j}\}$ (see the proof of Lemma 14), let let x have $\ell_{i,j}$

neighbours (excluding y) of degree i that are joined to x by $j \leq i$ edges. Then

$$\begin{aligned}
\mu' &= \mu - k - \ell + 1 && (127)\\
v'_{1,C} &= \sum_{i\geq 2} \ell_{i,i-1} && (128)\\
v'_{1,R} &= \sum_{i\geq 2} k_{i,i-1} && (129)\\
v'_{C} &= v_C - 1 - \sum_{i\geq 2} \ell_{i,i-1} - \sum_{i\geq 2} \ell_{i,i} && (130)\\
v'_{R} &= v_R - 1 - \sum_{i\geq 2} k_{i,i-1} - \sum_{i\geq 2} k_{i,i} && (131)\\
v_{0,R} &= \sum_{j\geq 1} k_{j,j} && (132)\\
v_{0,C} &= \sum_{j\geq 1} \ell_{j,j} && (133)
\end{aligned}$$

Now (117) still holds and there is an analogous expression for the $\ell_{i,j}$.

The rest of proof of the lemma follows the same pattern as that for Lemma 14 and is left to the reader, who should notice how close the claim is to two applications of the previous case. □

Observe that (120) holds in this case too.

Note also that in both cases

$$\mathbf{E}(v_0 \mid \boldsymbol{\nu}) = O\left(\frac{v_1 + (\log n)^2}{\mu}\right). \tag{134}$$

Remarkably, the equations involving $v_1, v_1', v, v', \mu, \mu'$ are up to the error terms, identical to those given in Lemmas 6 and 7 of [2].

4.3.2 Multi-step parameter changes

At this point let us try to summarise the (conditional) expected changes in the parameters v_1, γ, γ_1 as the algorithm proceeds. Here

$$\gamma_1 = |v_R - v_C|.$$

We need to show that v_1 does not grow large as v_1 determines the rate at which isolated vertices are created. We also need to show that γ_1 does not grow large so that we can for one thing approximate z_R, z_C by z.

Let us first consider the case where $v_1 > 0$. There is a *preferred* side from which to choose the vertex x of degree 1. This is the side with fewest vertices of degree 2 or more. v_1 is well behaved. When $v_1 > 0$ the expected change in v_1 is negative and so it is relatively easy to show that it is unlikely to get too large. γ_1 is not so well behaved. It has a positive expected change when we are forced to choose x on the less preferred side. However, this expected increase is compensated by the expected decrease in v_1. This is why we introduce the parameter γ which is much better behaved than γ_1. Indeed, if x is chosen on the preferred side then the expected change in γ is negative and if x is chosen on the less prefered side then the change is non-positive (deterministically).

Now consider what happens if $v_1 = 0$. There is only a small chance that $v_1' = 0$ too and the expected increase in γ_1 while v_1 remains zero turns out to be negligible. If $v_1' > 0$ then we see a rise in γ (due to the rise in v_1). One more step will show an expected decrease i γ_1. The rise in v_1 is handled by looking forward to the next time that $v_1 = 0$, see Lemma 19.

The main analysis of the chain $(\boldsymbol{\nu}(t))$ is restricted to the times when $\boldsymbol{\nu}(t) \in W_0$ or $\boldsymbol{\nu}(t) \in W_1$. where for $\sigma = 0, 1$,

$$W_\sigma = \{\boldsymbol{\nu} :\ z \geq n^{-\alpha_i},\, v \geq A_{135} n z^2,\, v_1 \leq B_{135} n^{2\alpha_\sigma} (\log n)^3,\, \gamma \leq C_{135} n^{2\alpha_\sigma} (\log n)^\xi\}. \tag{135}$$

Here $\alpha_0 = .14$, $\alpha_1 = .17$, $\xi =$ and $A_{135}, B_{135}, C_{135}$ are large constants and $W_0 \subseteq W_1$.

Note that (98) holds for $\boldsymbol{\nu}(t) \in W_1$.

For $\sigma = 0, 1$ we introduce stopping times

$$\mathcal{T}_\sigma = \begin{cases} \min\{t \leq \mathcal{S} :\ v_1(\tau) \notin W_\sigma\} \\ n \qquad \text{if no such } t \text{ exist} \end{cases}$$

Now it follows from (125) that we can find $K > 0$ such that with probability $1 - O(n^{-5})$ there is no sequence $t, t+1, \ldots, t + K \log n < n$ such that $v_1(\tau) = 0$ for $t \in [t, t + K \log n]$. So we introduce another stopping time

$$\mathcal{S}_1 = \begin{cases} \min\{t \leq \mathcal{S} :\ v_1(\tau) = 0, \tau \in [t - K \log n, t]\} \\ n \qquad \text{if no such } t \text{ exist} \end{cases}$$

and then

$$\mathcal{S} := \min\{\mathcal{S}, \mathcal{S}_1\}.$$

Lemma 16. *Let t be such that*

$$\boldsymbol{\nu}(t) \in W_1 \text{ and } v_1(t) = 0$$

and

$$\gamma_1(t) = v_C(t) - v_R(t) \gg (\log n)^3.$$

Let

$$t' = \begin{cases} \min\{t < \tau \leq \mathcal{T}_1 :\ v_1(\tau) > 0\} & \text{if such } \tau \text{ exist} \\ n & \text{otherwise} \end{cases}$$

Then

$$\mathbf{E}(\gamma_1(t'+1) - \gamma_1(t) \mid \boldsymbol{\nu}(t)) \leq -\beta(t)$$

where

$$\beta(t) = -\min\left\{\frac{z(t)^2}{C_{136}}, \frac{1}{6}\right\}. \tag{136}$$

where $C_{136} = 3C_{119}$

Proof We first observe that either $\mathcal{S}_1 \leq t + K \log n$ or $t' < t + K \log n$ and then $v_C(\tau) - v_R(\tau) \gg (\log n)^3$ for $t \leq \tau \leq t' + 1$.

We next assemble the following facts: Assume that $0 < \tau - t \leq K \log n$.

Claim 1

$$\mathbf{Pr}(t' = \tau \mid t' > \tau - 1, \boldsymbol{\nu}(\tau - 1)) = 1 - \left(1 - \frac{z}{e^z - 1}\right)^2 + \Theta \stackrel{def}{=} \pi_1 + \Theta.$$

This follows from (125). Here we can take $z = z(t)$ since z changes by $O\left(\frac{(\log n)^2}{v}\right)$ in $O(\log n)$ steps (see (93)) and this quantity is $o(z^3)$.

Claim 2

$$\mu_2 = \mathbf{E}(\gamma_1(\tau) - \gamma_1(\tau - 1)1_{t' \neq \tau} \mid \boldsymbol{\nu}(\tau - 1))) = \Theta.$$

This follows from (121).

Claim 3

$$\begin{aligned} \mathbf{Pr}(v_{1,R}(t) > 0 \mid v_1(t-1) = 0, v_1(t) > 0, \boldsymbol{\nu}(\tau - 1)) &= \frac{\frac{z}{e^z-1}}{1 - \left(1 - \frac{z}{e^z-1}\right)^2} + \Theta \\ &\stackrel{def}{=} \pi_3 + \Theta \\ &= 1 - \frac{z}{2} + O(z^2). \end{aligned}$$

This follows from (124) and (125).

Claim 4

$$\mathbf{E}(\gamma_1(\tau + 1) - \gamma_1(\tau) \mid t' = \tau, v_{1,R}(t') > 0, \boldsymbol{\nu}(\tau)) = -1 + \frac{ze^z}{(e^z - 1)^2} + \Theta \stackrel{def}{=} \mu_4 + \Theta.$$

This follows from (104).

Claim 5

$$\mathbf{E}(\gamma_1(\tau) - \gamma_1(\tau - 1) \mid t' = \tau, v_{1,R}(t') = 0, \boldsymbol{\nu}(\tau)) = 1 - \frac{ze^z}{(e^z - 1)^2} + \Theta = -\mu_4 + \Theta.$$

This follows from (104).

Putting these facts together we get

$$\mathbf{E}(\gamma_1(t' + 1) - \gamma_1(t) \mid \boldsymbol{\nu}(t)) = \frac{1}{\pi_1}\Theta + \pi_3\mu_4 + (\pi_1 - \pi_3)(-\mu_4) + \Theta. \tag{137}$$

Explanation of (137): The first term accounts for the expected increase in γ_1 between t and t'. The term $\pi_3\mu_5$ accounts for the expected increase in γ_1 when $v_{1,R}(t') > 0$ (this is negative) and the term $(\pi_1 - \pi_3)(-\mu_4)$ accounts for the expected increase in γ_1 when $v_{1,R}(t') = 0$. Included in here is a negligible $O(n \cdot n^{-5})$ for the case of $\mathcal{T}_1 \leq t$.

It can be shown that $2\pi_1 - \pi_3 > \frac{1}{2}$ and so

$$\mathbf{E}(\gamma_1(t' + 1) - \gamma_1(t) \mid \boldsymbol{\nu}(t)) \leq \frac{1}{2}\mu_4 \leq -\min\left\{\frac{z^2}{3C_{119}}, \frac{1}{5}\right\}. \tag{138}$$

The lemma follows. □

We put these ideas to work in the next few sections.

5 Number of vertices left isolated

Let $\alpha_\sigma, W_\sigma, \mathcal{T}_\sigma, i = 0, 1$ be as in (135).

The analysis is in two parts, $t = 1 \dots, \mathcal{T}_0$ and $t = \mathcal{T}_0 + 1 \dots \mathcal{T}_1$. The reason for this split will not become apparant until the middle of proof of Lemma 20 and so the reader will have to take the need for a split on trust.

Note that if $\boldsymbol{\nu} \in W_1$ then $vz^2 \geq A_{135} nz^4 \gg (\log v)^2$ and so the conclusions of Lemmas 14 and 15 are valid.

Fix $\sigma = 0$ or 1 Now let X_t, $t = 0, 1, \dots, \mathcal{T}_\sigma - 1$ be the number of isolated vertices created at time t and let $X_t = 0$ for $t \geq \mathcal{T}_\sigma$. Let also set $\boldsymbol{\nu}(t) \equiv \boldsymbol{\nu}(\mathcal{T}_\sigma - 1)$ for $t \geq \mathcal{T}_\sigma$. Then the random variables X_t satisfy

$$X_t \geq 0;\ \ X_t \leq \log n;\ \ \mathbf{E}(X_t \mid \cdot) \leq \frac{B_{135} C_{134} n^{2\alpha_\sigma} (\log n)^3}{\mu(t)},$$

where $\mid \cdot$ denotes conditioning with respect to $\{\boldsymbol{\nu}(\tau)\}_{0 \leq \tau < t}$, and C_{134} is the hidden constant in (134). Putting $\lambda = (\log n)^{-2}$ and using $\lambda X_t \leq (\log n)^{-1}$, $e^x \leq 1 + 1.5x$, $x \downarrow 0$, we see that

$$\mathbf{E}(e^{\lambda X_t} \mid \cdot) \leq 1 + 1.5\lambda \mathbf{E}(X_t \mid \cdot) \leq e^{1.5\lambda \mathbf{E}(X_t \mid \cdot)}.$$

Therefore, introducing $X = \sum_{t \geq 0} X_t$, and using the bound for $\mathbf{E}(X_t \mid \cdot)$ together with $\sum_t \mathbf{E}(m^{-1}(t)) \leq \log n$, we have

$$\mathbf{E}(e^{\lambda X}) \leq \exp(2 B_{135} C_{134} n^{2\alpha_\sigma} (\log n)^2).$$

Applying the Markov inequality, we obtain

$$\mathbf{Pr}(X \geq 3 B_{135} C_{134} n^{2\alpha_\sigma} (\log n)^4) \leq e^{-3 B_{135} C_{134} n^{2\alpha_\sigma} (\log n)^4 \lambda} \mathbf{E} e^{\lambda X} \leq e^{-B_{135} C_{134} n^{2\alpha_\sigma} (\log n)^3}. \quad (139)$$

This proves:

Lemma 17.

$$\textit{At most } C_{140} n^{2\alpha_\sigma} (\log n)^4 \textit{ isolated vertices are created up to time } \mathcal{T}_\sigma, \ \mathbf{qs} \quad (140)$$

where $C_{140} = 3 B_{135} C_{134}$.

Our next task is to get a good estimate of $\boldsymbol{\nu}(\mathcal{T}_\sigma)$ at the stopping time $\mathcal{T}_\sigma$.

Lemma 18. *With probability* $1 - O(n^{-2})$

$$v_1(t) \leq C_{141} n^{2\alpha_\sigma} (\log n)^3 \qquad \forall t \in [1, \mathcal{T}_\sigma], \quad (141)$$

where $C_{141} = 8 C_{119}$ *(and hence we can take* $B_{135} = 8 C_{119}$*).*

Proof Let $\Delta_\sigma = C_{141} n^{2\alpha_\sigma} (\log n)^3$. First of all, by (87), **qs** the conditions $v_1(t-1) = 0$, $v_1(t) > 0$ and $t \geq 1$ imply that $v_1(t) \leq \log n$. In view of this, for $t_1 < t_2$, define the event

$$\mathcal{E}_1(t_1, t_2) = \{v_1(t_2) - v_1(t_1) > C_{141} n^{2\alpha_\sigma} (\log n)^3\} \cap \{\forall\, t \in [t_1, t_2),\, v_1(t) > 0\}.$$

Clearly then it suffices to prove that

$$\mathbf{Pr}\left(\bigcup_{t_1 < t_2 \leq \mathcal{T}_\sigma} \mathcal{E}_1(t_1, t_2) \right) = O(n^{-2}).$$

Define

$$Y_t = \begin{cases} v_1(t+1) - v_1(t) & \text{if } v_1(t) > 0 \text{ and } t < \mathcal{T}_\sigma, \\ -\beta_{142} & \text{otherwise,} \end{cases}$$

where,

$$\beta_{142} = \frac{1}{C_{119} n^{2\alpha_\sigma}}. \tag{142}$$

We notice upfront that, for $t_1 < t_2 \leq \mathcal{T}_\sigma$, and $v_1(t) > 0$ for $t \in [t_1, t_2)$,

$$v_1(t_2) - v_1(t_1) = \sum_{t=t_1}^{t_2-1} Y_t.$$

Now, by the definition of $\mathcal{T}$, $|Y_t| \leq \log n$ for all t. Furthermore, if $\lambda = \frac{\beta_{142}}{2(\log n)^2}$ then

$$\begin{aligned} \mathbf{E}(e^{\lambda Y_u} \mid \boldsymbol{\nu}(\tau), \tau < t+u) &\leq 1 + \mathbf{E}(\lambda Y_u) + \sum_{i \geq 2} \frac{\lambda^i (\log n)^i}{i!} \\ &\leq 1 - \lambda\beta_{142} + \lambda^2 (\log n)^2 \\ &\leq 1. \end{aligned} \tag{143}$$

For $t < \mathcal{T}_\sigma$, (143) follows from the definition of Y_t, β_{142}, (119), and the definition of the stopping time $\mathcal{T}_\sigma$. For $t \geq \mathcal{T}_\sigma$, (143) holds trivially. Thus, the occurrence of the event $\mathcal{E}_1(t_1, t_2) \cap \{t_2 \leq \mathcal{T}_\sigma\}$ implies

$$\sum_{t=t_1}^{t_2} Y_t > \Delta_\sigma. \tag{144}$$

Now from (143),

$$\begin{aligned} \mathbf{Pr}\left(\sum_{t=t_1}^{t_2} Y_t > \Delta_\sigma \mid \boldsymbol{\nu}(\tau), \tau \leq t_1 \right) &\leq e^{-\lambda\Delta_\sigma} \mathbf{E}\left(\prod_{t=t_1}^{t_2} e^{\lambda Y_t} \mid \boldsymbol{\nu}(\tau), \tau \leq t_1 \right) \\ &\leq e^{-\lambda\Delta_\sigma}. \end{aligned} \tag{145}$$

Since the number of pairs (t_1, t_2) is $\binom{n}{2} = O(n^2)$, the statement follows. □

To account for the unlikely failure of (141) we introduce a stopping time

$$\mathcal{S}_2 = \begin{cases} \min\{t \leq \mathcal{T}_1 : \ v_1(t) > C_{141} n^{2\alpha_\sigma} (\log n)^3\} \\ n \qquad \text{if no such } t \text{ exist} \end{cases}$$

and let

$$\mathcal{S} := \min\{\mathcal{S}, \mathcal{S}_2\}.$$

Now to deal with γ.

Lemma 19. *With probability* $1 - O(n^{-2})$

$$\gamma(t) \leq 2C_{140} n^{2\alpha_\sigma} (\log n)^4 \qquad \forall t \in [1, \mathcal{T}_\sigma]. \tag{146}$$

(Thus we can take $C_{135} = 2C_{140}$*).*

Proof For $t_0 < t_2$ let

$$\mathcal{E}_2(t_0,t_2) = \{\gamma(t_2) - \gamma(t_0) > 2C_{140}n^{2\alpha_\sigma}(\log n)^4\}$$

and also for $t_0 < t_1 < t_2$ let

$$\mathcal{E}_3(t_0,t_1,t_2) = \{\gamma(t_1) - \gamma(t_0) < C_{140}n^{2\alpha_\sigma}(\log n)^3 \text{ and}$$
$$\gamma(t) - \gamma(t_0) \geq C_{140}n^{2\alpha_\sigma}(\log n)^3,\ t_1 < t < t_2 \text{ and } \gamma(t_2) - \gamma(t_0) > 2C_{140}n^{2\alpha_\sigma}(\log n)^4\}.$$

Note that if $t_2 < \mathcal{T}_\sigma$ and $\mathcal{E}_2(t_0,t_2)$ occurs then $\mathcal{E}_3(t_0,t_1,t_2)$ occurs for some $t_1 > t_0$.

$$\mathbf{Pr}\left(\bigcup_{0\leq t_0<t_1<t_2\leq \mathcal{T}_\sigma} \mathcal{E}_3(t_0,t_1,t_2)\right) = O(n^{-2}). \tag{147}$$

Fix $t_2 > t_1 > t_0$. If $\mathcal{E}_3(t_0,t_1,t_2)$ occurs for some $t_0 \leq \mathcal{T}_\sigma$ then the sign of $v_R - v_C$ does not change between t_1 and t_2. (If $v_R - v_C$ drops to zero then we will have $\gamma = v_1 \leq C_{141}n^{2\alpha_\sigma}(\log n)^3$). We will assume that $v_R(t_1) - v_C(t_1) < 0$ and introduce the stopping time

$$t_I = \begin{cases} \min\{t_1 \leq t \leq t_2 :\ v_R(t) \geq v_C(t) \\ t_2 \qquad \text{if no such } t \text{ exist} \end{cases}$$

We define a sequence of times $\tau_0 = t_1 \leq \tau_1 < \ldots < \tau_r \leq \tau_{r+1} = t_I$ as follows: $\tau_1 = \min\{t_1 \leq \tau < \min\{t_2,\mathcal{T}_3\} :\ v_1(\tau) = 0\}$. If such a τ does not exist then we take $\tau_1 = t_I$. Assume that we have defined τ_i with $v_1(\tau_i) = 0$. Define $\tau_i' = 1 + \min\{\tau_i < \tau \leq t_I :\ v_1(\tau) > 0\}$. If τ_i' does not exist then $r = i$. If τ_i' does exist then let $\tau_i^* = \min\{\tau_i \leq \tau \leq t_I :\ v_1(\tau) = 0\}$. If τ_i^* exists then $\tau_{i+1} = \tau_i^*$, otherwise $r = i$. We now bound the change in γ over these intervals.

(a) We first consider $\gamma(\tau_1) - \gamma(\tau_0)$. This is zero if $\tau_1 = \tau_0$ and so assume that $v_1(t_0) > 0$. If $v_{1,R}(t_1) = 0$ then (100) implies that $v_{1,R}(\tau) = 0$ for $t_0 \leq \tau \leq t_I\}$ and then (105) implies that $\gamma(\tau_1) \leq \gamma(\tau_0)$.

So assume that $v_{1,R}(t_1) > 0$. For $1 \leq u \leq n$ define

$$Y_u = \begin{cases} \gamma_1(t_1+u) - \gamma_1(t_1+u-1) & t_0 + u \leq \tau_1 \\ -\beta_{149} & \mathit{otherwise} \end{cases} \tag{148}$$

where

$$\beta_{149} = \frac{1}{C_{136}n^{2\alpha_\sigma}}. \tag{149}$$

Now we have $|Y_u| \leq \log n$ and (136) implies that

$$\mathbf{E}(Y_u \mid \boldsymbol{\nu}(\tau), \tau < t_1 + u) \leq -\beta_{149}.$$

So with $\lambda = \frac{\beta}{2\log n}$ we can argue as in (143) that

$$\mathbf{E}(e^{\lambda Y_u} \mid \boldsymbol{\nu}(\tau), \tau < t_1 + u) \leq 1.$$

We then argue as in (145) that for all $0 < T \leq n$,

$$\mathbf{Pr}\left(\sum_{u=1}^{T} Y_u \geq 10(\log n)^3\beta_{149}^{-1}\right) \leq n^{-5}.$$

It follows that

$$\mathbf{Pr}(\gamma(\tau_1) - \gamma(\tau_0) \geq 10(\log n)^3 \beta_{149}^{-1}) \leq n^{-4}. \tag{150}$$

(b) We now consider the random variables $\gamma(\tau_{i+1}) - \gamma(\tau_i)$ for $i \geq 1$. Fix $1 \leq i \leq r$. (When $i = r$ parts of the argument may have to be omitted or modified in a trivial way). It follows from Lemma 16 that

$$\mathbf{E}(\gamma_1(\tau_i') - \gamma_1(\tau_i) \mid \boldsymbol{\nu}(\tau), \tau \leq \tau_i)) \leq -\beta_{151} = -\frac{1}{C_{136} n^{2\alpha_\sigma}}. \tag{151}$$

Suppose that $v_{1,R}(\tau_i') = k$ and $v_{1,C}(\tau_i') = \ell$. Thus

$$\gamma(\tau_i') - \gamma(\tau_i) = \gamma_1(\tau_i') - \gamma_1(\tau_i) + k + \ell. \tag{152}$$

Next let $\tau_i'' = \min\{\tau \geq \tau_i' : \ v_{1,R} = 0\}$. It follows from (106) that

$$-2k - 2\ell \leq \gamma(\tau_i'') - \gamma(\tau_i') \leq -2k + Z_i$$

where

$$Z_i = \sum_{\tau=\tau_i'}^{\tau_i''} v_0(\tau).$$

This is because $v_R < v_C$ and $v_{1,R}(\tau_i'') = 0$ and $v_{1,C}(\tau_i'') \leq v_{1,C}(\tau_i')$ (by (100)).

Now (105) implies that

$$\gamma(\tau_{i+1}) = \gamma(\tau_i'') - Z_i'$$

where

$$Z_i' = \sum_{\tau=\tau_i''}^{\tau_{i+1}} v_0(\tau).$$

So if

$$\Gamma_i = \gamma(\tau_{i+1}) - \gamma(\tau_i) - Z_i + Z_i'$$

then

$$\Gamma_i \leq \gamma_1(\tau_i') - \gamma_1(\tau_i) + \ell - k \text{ and } |\Gamma_i| \leq 5 \log n.$$

So

$$\mathbf{E}(\Gamma_i \mid \boldsymbol{\nu}(\tau_i)) \leq -\beta_{151} + \Theta \leq -\beta_{151}/2.$$

Putting $\lambda = \frac{\beta_{151}}{25(\log n)^2}$ we can argue as in (143) that $\mathbf{E}(e^{\lambda \Gamma_i} \mid \boldsymbol{\nu}(\tau_i)) \leq 1$. Putting $\Gamma_i = 0$ for $i = r+2, \ldots, n$ we see that

$$\mathbf{Pr}\left(\sum_{i=1}^{r+1} \Gamma_i \geq 125(\log n)^3 \beta_{151}^{-1}\right) = \mathbf{Pr}\left(\sum_{i=1}^{n} \Gamma_i \geq 125(\log n)^3 \beta_{151}^{-1}\right) \leq e^{-125\lambda(\log n)^3 \beta_{151}^{-1}} = n^{-5}. \tag{153}$$

But

$$\gamma(\tau_{r+1}) - \gamma(\tau_1) = \sum_{i=1}^{r+1} \Gamma_i + \sum_{i=1}^{r+1} (Z_i - Z_i') \leq \sum_{i=1}^{r+1} \Gamma_i + \sum_{i=1}^{r+1} Z_i$$

and (140) implies that $\sum_{i=1}^{r+1} Z_i \leq C_{140} n^{2\alpha_\sigma} (\log n)^4$ **qs**. Therefore,

$$\mathbf{Pr}(\gamma(\tau_{r+1}) - \gamma(\tau_1) \geq 2C_{140} n^{2\alpha_\sigma} (\log n)^4) = O(n^{-5}). \tag{154}$$

The lemma follows from (150) and (154). □

We now check the second condition of W_σ, $v \geq A_{135} n z^2$.

Going back to (120), we are left to consider the differential equation,

$$\frac{dM}{dv} = \frac{1 + \frac{vz^2e^z}{mf} + \frac{v^2z^4e^z}{4m^2f^2}}{1 + \frac{v^2z^4e^z}{4m^2f^2}}.$$

The solution of this was obtained in [2]: Here z^* is the value of z at $t = 0$ and $M^* = 2m^* = 2cn$.

$$M = \frac{M^*(e^z - 1)z}{z^*(e^{z^*} - 1)} \exp\left\{ -\int_z^{z^*} \frac{\xi e^\xi}{e^\xi(1+\xi) - 1} d\xi \right\}. \tag{155}$$

Then (up to a v_1 error term)

$$\begin{aligned} v &= \frac{2mf(z)}{z(e^z - 1)} \\ &= \frac{2m^* f(z)}{z^*(e^{z^*} - 1)} \exp\left\{ -\int_z^{z^*} \frac{\xi e^\xi}{e^\xi(1+\xi) - 1} d\xi \right\}. \end{aligned} \tag{156}$$

So we define

$$J_1 = \frac{v}{nf(z)} \exp\left\{ \int_z^{z^*} \frac{\xi e^\xi}{e^\xi(1+\xi) - 1} d\xi \right\}$$

and

$$J_2 = \frac{m}{nz(e^z - 1)} \exp\left\{ \int_z^{z^*} \frac{\xi e^\xi}{e^\xi(1+\xi) - 1} d\xi \right\}.$$

Lemma 20. *Let $\mathcal{T}_{-1} = 0$. Then for $\sigma = 0, 1$,*

$$\mathbf{Pr}\left(\max_{\tau \in [\mathcal{T}_{\sigma-1}, \mathcal{T}_\sigma]} |J_i(\boldsymbol{\nu}(\tau)) - J_i(\boldsymbol{\nu}(\mathcal{T}_{\sigma-1}))| > n^{-\alpha_\sigma/4} \right) = O(n^{-4}), \qquad i = 1, 2.$$

Proof Now fix $i = 1$ or 2 and let $J(t) = J_i(\boldsymbol{\nu}(t))$. Let now $K = (\log n)^2$ and define $Q(t) = \exp\{K(J(t) - J(\mathcal{T}_{\sigma-1}))\}$ for $\mathcal{T}_{\sigma-1}) \leq t < \mathcal{T}_\sigma$. Let $Q(t) = 0$ for $t \geq \mathcal{T}_\sigma$.

We consider only $i = 1$ since the other case is very similar. For $t > \mathcal{T}_\sigma$, we obviously have $Q(t) = Q(t-1) = 0$. For $\mathcal{T}_{\sigma-1}) \leq t \leq \mathcal{T}_\sigma$ we can write

$$\mathbf{E}(Q(t) \mid \{\boldsymbol{\nu}(s)\}_{s<t}) \leq Q(t-1)\mathbf{E}\left\{ \mathbf{1}_{|\boldsymbol{\nu}(t) - \boldsymbol{\nu}(t-1)| \leq \log n} \exp\left[K(J(t) - J(t-1))\right] \middle| \, \boldsymbol{\nu}(t-1) \right\}. \tag{157}$$

Since $\boldsymbol{\nu}(t-1) \in W_\sigma$, each of $m(t-1)$ and $v(t-1)$ is of order $n^{1-2\alpha_\sigma}$ at least. The same holds then for $\boldsymbol{\nu}(t) \in B(\boldsymbol{\nu}(t-1), \log n) = \{\boldsymbol{\nu} : \ |\boldsymbol{\nu} - \boldsymbol{\nu}(t-1)| \leq \log n\}$. Consequently $v(t)z(t)$ is of order $n^{1-3\alpha_\sigma}$ at least. Moreover, it can be easily verified that, uniformly for such $\boldsymbol{\nu}$ and $i = 1, 2$, $x, y = v, m$,

$$\frac{\partial J}{\partial x} = O\left(\frac{1}{vz}\right), \tag{158}$$

$$\frac{\partial^2 J}{\partial x \partial y} = O\left(\frac{1}{v^2 z^2}\right). \tag{159}$$

Let $\hat{\boldsymbol{\nu}} = (v, m)$ i.e. drop all other parameters. Assuming $\boldsymbol{\nu}(t) \in B(\boldsymbol{\nu}(t-1), \log n)$, expanding the exponential function, and viewing $J(t)$ as a function of v, m only,

$$\exp\{K(J(t) - J(t-1))\} = [1 + K\nabla J(t)^T(\hat{\boldsymbol{\nu}}(t) - \hat{\boldsymbol{\nu}}(t-1)) + O(K^2(\log n)^2/(vz)^2)], \tag{160}$$

since

$$K \log n = o(vz). \tag{161}$$

Consequently, equation (157) becomes

$$\begin{aligned} \mathbf{E}(Q(t)|\{\boldsymbol{\nu}(s)\}_{s<t}) \quad &\leq \quad Q(t-1)\left\{1 + K\nabla J(t)^T \mathbf{E}[\hat{\boldsymbol{\nu}}(t) - \hat{\boldsymbol{\nu}}(t-1)|\boldsymbol{\nu}(t-1)]\right\} \\ &\quad\quad + O(Q(t-1)K^2(\log n)^2/(vz)^2). \end{aligned} \tag{162}$$

Putting

$$\mathbf{F}(\hat{\boldsymbol{\nu}}) = \begin{bmatrix} 1 + \frac{vz^2e^z}{mf} + \frac{v^2z^4e^z}{4m^2f^2} \\ 1 + \frac{v^2z^4e^z}{4m^2f^2} \end{bmatrix}$$

and using Lemmas 14 and 15,

$$\begin{aligned} \nabla J(t)^T \mathbf{E}[\hat{\boldsymbol{\nu}}(t) - \hat{\boldsymbol{\nu}}(t-1)|\boldsymbol{\nu}(t-1)] \quad &= \quad \nabla J(t-1)^T \left[\mathbf{F}(\hat{\boldsymbol{\nu}}(t-1)) + \Theta\right] \\ &= \quad O\big(\|\nabla J(t-1)\|\Theta\big) \\ &= \quad O\left(\frac{\log n}{vz}\Theta\right). \end{aligned} \tag{163}$$

($\nabla J(\hat{\boldsymbol{\nu}}) \perp \mathbf{F}(\hat{\boldsymbol{\nu}})$ since $J(\hat{\boldsymbol{\nu}})$ is constant along the trajectory of $d\hat{\boldsymbol{\nu}}/dt = \mathbf{F}(\hat{\boldsymbol{\nu}})$!)

Therefore, for $t - 1 < \mathcal{T}_\sigma$ and hence for all $t \geq \mathcal{T}_{\sigma-1}$,

$$\mathbf{E}(Q(t) \mid \{\boldsymbol{\nu}(s)\}_{s<t}) \leq Q(t-1)(1 + O(K(\log n)\Theta/(vz))) = Q(t-1)(1 + O(n^{7\alpha_0 - 2}(\log n)^4).$$

So for any positive ϵ, the random sequence

$$\{R(t)\} := \{(1 + n^{7\alpha_0 + \epsilon - 2})^{-t}Q(t)\}$$

is a *supermartingale.*

Introduce a stopping time

$$\mathcal{T}'_\sigma = \begin{cases} \min\{\mathcal{T}_{\sigma-1} \leq t < \mathcal{T}_\sigma : \ J(t) - J(\mathcal{T}_{\sigma-1}) > n^{-\alpha_\sigma/4}/2\}, & \text{if such } t \text{ exist,} \\ \mathcal{T}_\sigma, & \text{otherwise.} \end{cases}$$

For the reminder of the proof of the lemma, we take $\sigma = 0$. We will continue to use the subscript σ because we will return and finish the case $\sigma = 1$ later. Let $n_\sigma = \mathcal{T}_\sigma - \mathcal{T}_{\sigma-1}$.

Now, applying the Optional Sampling Theorem (Durrett [7]) to the supermartingale $\{R(t)\}$ and the stopping time $\mathcal{T}'_0$ we get

$$\begin{aligned} \mathbf{E}[Q(\mathcal{T}'_\sigma)] \quad &\leq \quad \mathbf{E}(1 + n^{7\alpha_\sigma + \epsilon - 2})^{n_\sigma} \cdot \mathbf{E}[Q(\mathcal{T}_{\sigma-1})] && (164) \\ &\leq \quad \mathbf{E}(1 + n^{7\alpha_\sigma + \epsilon - 2})^{n} \cdot \mathbf{E}[Q(\mathcal{T}_{\sigma-1})] \\ &= \quad (1 + n^{7\alpha_\sigma + \epsilon - 2})^n \\ &= \quad 1 + o(1), \ \text{as } n \to \infty, && (165) \end{aligned}$$

for ϵ sufficiently small.

Remark 2. *Note that in the case $\sigma = 1$ we have $7\alpha_1 - 2 > -1$ and we cannot argue that $(1+n^{7\alpha_1+\epsilon-2})^n = 1+o(1)$. We will have to argue instead that* **whp** *$n_1 = O(n^{1-2\alpha_0})$ and then all we need is that $7\alpha_1 - 2 < 2\alpha_0 - 1$.*

Since

$$\mathbf{E}[Q(\mathcal{T}'_\sigma)] \geq e^{n^{\alpha_\sigma/4}/2} \cdot \mathbf{Pr}\{\mathcal{T}'_\sigma < \mathcal{T}_\sigma\},$$

we have

$$\mathbf{Pr}\{\max_{\mathcal{T}_{\sigma-1}\leq t<\mathcal{T}_\sigma} [J(t) - J(\mathcal{T}_{\sigma-1})] > n^{-\alpha_\sigma/4}/2\} = \mathbf{Pr}\{\mathcal{T}'_\sigma < \mathcal{T}_\sigma\} = O(e^{-n^{\alpha_\sigma/4}/2}).$$

Analogously,

$$\mathbf{Pr}\{\min_{\mathcal{T}_{\sigma-1}\leq t<\mathcal{T}_\sigma} [J(t) - J(\mathcal{T}_{\sigma-1})] < -n^{-\alpha_\sigma/4}/2\} = O(e^{-n^{\alpha_\sigma/4}/2}).$$

So **qs**

$$\max_{\mathcal{T}_{\sigma-1}\leq t<\mathcal{T}_\sigma} |J(t) - J(\mathcal{T}_{\sigma-1})| \leq n^{-\alpha_\sigma/4}/2.$$

It only remains to note the equation (92).

This completes the proof of the lemma for $\sigma = 0$. □

At time $\mathcal{T}_0$ either (i) $z \leq n^{-\alpha_0}$, (ii) $z > 3c$, (iii) $v < A_{135}nz^2$ or (iv) $v_1 > B_{135}n^{2\alpha_0}(\log n)^3$ or (v) $\gamma > C_{135}n^{2\alpha_0}(\log n)^4$.

Possibility (ii) is ruled out by (94). (156) and Lemma 20 show that for $t \in [0, \mathcal{T}_0]$, $v(t) \approx A_2nz^2$ where

$$A = \frac{m^*}{nz^*(e^{z^*}-1)} \exp\left\{-\int_0^{z^*} \frac{\xi e^\xi}{e^\xi(1+\xi)-1} d\xi\right\}.$$

This rules out possibility (iii) if we take

$$A_{135} = A/2. \tag{166}$$

Possibility (iv) is ruled out by Lemma 18 and possibility (v) is ruled out by Lemma 19. So we can assume that at time $\mathcal{T}$,

$$\begin{aligned} z &\approx n^{-\alpha_0} & (167)\\ v &\approx Anz^2 & (168)\\ m &\approx Anz^2 & (169)\end{aligned}$$

$$z(\mathcal{T}_0) - z(\mathcal{T}_0 - 1) = O((\log n)/v) \tag{170}$$

is the justification for (167), $m \approx v$ comes from $z = o(1)$ and

$$2 \leq \frac{2m - v_1}{v} = \frac{z(e^z - 1)}{f(z)} = 2\left(1 + \frac{z}{6} + O(z^2)\right). \tag{171}$$

So if we condition on (167)–(169) then we can go back to the proof of Lemma 20 at equation (164) and take $n_1 = O(n^{1-2\alpha_0}$ and now find that (165) holds. We have fulfilled the condition laid out in Remark 2 and finish the proof of the lemma.

We can therefore argue that at time $\mathcal{T}_1$, (167,168,167) hold with α_0 replaced by α_1. The number of isolated vertices that are created from $\mathcal{T}_1$ onwards is bounded by the sum of (i) $\sum_{k\geq 3} kv_k(\mathcal{T}_1)$ and (ii) the number κ_1 of components of $G(\mathcal{T}_1)$ which are paths of odd length. Now **qs**

$$\sum_{k\geq 3} kv_k(\mathcal{T}_1) \approx v(\mathcal{T}_1)z(\mathcal{T}_1)/3 = O(n^{1-3\alpha-1}) = O(n^{.49}).$$

The number of paths of odd length is bounded by the number of vertices of degree 1 which is $O(n^{2\alpha_1+o(1)})$.

This completes the proof of Lemma 5.

6 Proof of Theorem 3

Our proof will follow the lines of the proof of Theorem 2. We will prove in Lemma 22 that adding a few random edges is likely to increase the size of a maximum matching. We then state a lemma for $G_{n,m}^{\delta\geq 2}$ which corresponds to Lemma 4. The final piece, the analogy to Lemma 5 can now be claimed from [2].

Lemma 21. *Let $G = (V,E)$ be a graph with $|V|$ even, $\delta(G) \geq 2$, with no isolated odd cycles and which does not have a perfect matching. For every x which is not covered by* every *perfect matching, there exists $K = K(x), L = L(x)$ such that*

(i) *$x \notin K \cup L$, and $(x,y) \notin E$ for every $y \in K$.*

(ii) $|K| = |L| + 1$.

(iii) *$N_G(K) = L$.* $\qquad$ *($N_G(K) = \{w \notin K : \exists (v,w) \in E_G, v \in K\}$).*

(iv) $|E_G(K \cup L)| \geq |K| + |L| + 1$.

(v) *Each $v \in L$ has at least 2 neighbours in K.*

(vi) *For every $y \in K$ there exists a maximum matching that does not cover y.*

(vii) *Adding any (x,y), $y \in K(x)$, to E increases the size of a maximum matching.*

Proof Let $x \in V$ and let M be a maximum matching which does not cover x. Since $|V|$ is even, there exists $s \neq x$ which is also left isolated by M. Now let T be a tree of maximal size which is rooted at s and such that for each $v \in T$, the path from s to v in T is alternating with respect to M. Let K, L be the set of vertices at even and odd distance respectively from s in T. For every $y \in K$, we can switch edges on the even path from the root to y to obtain another maximum matching that does not cover y. Furthermore, by maximality of T, all the leaves of T are in K, so that all the vertices of T, except s, are covered by M. Furthermore, if a neighbor u of a vertex from K is not in $K \cup L$, then u must be covered by M, which contradicts maximality of T. Therefore the pair (K,L) meets all the conditions, except possibly (iv). Using $\delta(G) \geq 2$, we can only assert that $|E_G(K \cup L)| \geq |K| + |L|$. But if $|E_G(K \cup L)| = |K| + |L|$ then $K \cup L$ induces an odd cycle and s is of degree 2 in G. Since there are no isolated odd cycles in G, there must be some edge (v,w), $v \in L$, $w \notin K \cup L$. Since v is covered by M, $(v,w) \notin M$ and we can alter M solely on $K \cup L$ to obtain a maximum matching M' in which v is isolated. Now the degree of v is at least 3 and so if we re-compute T with v in place of s, T will have at least 3 leaves and (iv) will be satisfied. □

So now let $\mathcal{A}_n(\epsilon)$ denote the event that there exist K, L satisfying (ii)–(v), and such that $|K| \leq \epsilon n$.

Lemma 22. *There exists an $\epsilon > 0$ such that $\mathbf{Pr}(\mathcal{A}_n(\epsilon)) = O(n^{-1})$.*

Proof First of all, [19], $N(n,m)$ the total number of graphs with minimum degree at least 2 is asymptotic to

$$N_0(n,m) = \frac{(2m-1)!!}{\sqrt{2\pi n \mathrm{Var} Z}} \cdot \frac{f_2(\rho)^n}{\rho^{2m}} \cdot \exp(-\hat{\rho}/2 - \hat{\rho}^2/4), \tag{172}$$

where $\rho, \hat{\rho}$ satisfy

$$\frac{\rho f_1(\rho)}{f_2(\rho)} = \frac{2m}{n}, \quad \hat{\rho} = \frac{\rho f_0(\rho)}{f_1(\rho)}. \tag{173}$$

and Z is Poisson(ρ), conditioned on $Z \geq 2$. In fact, for all $a, b, x > 0$,

$$N(a,b) \leq c^* \frac{(2b-1)!!}{\sqrt{nx}} \cdot \frac{f_2(x)^a}{x^{2b}}, \tag{174}$$

where c^* does not depend on a, b, x. (The attentive reader certainly notices direct analogy between these formulas and their counterparts for the bipartite case in Section 2.) The independent copies $Z_1, \dots, Z_n$ of Z provide an approximation to $\mathbf{deg}(\Gamma)$, the degree sequence of the random graph Γ, in the following sense:

$$\mathbf{Pr}(\mathbf{deg}(\Gamma) \in B) = O\big(n^{1/2}\mathbf{Pr}(\mathbf{Z} \in B)\big), \tag{175}$$

uniformly for all sets B of n-tuples. Consequently, if B is such that $\mathbf{Pr}(\mathbf{Z} \in B)$ is $O(n^{-b})$ for some $b > 1/2$, then $\mathbf{Pr}(\mathbf{deg}(\Gamma) \in B) = O(n^{-(b-1/2)})$, which goes to zero, too! A particular event B which will come in handy is defined as follows. Let $d(j) = d(j, \Gamma)$ denotes the j-th largest degree of Γ. Pick $a > e^{5+\rho}(h(\rho)+1)^2$ where $h(\rho) = \frac{2}{f_2(\rho)}$ and define $\ell(n,j) = \lceil \log \frac{e^a n}{j} \rceil$. Let us show that

$$\mathbf{Pr}(\exists j \in [1,n]: \ d(j) > \ell(n,j)) = O(n^{-1}). \tag{176}$$

To prove this, consider first $Z(j)$, the j-th largest among $Z_1, \dots, Z_n$. Clearly

$$\mathbf{Pr}(Z(j) > \ell(n,j)) \leq \binom{n}{j} \mathbf{Pr}^j(Z_1 > \ell(n,j)) \leq \exp\left(j \log \frac{en}{j} + j \log \mathbf{Pr}(Z_1 \geq \ell(n,j))\right),$$

and, using the definition of $\ell(n,j)$ and a,

$$\begin{aligned}
\mathbf{Pr}(Z_1 > \ell(n,j)) &\leq h(\rho)\frac{\rho^{\ell(n,j)}}{\ell(n,j)!} \leq \exp\left(\log h(\rho) - \ell(n,j) \log \frac{\ell(n,j)}{e\rho}\right) \\
&\leq \exp\left(-a \log \frac{e^a n}{j}\right).
\end{aligned}$$

Consequently

$$\mathbf{Pr}(Z(j) > \ell(n,j)) \leq \exp\left(-j(a-1)\log \frac{en}{j}\right),$$

so that

$$\mathbf{Pr}(\exists j \in [1,n]: \ Z(j) > \ell(n,j)) \leq \sum_{j=1}^{n} \mathbf{Pr}(Z(j) > \ell(n,j)) = O(n^{-2}),$$

whence the probability in (176) is $O(n^{-3/2})$. Now, for a given vertex subset S,

$$\sum_{j \in S} d_j \leq \sum_{j=1}^{|S|} d(j),$$

and on the event in (176)

$$\sum_{j=1}^{s} d(j) \leq \sum_{j=1}^{s} \lceil \log \frac{e^a n}{j} \rceil \leq (2+a)s + s \log \frac{n}{s}.$$

We conclude that

$$\mathbf{Pr}\big(\exists S \subset [n] \,:\, \sum_{j\in S} d_j > (2+a)|S| + |S|\log(n/|S|)\big) = O(n^{-1}). \tag{177}$$

Now, with k as a hidden parameter, let T_{μ,ν,ν_1} denote the total number of pairs (K,L) consisting of disjoint subsets $K, L \subset [n]$ such that $|K| = k$, $|L| = k-1$, **(ii)**–**(v)** hold and μ, ν, ν_1 are given by

$$\begin{aligned}
&|E(K)| + |E(L)| = \mu,\\
&|\{(u,w) \in E(\Gamma) : u \in K, w \in L\}| = \nu,\\
&|\{(u,w) \in E(\Gamma) : u \in L, w \in (K \cup L)^c\}| = \nu_1.
\end{aligned}$$

Note that by **(iii)**

$$\nu + \mu \geq 2k. \tag{178}$$

We want to show that

$$\mathbf{Pr}\left(\sum_{2\leq k\leq \epsilon n} \sum_{\{\mu,\nu,\nu_1\}} T_{\mu,\nu,\nu_1} > 0\right) = O(n^{-1}),$$

provided that $\epsilon > 0$ is sufficiently small. By the above discussion we may and will confine ourselves to μ, ν, ν_1 such that

$$\mu, \nu, \nu_1 \leq A(k + k\log n/k), \tag{179}$$

for a large enough constant A. All we need to show is that

$$\sum_{k\leq \epsilon n} \sum_{\substack{\mu,\nu,\nu_1:\\ (179)\text{ holds}}} E_{\mu,\nu,\nu_1} = O(n^{-1}), \quad E_{\mu,\nu,\nu_1} := \mathbf{E}(T_{\mu,\nu,\nu_1}). \tag{180}$$

By symmetry,

$$E_{\mu,\nu,\nu_1} = \binom{n}{k, k-1, n-2k+1} P_{\mu,\nu,\nu_1}, \tag{181}$$

where P_{μ,ν,ν_1} is the probability that the subsets $K^* = \{1,\dots,k\}$ and $L^* = \{k+1,\dots,2k-1\}$ form such a pair. To bound this probability we need to bound N_{μ,ν,ν_1}, the total number of graphs in question in which the pair $\{K^*, L^*\}$ has the prescribed properties.

Let $(\delta_j)_{j\in K^*}$, $(\delta_j)_{j\in L^*}$ and $(\delta_j)_{j\in (K^*\cup L^*)^c}$ be the degree sequences for subgraphs $G(K^*)$, $G(L^*)$ and $G((K^*\cup L^*)^c)$ respectively. For $j \in K^*$ ($j \in L^*$ resp.) let Δ_j denote the total number of neighbors of j in L^* (in K^* resp.). For $j \in L^*$ ($j \in (K^*\cup L^*)^c$ resp.) let ∂_j denote the total number of neighbors of j in $(K^*\cup L^*)^c$ (in L^* resp.). Then

$$\begin{aligned}
&\sum_{j\in K^*\cup L^*} \delta_j = 2\mu, \qquad \sum_{j\in (K^*\cup L^*)^c} \delta_j = 2(m-\mu-\nu-\nu_1),\\
&\sum_{j\in K^*} \Delta_j = \sum_{j\in L^*} \Delta_j = \nu, \quad \sum_{j\in L^*} \partial_j = \sum_{j\in (K^*\cup L^*)^c} \partial_j = \nu_1.
\end{aligned} \tag{182}$$

In addition,

$$\begin{aligned}\delta_j+\Delta_j\geq 2,\quad & j\in K^*\cup L^*,\\ \delta_j+\partial_j\geq 2,\quad & j\in (K^*\cup L^*)^c.\end{aligned}\tag{183}$$

It is worth noticing that (183) is a relaxed version of the actual restrictions. Also, lumping together δ_j for $j \in K^*$ and $j \in L^*$, we effectively ignore the fact that the graphs $G(K^*)$ and $G(L^*)$ are disjoint.

Denoting the total number of graphs with the given $\mathbf{D} = (\boldsymbol{\delta}, \boldsymbol{\Delta}, \boldsymbol{\partial})$ by $N(\mathbf{D})$, and using the degree-dependent bounds for the counts of graphs, both general and bipartite, we obtain

$$\begin{aligned}
&N(\mathbf{D}) \leq \\
&\left((2\mu-1)!!\prod_{j\in K^*\cup L^*}\frac{1}{\delta_j!}\right)\times\left(\nu!\prod_{j\in K^*\cup L^*}\frac{1}{\Delta_j!}\right)\times\left(\nu_1!\prod_{j\in(K^*\cup L^*)^c}\frac{1}{\partial_j!}\prod_{j\in L^*}\frac{1}{\partial_j!}\right)\times\\
&\left((2(m-\mu-\nu-\nu_1)-1)!!\prod_{j\in(K^*\cup L^*)^c}\frac{1}{\delta_j!}\right)\\
&=(2\mu-1)!!\nu!\nu_1!(2(m-\mu-\nu-\nu_1)-1)!!\cdot\Phi(\mathbf{D});\\
&\Phi(\mathbf{D})=\prod_{j\in K^*\cup L^*}\frac{1}{\delta_j!\Delta_j!}\cdot\prod_{j\in L^*}\frac{1}{\partial_j!}\cdot\prod_{j\in(K^*\cup L^*)^c}\frac{1}{\partial_j!\delta_j!}.
\end{aligned}\tag{184}$$

Our task now is to evaluate S_{μ,ν,ν_1}, the sum of $\Phi(\mathbf{D})$, for all $\mathbf{D}$ that meet (182) and (183). To do so, let us first determine a *multivariate* generating function of $\Phi(\mathbf{D})$, for $\mathbf{D}$ satisfying (183) only:

$$\begin{aligned}
&\sum_{\mathbf{D}\text{ satisfies (183)}} y_1^{\sum_{j\in K^*\cup L^*}\delta_j}y_2^{\sum_{j\in K^*\cup L^*}\Delta_j}y_3^{\sum_{j\in L^*}\partial_j}y_4^{\sum_{j\in(K^*\cup L^*)^c}\partial_j}y_5^{\sum_{j\in(K^*\cup L^*)^c}\delta_j}\cdot\Phi(\mathbf{D})\\
&\qquad=\left(\sum_{\delta+\Delta\geq 2}\frac{y_1^{\delta}y_2^{\Delta}}{\delta!\Delta!}\right)^{2k-1}\left(\sum_{\partial\geq 0}\frac{y_3^{\partial}}{\partial!}\right)^{k-1}\left(\sum_{\partial+\delta\geq 2}\frac{y_4^{\partial}y_5^{\delta}}{\partial!\delta!}\right)^{n-2k+1}\\
&\qquad=f_2(y_1+y_2)^{2k-1}f_0(y_3)^{k-1}f_2(y_4+y_5)^{n-2k+1};
\end{aligned}\tag{185}$$

So now S_{μ,ν,ν_1} is the coefficient of $y_1^{2\mu}y_2^{2\nu}(y_3y_4)^{\nu_1}y_5^{2(m-\mu-\nu-\nu_1)}$ in the function on the right hand of (185). Using

$$[z_1^{a_1}z_2^{a_2}]F(z_1+z_2)=\binom{a_1+a_2}{a_1}[z^{a_1+a_2}]F(z),$$

and $f_0(y_3) = e^{y_3}$, we obtain then:

$$\begin{aligned}
S_{\mu,\nu,\nu_1}=&\binom{2(\mu+\nu)}{2\mu}\binom{2(m-\mu-\nu)-\nu_1}{\nu_1}\frac{(k-1)^{\nu_1}}{\nu_1!}\\
&\times[x_1^{2(\mu+\nu)}]f_2(x_1)^{2k-1}\cdot[x_2^{2(m-\mu-\nu)-\nu_1}]f_2(x_2)^{n-2k+1}.
\end{aligned}\tag{186}$$

Here

$$[x_1^{2(\mu+\nu)}]f_2(x_1)^{2k-1} \le \frac{f_2(x_1)^{2k-1}}{x_1^{2(\mu+\nu)}}, \quad \forall\, x_1 > 0,$$

and we will see that a sufficiently small x_1 will do the job. We can write an analogous bound for the last factor in (186), and (in the light of (172) and the relative smallness of our parameters μ, ν, ν_1) $x_2 = \rho$ is a natural choice. In fact, we can do a bit better and get an extra factor $n^{-1/2}$, by applying the Cauchy (circular) contour formula, cf. (14), in combination with (15). Using

$$\begin{aligned} P_{\mu,\nu,\nu_1} =& \frac{N_{\mu,\nu,\nu_1}}{N(n,m)}, \\ N_{\mu,\nu,\nu_1} =& \sum_{\mathbf{D}\text{ satisfies (182),(183)}} N(\mathbf{D}), \end{aligned}$$

(172),(184), and an inequality

$$\binom{2u-v}{v}(2(u-v)-1)!! \le \frac{(2u-1)!!}{v!},$$

we obtain then

$$\begin{aligned} P_{\mu,\nu,\nu_1} \le_b & (2\mu-1)!!\nu!\binom{2(\mu+\nu)}{2\mu}\frac{(2(m-\mu-\nu)-1)!!}{(2m-1)!!} \\ & \times \frac{f_2(x_1)^{2k-1}}{x_1^{2(\mu+\nu)}} \cdot \frac{\rho^{2(\mu+\nu)}}{f_2(\rho)^{2k-1}} \cdot \frac{(\rho(k-1))^{\nu_1}}{\nu_1!}. \end{aligned} \tag{187}$$

Then since

$$\sum_{\nu_1 \ge 0} \frac{(\rho(k-1))^{\nu_1}}{\nu_1!} = e^{\rho(k-1)} < e^{\rho k},$$

we get the bound (call it $Q_{\mu,\nu}$) for $\sum_{\nu_1} P_{\mu,\nu,\nu_1}$, which is (187) with the last factor replaced by $e^{\rho k}$.

Next, for $\nu \ge \nu_0 = \nu_0(\mu) := \max\{2(k-1), 2k-\mu\}$, using (179),

$$\begin{aligned} \frac{Q_{\mu,\nu+1}}{Q_{\mu,\nu}} \le_b & \frac{\rho^2}{x_1^2} \cdot \frac{\mu^2+\nu^2}{\nu m} \\ \le_b & \frac{\rho^2}{x_1^2} \cdot \left(\frac{k^2 + k^2\log^2(n/k)}{km} + \frac{\nu}{m}\right) \\ \le_b & \frac{\rho^2}{x_1^2} \cdot (\epsilon + \epsilon\log^2(1/\epsilon)) \\ \le & 1/2, \end{aligned}$$

if $k \le \epsilon n$ and $0 < \epsilon \le \epsilon_1(x_1)$ is chosen sufficiently small. For this choice of ϵ,

$$\sum_{\nu \ge \nu_0(\mu)} Q_{\mu,\nu} = O(Q_{\mu,\nu_0(\mu)}). \tag{188}$$

Furthermore, if $\mu \geq 2$ then $\nu_0 = \nu_0(\mu) = 2(k-1)$ and we have

$$\begin{aligned}\frac{Q_{\mu+1,\nu_0(\mu+1)}}{Q_{\mu,\nu_0(\mu)}} &\leq_b \frac{\rho^2}{x_1^2} \cdot \frac{\mu^2+k^2}{\mu m} \\ &\leq_b \frac{\rho^2}{x_1^2} \cdot \left(\frac{k}{m} + \frac{k}{m}\log^2(n/k)\right)\frac{k}{\mu} \\ &\leq_b \frac{\rho^2}{x_1^2} \cdot (\epsilon + \epsilon \log^2(1/\epsilon)) \cdot \frac{k}{\mu} \\ &\leq 1/2,\end{aligned}$$

if $\mu \geq \epsilon^{1/2}k$, and $0 < \epsilon \leq \epsilon_2(x_1) < \epsilon_1(x_1)$ is chosen sufficiently small. If so,

$$\sum_{\mu \geq 2} Q_{\mu,\nu_0} \leq_b k \max_{\mu}\{Q_{\mu,\nu_0} : 2 \leq \mu \leq k\epsilon^{1/2}\}. \tag{189}$$

To make the last bound explicit, we use $(2a-1)!! = (2a)!/(2^a a!)$ and the Stirling formula for factorials to bound, for $\nu = \nu_0$ and $2 \leq \mu \leq \epsilon^{1/2}k$, the combinatorial factors in (187) as follows:

$$\begin{aligned}(2\mu-1)!! &\leq_b k^2\left(\frac{2\mu}{e}\right)^{\mu-2}; \\ \binom{2(\mu+\nu_0)}{2\mu} &\leq_b \exp(O(k\epsilon^{1/2}\log\epsilon^{-1})); \\ \frac{(2(m-\mu-\nu_0)-1)!!}{(2m-1)!!} &\leq_b (2m)^{-(\mu-2)-2k}\exp(O(\epsilon k)).\end{aligned}$$

Using these bounds and $k^2\nu_0! \leq (2k)!$, we obtain: for $2 \leq \mu \leq \epsilon^{1/2}k$ and $x_1 < \lambda$,

$$Q_{\mu,\nu_0} \leq_b n^{1/2}(2k)!(2m)^{-2k}\left(\frac{f_2(x_1)}{x_1^2}\right)^{2k}\left(\frac{\rho^2}{f_2(\rho)}\right)^{2k} e^{\rho k}\left(\frac{\rho}{x_1}\right)^{O(\epsilon^{1/2}k)} \exp(O(k\epsilon^{1/2}\log\epsilon^{-1})). \tag{190}$$

If $\mu = 0$ then $\nu_0 = 2k$, and if $\mu = 1$ then $\nu_0 = 2k-1$. The direct computation shows that the bound (190) holds in these two remaining cases as well. So, collecting the pieces and using

$$\frac{(2k)!}{k!(k-1)!} \leq_b k2^{2k},$$

we get

$$\sum_{\mu,\nu,\nu_1} E_{\mu,\nu,\nu_1} \leq_b n^{-1/2}q^{2k}\exp(O(k\epsilon^{1/2}\log\epsilon^{-1})), \tag{191}$$

where

$$q = 2\frac{n}{2m} \cdot \frac{f_2(x_1)}{x_1^2} \cdot \frac{\rho^2}{f_2(\rho)} \cdot e^{\rho/2}. \tag{192}$$

Using (173), we transform (192) into

$$q = \frac{\rho}{e^{\rho/2} - e^{-\rho/2}} \cdot \frac{2f_2(x_1)}{x_1^2}.$$

The first fraction is strictly less than 1, while the second fraction approaches 1 from above when $x_1 \downarrow 0$. So we can pick x_1 small enough to make $\rho < 1$. For this choice of x_1, and the corresponding $\epsilon = \epsilon(x_1) < \epsilon_2(x_1)$, we have

$$q^2 \exp(O(\epsilon^{1/2} \log \epsilon^{-1})) \le q_1 := \frac{1+q^2}{2} < 1.$$

Then (191) implies that

$$\sum_{k \le \epsilon n} \sum_{\mu,\nu,\nu_1} E_{\mu\nu\nu_1} \le_b n^{-1/2} \sum_{k \ge 0} q_1^k = O(n^{-1/2}).$$

Thus (180) is completely proved, and so is Lemma 22. □

Introduce ∂E, the set of *non-edges* (x, y) such that adding (x, y) to E, the edge set of the random graph, increases the maximum matching number.

Lemma 23.

$$\mathbf{Pr}(0 < |\partial E| < \epsilon^2 n/2) = O(n^{-1}).$$

Proof Suppose the event $\{|\partial E| > 0\} \cap \mathcal{A}_n(\epsilon)^c$ happens. Then, by Lemma 21, for every vertex x not covered by at least one maximum matching there exists a vertex set K of cardinality ϵn or more, such that (1) $x \notin K$, $x \notin N_G(K)$; (2) adding any (x, y), $y \in K$, to the edge set increases the maximum matching number; (3) for every vertex $y \in K$ there exists a maximum matching that does not cover y. This implies existence of the vertices $x_1, \dots, x_{\nu_n}$, ($\nu_n := [\epsilon n]$), such that for every x_j there is a corresponding vertex subset Y_j satisfying (1) $x_j \notin Y_j$, $x_j \notin N_G(Y_j)$; (2) for every $y \in Y_j$, adding (x_j, y) to the edge set increases the maximum matching number; (3) $|Y_j| \ge \nu_n$. Consequently the edge set of the graph is missing at least

$$\sum_{j=1}^{\nu_n} (\nu_n - j + 1) = \frac{\nu_n(\nu_n + 1)}{2} \ge \frac{\epsilon^2 n^2}{2}$$

pairs (x, y) such that adding any such pair to the edge set would increase the maximum matching. Therefore

$$\{0 < |\partial E| \le \epsilon^2 n^2/2\} \subseteq \mathcal{A}_n(\epsilon),$$

and the claim follows from Lemma 22. □

The next ingredient is

Lemma 24. *Let P_n denote the probability that the random graph has no isolated odd cycles. Then*

$$\lim_{n\to\infty} P_n = e^{\sigma/2} \left(\frac{1-\sigma}{1+\sigma}\right)^{1/4}, \tag{193}$$

where

$$\sigma = \frac{\rho}{e^\rho - 1},$$

and ρ satisfies

$$\frac{\rho(e^\rho - 1)}{e^\rho - 1 - \rho} = 2c.$$

Proof Let $X_{n,\ell}$ denote the total number of isolated cycles of length $\ell \geq 3$. Then, given $L > 3$,

$$\mathbf{E}\left(\sum_{\ell \geq L} X_{n,\ell}\right) = \sum_{\ell \geq L} \binom{n}{\ell} \cdot \frac{(\ell-1)!}{2} \frac{N(n-\ell, m-\ell)}{N(n,m)}. \tag{194}$$

Using (172) and (174) with $x = \rho$, and

$$\prod_{j=0}^{\ell-1} \frac{n-j}{2(m-j)-1} \leq \left(\frac{n}{2m-1}\right)^{\ell},$$

we see that the generic term in the sum is of order at most

$$\frac{1}{2\ell}\sqrt{\frac{n}{n-\ell}} \cdot \left(\frac{n}{2m} \cdot \frac{\rho^2}{f_2(\rho)}\right)^{\ell} = \frac{1}{2\ell} \cdot \sqrt{\frac{n}{n-\ell}} \cdot \sigma^{\ell}.$$

Since $\sigma < 1$, it easily follows then that $\mathbf{E}\left(\sum_{\ell \geq L} X_{n,\ell}\right) \to 0$ if $L = L(n) \to \infty$ however slowly. Consequently, whp there are no isolated cycles of length exceeding $L = L(n)$. Introduce

$$X_n = \sum_{\substack{\ell \leq L(n) \\ \ell \text{ odd}}} X_{n,\ell}.$$

Then, for every fixed $k \geq 1$,

$$\begin{aligned}
\mathbf{E}[(X_n)_k] &= \sum_{\ell \leq kL} R_{n,m}(\ell) \sum_{\substack{\ell_1,\ldots,\ell_k \in [3,L] \\ \ell_1+\cdots+\ell_k=\ell \\ \ell_1,\ldots,\ell_k \text{ odd}}} \prod_{j=1}^{k} \frac{(\ell_j-1)!}{2(\ell_j!)} \\
&= \sum_{\ell \leq kL} R_{n,m}(\ell) \cdot [x^{\ell}] \left(\sum_{j \in [3,L], j \text{ odd}} \frac{x^j}{2j}\right)^K; \\
R_{n,m}(\ell) &:= \frac{n! N(n-\ell, m-\ell)}{(n-\ell)! N(n,m)}.
\end{aligned}$$

For $\ell \leq L(n)$ and $L(n) \to \infty$ sufficiently slowly, $N(n-\ell, m-\ell)$ is asymptotic the RHS in (172), with n and m replaced by $n-\ell$ and $m-\ell$. (The point here is that the difference between ρ and $\rho(\ell)$ corresponding to $n-\ell$, $m-\ell$ is of order $O(L/n)$, and this difference leads to an extra factor $\exp(O(L^2/n)) \to 1$, provided that $L = o(n^{1/2})$.) Consequently $R_{n,m}(\ell) \sim \sigma^{\ell}$, uniformly for $\ell \leq L$. Therefore, using $\sigma < 1$,

$$\begin{aligned}
\mathbf{E}[(X_n)_k] \sim &\quad \textstyle\sum_{\ell \leq kL} \sigma^{\ell} \cdot [x^{\ell}] \left(\displaystyle\sum_{j \in [3,L], j \text{ odd}} \frac{x^j}{2j}\right)^k \\
= &\quad \sum_{\ell \leq kL} [x^{\ell}] \left(\sum_{j \in [3,L], j \text{ odd}} \frac{(\sigma x)^j}{2j}\right)^k \\
\sim &\quad \sum_{\ell} [x^{\ell}] \left(\sum_{j \in [3,L], j \text{ odd}} \frac{(\sigma x)^j}{2j}\right)^k \\
= &\quad \left(\sum_{j \in [3,L], j \text{ odd}} \frac{\sigma^j}{2j}\right)^k.
\end{aligned}$$

Thus X_n is in the limit Poisson with parameter

$$\sum_{j\in[3,L],j \text{ odd}} \frac{\sigma^j}{2j} = \frac{1}{4}\log\frac{1+\sigma}{1-\sigma} - \frac{\sigma}{2} + o(1),$$

so that

$$\lim_{n\to\infty} \mathbf{Pr}(X_n = 0) = e^{\sigma/2}\left(\frac{1-\sigma}{1+\sigma}\right)^{1/4}.$$

□

We can now quickly finish the proof of Theorem 3. First of all, Lemma 24 implies that

$$\mathbf{Pr}(G_{n,m}^{\delta\geq 2} \text{ has no perfect matching}) \geq 1 - e^{\sigma/2}\left(\frac{1-\sigma}{1+\sigma}\right)^{1/4} - o(1).$$

With probability $\geq e^{\sigma/2}\left(\frac{1-\sigma}{1+\sigma}\right)^{1/4} - o(1)$ the random graph $G_{n,m-n^{1/4}}^{\delta\geq 2}$ has no odd isolalated cycles. It follows from the analysis in [2] that with probability $1 - O(n^{-2})$, $G_{n,m-n^{1/4}}^{\delta\geq 2}$ has a matching of size $n/2 - n^{1/5}(\log n)^{O(1)}$. Lemma 4 can easily be extended to $G_{n,m}^{\delta\geq 2}$. So with the aid of Lemmas 22, 23 we see that

$$\begin{aligned}
&\mathbf{Pr}(G_{n,m}^{\delta\geq 2} \text{ has a perfect matching}) \geq \\
&\mathbf{Pr}(G_{n,m-n^{1/4}}^{\delta\geq 2} \text{ has a matching of size } n/2 - n^{1/5}(\log n)^{O(1)} \text{ and no odd isolated cycles}) - o(1) \\
&\qquad = e^{\sigma/2}\left(\frac{1-\sigma}{1+\sigma}\right)^{1/4} - o(1).
\end{aligned}$$

□

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A Proof of Lemma 1

Let $W=\sum_\ell Y_\ell$. As usual, we start with the inversion formula

$$\begin{aligned}\mathbf{Pr}\,(W=\tau) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\tau x}\mathbf{E}\left(e^{ix\sum_\ell Y_\ell}\right)\,dx\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\tau x}\prod_{\ell=1}^{\nu_1}\mathbf{E}(e^{ixY_\ell})^v\,dx,\end{aligned}\tag{195}$$

where $\tau=\mu+a$. Let

$$\Sigma_1=\sum_\ell\frac{\rho_1}{c_\ell+1}=\Theta(\nu_1\rho_1)$$

and consider first $|x|\ge\Sigma_1^{-5/12}$. Using inequality (15) we estimate

$$\begin{aligned}\frac{1}{2\pi}\int_{|x|\ge\Sigma_1^{-5/12}}\left|e^{-i\tau x}\prod_{\ell=1}^{\nu_1}\left(\frac{f_{c_\ell}(e^{ix}\rho_1)}{f_{c_\ell}(\rho_1)}\right)\right|\,dx &\le \frac{1}{2\pi}\int_{|x|\ge\Sigma_1^{-5/12}}\prod_{\ell=1}^{\nu_1}e^{\rho_1(\cos x-1)/(c_\ell+1)}\,dx\\ &\le e^{-\Sigma_1^{1/6}/3}.\end{aligned}\tag{196}$$

For $|x| \leq \Sigma_1^{-5/12}$, putting $\eta = \rho_1 e^{ix}$ and using $\sum_\ell \rho_1 f'_{c_\ell}(\rho_1)/f_{c_\ell}(\rho_1) = \mu$, $d/dx = i\eta d/d\eta$ we expand as a Taylor series around $x = 0$ to obtain

$$\begin{aligned}
-i\tau x + \sum_\ell \log\left(\frac{f_{c_\ell}(e^{ix}\rho_1)}{f c_\ell(\rho_1)}\right) &= -iax - \frac{x^2}{2}\sum_\ell \mathcal{D}\left(\frac{\eta f'_{c_\ell}(\eta)}{f_{c_\ell}(\eta)}\right)\bigg|_{\eta=\rho_1} \\
&\quad -\frac{ix^3}{3!}\sum_\ell \mathcal{D}^2\left(\frac{\eta f'_{c_\ell}(\eta)}{f_{c_\ell}(\eta)}\right)\bigg|_{\eta=\rho_1} \\
&\quad +O\left[x^4\sum_\ell \mathcal{D}^3\left(\frac{\eta f'_{c_\ell}(\eta)}{f_{c_\ell}(\eta)}\right)\bigg|_{\eta=\tilde\eta}\right]; \qquad (197)
\end{aligned}$$

here $\tilde\eta = \rho_1 e^{i\tilde x}$, with $\tilde x$ being between 0 and x, and $\mathcal{D} = \eta(d/d\eta)$. Now, the coefficients of $x^2/2$, $x^3/3!$ and x^4 are $\mathbf{Var}(W)$, $O(\mathbf{Var}(W))$, $O(\mathbf{Var}(W))$ respectively, and $\mathbf{Var}(W)$ is of order Σ_1. So the second and the third terms in (197) are $o(1)$ uniformly for $|x| \leq \Sigma_1^{-5/12}$. Therefore

$$\frac{1}{2\pi}\int_{|x|\leq\Sigma_1^{-5/12}} = \int_1 + \int_2 + \int_3, \qquad (198)$$

where

$$\begin{aligned}
\int_1 &= \frac{1}{2\pi}\int_{|x|\leq\Sigma_1^{-5/12}} e^{-iax-\mathbf{Var}(W)x^2/2}\,dx \\
&= \frac{1}{\sqrt{2\pi\mathbf{Var}(W)}} + O\left(\frac{a^2+1}{\Sigma_1^{3/2}}\right), \qquad (199) \\
\int_2 &= O\left(\sum_\ell \mathcal{D}^2\left(\frac{\rho_1 f'_{c_\ell}(\rho_1)}{f(\rho_1)}\right)\int_{|x|\leq\Sigma_1^{-5/12}} x^3 e^{-\mathbf{Var}(W)x^2/2}\,dx\right) \\
&= O\left(\Sigma_1\int_{|x|\geq\Sigma_1^{-5/12}} |x|^3 e^{-\mathbf{Var}(W)x^2/2}\,dx\right) \\
&= O(e^{-\alpha\Sigma_1^{1/6}}), \qquad (200)
\end{aligned}$$

($\alpha > 0$ is an absolute constant), and

$$\begin{aligned}
\int_3 &= O\left[\Sigma_1\int_{|x|\leq\Sigma_1^{-5/12}} x^4 e^{-\mathbf{Var}(W)x^2/2}\,dx\right] \\
&= O\left(\frac{1}{\Sigma_1^{3/2}}\right). \qquad (201)
\end{aligned}$$

Using (195)-(201), we arrive at

$$\mathbf{Pr}(W=\tau) = \frac{1}{\sqrt{2\pi v\mathbf{Var}(W)}} \times \left[1 + O\left(\frac{a^2+1}{\Sigma_1}\right)\right].$$

□

B Proof of Lemma 13

For $\boldsymbol{\nu} = (v_{1,R}, v_{1,C}, v_R, v_C, m)$ let $Z_{\boldsymbol{\nu}} = \{(\mathbf{x},\mathbf{y}) \in ([n]\cup\{\star\})^{2cn}$ with m pairs $x_i, y_i \neq \star$ etc..

Lemma 25. *Each $(\mathbf{x}', \mathbf{y}') \in Z_{\boldsymbol{\nu}'}$ arises by a transition of KSGREEDY from the same number $D(\boldsymbol{\nu}, \boldsymbol{\nu}')$ of $(\mathbf{x}', \mathbf{y}') \in Z_{\boldsymbol{\nu}}$.*

Proof **Case 1:** $v_1 > 0$ and an R-vertex x of degree 1 is selected and its neighbour y is of degree at least 2.

Let y be the C-neighbour of x in $G_{\mathbf{x},\mathbf{y}}$ and suppose y has $k_{i,j}$ neighbours of degree i that are incident $j \le i$ times with y (multiple edges). Then

$$\begin{aligned}
m' &= m - 1 - \sum_{i,j} jk_{i,j} \\
v'_{1,C} &= v_{1,C} \\
v'_{1,R} &= v_{1,R} - 1 - k_{1,1} + \sum_{i \ge 2} k_{i,i-1} \\
v'_C &= v_C - 1 \\
v'_R &= v_R - \sum_{i \ge 2} k_{i,i-1} - \sum_{i \ge 2} k_{i,i}
\end{aligned}$$

Given $\mathbf{x}', \mathbf{y}'$, our choices for $\mathbf{x}, \mathbf{y}$ are determined as follows: Observe that always the number of choices depends only on $\boldsymbol{\nu}, \boldsymbol{\nu}'$. Also let $n_R = n - v'_{1,R} - v'_R, n_C = n - v'_{1,C} - v'_C$ be the number of vertex labels missing from $\mathbf{x}', \mathbf{y}'$. Each quantity below should be multiplied by the number of choices of where to replace $\star$'s by vertex labels.

- Choose x, y in $n_R n_C$ ways.
- Choose the sequence $k_{i,j}$ such that the above equations hold.
- Choose the labels for the $\sum_i k_{i,i}$ new isolated R vertices in $\binom{n_R - 1}{k_{1,1}, k_{2,2}, \ldots}$ ways.
- Choose the labels for the $\sum_{i \ge 2} k_{i,i-1}$ vertices which become degree 1 in $\binom{v'_{1,R}}{k_{2,1}, k_{3,2}, \ldots}$ ways.
- Now let

  $$\mu = \sum_{i,j} jk_{i,j} - \sum_{i \ge 1} k_{i,i} - \sum_{i \ge 2} k_{i,i-1} = m - m' - 1 - \sum_{i \ge 1} ik_{i,i} - \sum_{i \ge 2} (i-1) k_{i,i-1}$$

  be the number of unaccounted for edges. These edges join y to vertices which remain of degree at least 2. Assign labels to these edges in $(v'_R)^\mu$ ways.

Case 2: $v_1 > 0$ and an R-vertex x of degree 1 is selected and its neighbour y is of degree 1.

Here there are $n_R n_C$ times the number of choices of where to replace $\star$'s by vertex labels.

Case 3: $v_1 = 0$

We add the parameters $\ell_{i,j}$ for the number of neighbours of x, other than y, which are of degree

i and are incident $j \le i$ times with x. We then have

$$\begin{aligned}
m' &= m - 1 - \sum_{i,j} j k_{i,j} \\
v'_{1,C} &= \sum_{i \ge 2} \ell_{i,i-1} \\
v'_{1,R} &= \sum_{i \ge 2} k_{i,i-1} \\
v'_C &= v_C - \sum_{i \ge 2} \ell_{i,i-1} - \sum_{i \ge 2} \ell_{i,i} \\
v'_R &= v_R - \sum_{i \ge 2} k_{i,i-1} - \sum_{i \ge 2} k_{i,i}
\end{aligned}$$

The number of choices for $\mathbf{x}, \mathbf{y}$ can now be enumerated:

- Choose x, y in $n_R n_C$ ways.
- Choose the sequences $k_{i,j}, \ell_{i,j}$ such that the above equations hold.
- Choose the labels for the $\sum_i k_{i,i}$ new isolated R vertices in $\binom{n_R - 1}{k_{1,1}, k_{2,2}, \ldots}$ ways.
- Choose the labels for the $\sum_i \ell_{i,i}$ new isolated C vertices in $\binom{n_C - 1}{k_{1,1}, k_{2,2}, \ldots}$ ways.
- Choose the labels for the $\sum_{i \ge 2} k_{i,i-1}$ R vertices which become degree 1 in $\binom{v'_{1,R}}{k_{2,1}, k_{3,2}, \ldots}$ ways.
- Choose the labels for the $\sum_{i \ge 2} \ell_{i,i-1}$ C vertices which become degree 1 in $\binom{v'_{1,C}}{k_{2,1}, k_{3,2}, \ldots}$ ways.
- Now let

  $$\mu = m - m' - \rho - \sum_{i \ge 2} i k_{i,i} - \sum_{i \ge 2} (i-1) k_{i,i-1} - \sum_{i \ge 2} i \ell_{i,i} - \sum_{i \ge 2} (i-1) \ell_{i,i-1}$$

  be the number of unaccounted for edges. Let $\mu_R = \sum_{j \le i-2} j k_{i,j}$ and $\mu_C = \mu - \mu_R$. Assign labels to these edges in $(v'_R)^{\mu_R} (v'_C)^{\mu_C}$ ways.

$\square$

The rest of the proof of Lemma 13 is essentially identical to that of Lemma 3 of [2]. $\square$