The concentration of the maximum degree in the duplication-divergence models

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Abstract

We pursue the analysis of the maximum degree in a dynamic duplication-divergence graph model defined by Solé et al. in which a new node arriving at time t first randomly selects an existing node and connects to its neighbors with probability p, and then connects to the other nodes with probability r/t. This model is often said to capture the growth of some real-world processes e.g. biological or social networks. However, there are only a handful of rigorous results concerning this model. In this paper we present rigorous results concerning the distribution of the maximum degree of a vertex in graphs generated by this model.

In this paper we solve an open problem by proving that for $\frac{1}{2} < p < 1$ with high probability the maximum degree is asymptotically concentrated around t^p, i.e. it deviates from this value by at most a polylogarithmic factor. Our findings are a step towards a better understanding of the overall structure of graphs generated by this model, especially the degree distribution, compression, and symmetry, which are important open problems in this area.

2012 ACM Subject Classification Mathematics of computing → Random graphs; Theory of computation → Random network models

Keywords and phrases random graphs, duplication-divergence model, degree distribution, maximum degree, large deviation

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Funding This work was supported by NSF Center for Science of Information (CSoI) Grant CCF-0939370, by NSF Grants CCF-1524312, CCF-2006440, CCF-2007238, DMS1952285 and, in addition, by the National Science Center, Poland, Grant 2018/31/B/ST6/01294.

1 Introduction

Studying structural properties of graphs (e.g., symmetry, compressibility, vertex degree) is a popular topic of research in computer science and discrete mathematics ever since the seminal work of Paul Erdős and Alfréd Rényi [8] . Recently attention has turned to dynamic graphs such as preferential attachment (Barabási-Albert) graphs [1], Watts-Strogatz small world graphs [25] or duplication-divergence graphs. Dynamic graphs, in which the edge- and/or vertex-sets are functions of time, are ubiquitous in diverse application domains ranging from biology to finance to social science. Deriving novel insights and knowledge from dynamic structures is a key challenge and understanding the structural properties of such dynamic graphs is critical for new characterizations and insights of the underlying dynamic processes.

Numerous networks in the real world change over time, in the sense that nodes and edges enter and leave the networks. To explain their macroscopic properties (e.g., subgraph frequencies, diameter, degree distribution, symmetry) and to make predictions and other inferences (such as link prediction, community detection, graph compression, order of node arrivals), several generative models have been proposed [19, 24]. Typically, one tries to capture the behavior of well-known graph parameters under probability distributions induced by the models, e.g. the distribution of the number of vertices with a given degree, the number of connected components, the existence of Hamiltonian paths or other parameters like clique number and chromatic number (see [3, 9, 13] for overviews of the main results in the area).

In this paper we make further progress on structural properties of the *duplication-divergence* graph models, in which vertices arrive one by one, select an existing node as a parent, connect to the some neighbors of its parent and other vertices according to some pre-defined rule. More precisely, a newly arriving node at time t first selects randomly an existing node and connects to its neighbors with probability p; and then connects to other nodes with probability r/t. The particular model which we bring under consideration is a duplication-divergence model, first defined by Solé, Pastor-Satorras et al. [21]. It has been a popular object of study because it has been shown empirically that its degree distribution, small subgraph (graphlets) counts and number of symmetries fit very well with the structure of some real-world biological and social networks, e.g. protein-protein and citation networks [5, 20, 22]. This suggests a possible real-world significance for the duplication-divergence model, which further motivates the studies of its structural properties. However, it is also one of the least understood models, much less so than the Erdős-Rényi or preferential attachment models. At the moment there exist only a handful of results related to the behavior of the degree distribution of the graphs generated by this model. Unlike other dynamic graphs such as the preferential attachment model, the graphs generated by the duplication-divergence model can be very symmetric or quite asymmetric. In Figure 1 it is shown that there exist certain ranges of the model parameters p and r such that the graphs generated from the model are high symmetric, and certain ranges such that the graphs are asymmetric. Here the symmetry is measured by the size of the automorphism group $|\mathrm{Aut}(G)|$, i.e. the number of distinct mapping of vertices onto themselves preserving the adjacency matrix. Still the basic

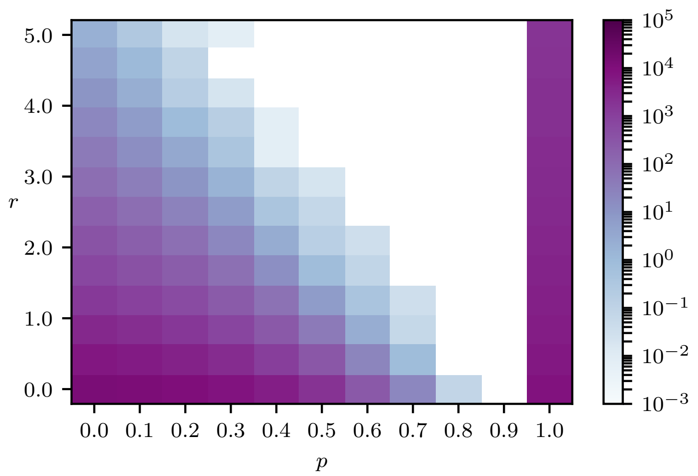

Figure 1 Symmetry of graphs ($\log |\mathrm{Aut}(G)|$) generated by the Solé-Pastor-Satorras duplication-divergence model, as presented in [22].

question about the conditions under which the generated graph is symmetric or not remains unanswered. We believe that proving results about the range of the maximum degree can be a stepping stone for rigorous general results regarding symmetries and compression, just as it has been in the case for other random graph models.

In particular, the parameters such as the maximum degree of a random graph and the degree of a given vertex are parameters that are studied not only for their own sake, but it turns out that their analysis opens the way to further results. Let us recall here two examples of these insights related to the questions of graph asymmetry and incompressibility.

First, Łuczak et al. [17] used the estimation of these parameters to prove that the preferential attachment model with $m \geq 3$ (where m is the number of edges added when a new node arrives) generates asymmetric graphs (i.e. graphs with only one automorphism) with high probability. This was achieved by proving two properties: (A) for any pair of early vertices t_1 and t_2 the degrees of both nodes t_1 and t_2 are distinct, and (B) for any pair of late vertices their corresponding neighbors are not the same, in particular, they have different sets of early neighbors (and therefore, a permutation of t_1 and t_2 does not produce symmetry). We believe that this approach to asymmetry analysis can be extended to the dupication-divergence model and it requires knowledge of the maximum degree which is exactly the topic of this paper.

A second usage of these parameters was presented by Chierichetti et al. in [4]. For example, for the preferential attachment model they used an upper bound on the maximum degree and the degree of a vertex arriving at time s to show that the entropy over all graphs on t vertices generated by this model is bounded by $\Omega(t \log t)$. They also used their bound on vertex degrees to provide the lower bounds on graph entropy for several other random graph models known in the literature, e.g. copying model or ACL model (see also [18] for the preferential attachment graph compression algorithm).

Therefore, we turn our attention to the asymptotic behavior of the distribution of degrees of vertices in random graphs generated by the duplication-divergence model. Let us recall that, for example, for Erdős-Rényi model $\texttt{ER}(t, p)$ it is known that the degree distribution approximately follows the Poisson distribution with a tail decreasing exponentially [2]. Clearly, the degree of each vertex is a random variable with the binomial distribution, so it is highly concentrated around its mean $(t-1)p$. Moreover, the maximum degree is also highly concentrated around $(t-1)p + \sqrt{2p(1-p)(t-1)\log t}$ [9, Theorem 3.5]. For preferential attachment model $\texttt{PA}(t, m)$ it was proved that the degree distribution exhibits scale-free behaviour, i.e. the number of vertices with degree k is proportional to k^{-3} [3]. In addition, if we consider a vertex arriving at time s, its degree in graph on t vertices is proportional to $\sqrt{t/s}$ on average and with high probability it does not exceed $\sqrt{t/s}\log^3 t$ [6]. Recently, in [10] some large deviation results for the degree distribution were presented.

Here we provide analogous results for duplication-divergence model. The paper is organized as follows: in Section 2 we present a formal definition of the duplication-divergence model, recall previous results related to the properties of the degree distribution and introduce our main results. In Section 3.1 and Section 3.2 we prove upper bounds for the degrees for earlier and later vertices arriving in the graph, respectively. Finally, in Section 3.3 we give a proof of the lower bound for the maximum degree in the graph.

2 Model definition and main results

We formally define the duplication-divergence model $\texttt{DD}(t, p, r)$, introduced by Solé et al. [21]. Then we summarize our main results about the high-probability bounds on the the maximum degree.

Throughout the paper we use standard graph notation from [7], e.g. $V(G)$ denotes the vertex set of a graph G, $\deg_G(s)$ – the degree of node s in G and $\Delta(G)$ – the maximum degree of a vertex in G. All graphs considered in the paper are simple.

By G_t we denote a graph on t vertices. Since in the paper we deal with the graphs that are dynamically generated, we assume that the vertices are identified with the natural numbers according to their arrival time. We use the notation $\deg_t(s)$ for the random variable denoting the degree of vertex s at time t i.e. after t vertices have been added in total.

Let us now formally define the model $\mathtt{DD}(t, p, r)$ as follows: let $0 \leq p \leq 1$ and $0 \leq r \leq T$ be the parameters of the model. Let also G_T be a graph on $T \leq t$ vertices, with vertices having distinct labels from 1 to T. Now, for every $t = T, T+1, \ldots$ we create G_{t+1} from G_t according to the following rules:

1. we add a new vertex $t+1$ to the graph,
2. we choose a vertex u from G_t uniformly at random – and we denote u as parent$(t+1)$,
3. for every vertex v:
   a. if v is adjacent to u in G_t, then add an edge between v and $t+1$ with probability p,
   b. if v is not adjacent to u in G_t, then add an edge between v and $t+1$ with probability $\frac{r}{t}$.

All edge additions are independent random Bernoulli variables.

We now review in some details recent results on the degree distribution. For example, for $p < 1$ and $r = 0$, it is shown in [11] that even for large p the limiting distribution of degree frequencies indicates that almost all vertices are isolated as $t \to \infty$. Moreover, from [16] we know that the number of vertices of degree one is $\Omega(\log t)$ but again the precise rate of growth of the number of vertices with any fixed degree $k > 0$ is currently unknown. Recently, also for $r = 0$, in [14, 12] authors showed that for $0 < p < e^{-1}$ the non-trivial connected component has a degree distribution that has a power-law behavior with the exponent is equal to γ satisfying $3 = \gamma + p^{\gamma-2}$.

Now let us turn to results directly related to the question of maximum degree. For example, in [23] it was shown that for any fixed s asymptotically as $t \to \infty$ it holds that

$$\mathbb{E}[\deg_t(s)] = \begin{cases} \Theta(\ln t) & \text{if } p = 0 \text{ and } r > 0, \\ \Theta(t^p) & \text{otherwise.} \end{cases}$$

Note that by the close relation between parameters $\Delta(G_t)$ and $\deg_t(s)$ we can establish easily that $\mathbb{E}[\Delta(G_t)] = \Omega(t^p)$ when $p > 0$ or $r = 0$, and $\mathbb{E}[\Delta(G_t)] = \Omega(\ln t)$ otherwise.

It turns out that a lower bound on maximum degree is easily established as a byproduct of existing result by Frieze et al. [10]: for $\frac{1}{2} < p < 1$ and $G_t \sim \mathtt{DD}(t, p, r)$ with $p > 0$ and $s = O(1)$ it holds that

$$\Pr\left[\deg_t(s) \leq \frac{C}{A} t^p \log^{-3-\varepsilon}(t)\right] = O(t^{-A})$$

for some fixed constant $C > 0$ and any $A > 0$. This is obviously the case because for any s it holds that $\deg_t(s) \leq \Delta(G_t)$. In the same paper, Frieze et al. also proved that for $\frac{1}{2} < p < 1$, $G_t \sim \mathtt{DD}(t, p, r)$ and $s = O(1)$ it holds asymptotically that

$$\Pr[\deg_t(s) \geq A\,C\,t^p \log^2(t)] = O(t^{-A})$$

for some fixed constant $C > 0$ and any $A > 0$. They also left as an open problem the question of the behavior of the right tail of the maximum degree distribution or, equivalently, of the upper bound on $\deg_t(s)$ for larger s that holds with high probability.

In this paper, we solve this problem. More precisely, we obtain two major results: first, we provide a bound $\deg_t(s) \leq (1+\varepsilon)t^p \mathrm{polylog}(t)$ which holds quite surely (i.e. at least $1-O(t^{-A})$ for any given $A > 0$ [15]) for any $\varepsilon > 0$. We prove that this bound is valid for all vertices in G_t, not only for $s = O(1)$ as before leading to the estimate $\Delta(G_t) \leq (1+\varepsilon)t^p \mathrm{polylog}(t)$ for any $\varepsilon > 0$ with high probability. Next, we provide a precise lower bound and we show that there exists an early vertex s such that $\deg_t(s) \geq (1-\varepsilon)t^p$ for any $\varepsilon > 0$ quite surely. Putting everything together we obtain the main result of this paper, that is:

▶ **Theorem 1.** *Let $\frac{1}{2} < p < 1$. Asymptotically for $G_t \sim DD(t, p, r)$*

$$\Pr[(1-\varepsilon)t^p \leq \Delta(G_t) \leq (1+\varepsilon)t^p \log^{5-4p}(t)] = O(t^{-A})$$

for any constants $\varepsilon > 0$ and $A > 0$.

In other words, we are now certain that the maximum degree of the graph is concentrated in the sense that by moving only by some polylogarithmic factor from the mean to both left and right we observe the polynomial tail decay.

3 Analysis and proofs

3.1 Upper bound, early vertices

The main idea of the proof of the upper bound of the maximum degree is as follows: we first find for small s (i.e. $s \leq t_0$) a Chernoff-type bound on the growth of $\deg_\tau(s)$ over an interval of certain length h.

Then, we introduce an auxiliary deterministic sequences t_i and X_{t_i} such that $t_0 < \ldots < t_{k-1} < t \leq t_k$. The definition of these sequences stems from the bound mentioned above, in particular from the relation between h and the growth of the degree, guaranteed with high probability. Ultimately, we prove $\deg_\tau(s) \leq X_\tau$ with high probability for all $s \leq t_0$.

Let us start with providing a Chernoff-type bound on the growth of the degree of a given early vertex:

▶ **Lemma 2.** *Let $1 \leq s \leq \tau \leq t$. Let X_τ be any value such that $\deg_\tau(s) \leq X_\tau$. Then for any $h \leq \varepsilon X_\tau$ with $\varepsilon \in (0,1)$ it is true that*

$$\Pr\left[\deg_{\tau+h}(s) \geq \deg_\tau(s) + (1+3\varepsilon)\frac{h(pX_\tau + r)}{\tau}\right] \leq \exp\left(-\frac{h\varepsilon^2(1+\varepsilon)(pX_\tau + r)}{3\tau}\right).$$

Proof. First, recall that for $i = 0, 1, \ldots, h-1$ we have $\deg_{\tau+i+1}(s) = \deg_{\tau(s)+i} + I_{\tau+i}$ where $I_{\tau+i} \sim Be\left(\frac{p\deg_{\tau+i}(s)+r}{\tau+i}\right)$. Also clearly $\deg_{\tau+i}(s) \leq \deg_\tau(s) + i$ for any $i = 0, 1, \ldots, h$, so we have

$$\frac{\deg_{\tau+i}(s)}{\tau+i} \leq \frac{\deg_\tau(s)+i}{\tau} \leq \left(1 + \frac{i}{X_\tau}\right)\frac{X_\tau}{\tau} \leq \left(1 + \frac{h}{X_\tau}\right)\frac{X_\tau}{\tau} \leq (1+\varepsilon)\frac{X_\tau}{\tau}.$$

Therefore for any $i = 0, 1, \ldots, h-1$ we know that $I_{\tau+i}$ is stochastically dominated by $I^*_{\tau+i} \sim Be\left((1+\varepsilon)\frac{pX_\tau+r}{\tau}\right)$.

Now, from the well known Chernoff bound formula we know that for any $\varepsilon \in (0,1)$

$$\Pr\left[\deg_{\tau+h}(s) - \deg_\tau(s) \geq (1+\varepsilon)\,\mathbb{E}\left[\sum_{i=0}^{h-1} I^*_{\tau+i}\right]\right] \leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}\left[\sum_{i=0}^{h-1} I^*_{\tau+i}\right]\right)$$

and therefore

$$\Pr\left[\deg_{\tau+h}(s) \geq \deg_\tau(s) + (1+3\varepsilon)\,\frac{h(pX_\tau + r)}{\tau}\right]$$
$$\leq \Pr\left[\deg_{\tau+h}(s) \geq \deg_\tau(s) + (1+\varepsilon)^2\,\frac{h(pX_\tau + r)}{\tau}\right] \leq \exp\left(-\frac{h\varepsilon^2(1+\varepsilon)(pX_\tau + r)}{3\tau}\right).$$

◀

Immediately we can infer how large h has to be to get the polynomial tail:

▶ **Corollary 3.** *Let $1 \leq s \leq \tau \leq t$. Let $X_\tau \geq 0$, $\varepsilon \in (0,1)$ be values such that asymptotically for any $A > 0$, it holds that $\deg_\tau(s) \leq X_\tau$ and $3A\tau \log t \leq \varepsilon^3 X_\tau(pX_\tau + r)$. Then for any $h \in \left[\frac{3A\tau\log t}{\varepsilon^2(pX_\tau+r)}, \varepsilon X_\tau\right]$ it is true that*

$$\Pr\left[\deg_{\tau+h}(s) > \deg_\tau(s) + (1+3\varepsilon)\,\frac{h(pX_\tau + r)}{\tau}\right] = O(t^{-A}).$$

Now we provide the definitions for two auxiliary sequences that we mentioned earlier:

▶ **Definition 4.** *Let $0 < p < 1$ be fixed with certain $\alpha \geq \beta_i$ and $\phi < t$. We define the increasing sequences $(t_i)_{i=0}^k$ and $(X_{t_i})_{i=0}^k$ and a number k in the following way:*

$$t_0 = \phi, \qquad t_{i+1} = t_i + \frac{\alpha\, t_i \log t_i}{X_{t_i}}, \qquad t_{k-1} < t \leq t_k,$$
$$X_{t_0} = t_0, \qquad X_{t_{i+1}} = X_{t_i} + \beta_i \log t_i.$$

Observe that directly from the definition we know that $X_{t_i} \leq t_i$ for all $i = 0, 1, \ldots, k$.

Moreover, note that we do not specify the values of X_τ for τ other than $\{t_0, t_1, \ldots, t_k, \ldots\}$. However, in this section we will be using precisely these values in the following proofs, so such definition is sufficient for our purposes.

Now we analyze the asymptotic properties of these sequences. We start with a simple lower bound:

▶ **Lemma 5.** *Assume that $\phi \geq \log^2 t$, $\alpha \leq \sqrt{\phi}$ and $\beta_i \geq \alpha(p-\delta)$ for some $\delta \in [0,p)$. Asymptotically as $t \to \infty$ for any $i = 0, 1, \ldots, k$ we have $X_{t_i} \geq t_i^{p-\delta}$.*

Proof. Let us define $Y_\tau = \tau^{p-\delta}$. By definition we know that $X_{t_0} = \phi \geq Y_{t_0}$.

Now, let us assume that $X_{t_i} \geq Y_{t_i}$ holds for some $i \geq 0$. Let us also denote by $h = t_{i+1} - t_i = \frac{\alpha\, t_i \log t_i}{X_{t_i}}$. Then we have asymptotically

$$Y_{t_{i+1}} - Y_{t_i} = (t_i + h)^{p-\delta} - t_i^{p-\delta} = t_i^{p-\delta}\left(\left(1 + \frac{h}{t_i}\right)^{p-\delta} - 1\right) \leq t_i^{p-\delta}\,\frac{(p-\delta)h}{t_i},$$

for any $\delta \in [0,p)$, because $X_{t_i} \geq \phi \geq \log^2 t$, so $\frac{h}{t_i} = \frac{\alpha \log t_i}{X_{t_i}} \leq \frac{\alpha \log t_i}{\phi} \leq \frac{\log t}{\sqrt{\phi}} \leq 1$. Thus,

$$Y_{t_{i+1}} - Y_{t_i} \leq Y_{t_i}\,\frac{(p-\delta)\,h}{t_i} \leq X_{t_i}\,\frac{(p-\delta)\,h}{t_i} = \alpha(p-\delta)\log t_i \leq \beta_i \log t_i = X_{t_{i+1}} - X_{t_i},$$

so clearly $X_{t_{i+1}} \geq Y_{t_{i+1}}$ holds as well, which completes the inductive step. ◀

Now we prove the upper bound:

▶ **Lemma 6.** *Assume that $\phi \geq \log^3 t$, $\alpha(p-\delta) \leq \beta_i \leq \alpha p + \frac{\alpha}{2\log t_i}$ for some $\delta \in [0,p)$. It holds asymptotically as $t \to \infty$ that $X_{t_i} \leq \phi^{1-p} t_i^p \log t_i$ for all $i = 0, 1, \ldots, k$.*

Proof. We again proceed by induction. Clearly, $X_{t_0} = t_0 \leq t_0 \log t_0$.

Directly from the definition we get

$$\begin{aligned}
\phi^{1-p} t_{i+1}^p \log t_{i+1} - X_{t_{i+1}} &= \phi^{1-p} t_{i+1}^p \log t_{i+1} - X_{t_i} - \beta_i \log t_i \\
&\geq \phi^{1-p} t_{i+1}^p \log t_{i+1} - \phi^{1-p} t_i^p \log t_i - \beta_i \log t_i \\
&\geq \phi^{1-p} t_i^p \log t_i \left(\left(\frac{t_{i+1}}{t_i} \right)^p \left(\frac{\log t_{i+1}}{\log t_i} \right) - 1 \right) - \beta_i \log t_i \\
&= \phi^{1-p} t_i^p \log t_i \left(\left(1 + \frac{\alpha \log t_i}{X_{t_i}} \right)^p \left(1 + \frac{\log(1 + \alpha \log t_i / X_{t_i})}{\log t_i} \right) - 1 \right) - \beta_i \log t_i.
\end{aligned}$$

Now we use the inequalities $(1+x)^p \geq 1 + px - \frac{p(1-p)x^2}{2} + O(x^3)$ and $\log(1+x) \geq x - O(x^2)$, true for any $p \in [0,1]$ and any $x \to 0$. In particular, in our case $x = \frac{\alpha \log t_i}{X_{t_i}} \leq \frac{1}{\sqrt{\log t}}$ since $\alpha \leq \sqrt{\phi}$ and $\phi \geq \log^3 t$. Therefore

$$\begin{aligned}
&\phi^{1-p} t_{i+1}^p \log t_{i+1} - X_{t_{i+1}} \\
&\geq \phi^{1-p} t_i^p \log t_i \left(\frac{\alpha\, p \log t_i}{X_{t_i}} + \frac{\alpha}{X_{t_i}}(1 - o(1)) - \frac{\alpha^2\, p(1-p) \log^2 t_i}{2 X_{t_i}^2}(1 - o(1)) \right) - \beta_i \log t_i \\
&\geq \alpha\, p \log t_i + \alpha(1 - o(1)) - \frac{\alpha^2\, p(1-p) \log^2 t_i}{2 X_{t_i}}(1 - o(1)) - \beta_i \log t_i \\
&\geq \alpha \log t_i \left(p + \frac{1}{\log t_i}(1 - o(1)) - \frac{p(1-p) \log t_i}{2\sqrt{t_i^{p-\delta}}}(1 - o(1)) \right) - \beta_i \log t_i,
\end{aligned}$$

where in the last line we used the fact that $X_{t_i} \geq \sqrt{\phi t^{p-\delta}} \geq \alpha \sqrt{t^{p-\delta}}$ – itself derived as a geometric mean between the bounds from Definition 4 and Lemma 5.

Finally, we note that for a series $\beta_i \leq \alpha p + \frac{\alpha}{2\log t_i}$ and for sufficiently large t clearly the last expression is non-negative, which completes the proof. ◀

▶ **Corollary 7.** *If $\alpha \leq \phi$, then for the value of k such that $t_{k-1} < t \leq t_k$ it is true that $\alpha k < t$.*

Proof. We know from the definition of t_i and Lemma 6 that

$$t > t_{k-1} - t_0 \geq t_0 + \sum_{i=0}^{k-2} \frac{\alpha t_i \log t_i}{\phi^{1-p} t_i^p \log t_i} \geq t_0 + \sum_{i=0}^{k-2} \alpha \geq \phi + (k-1)\alpha > \alpha\, k.$$

◀

Here let us note the relation between the last elements of the sequences $(t_i)_{i=0}^k$, $(X_{t_i})_{i=0}^k$ and the final values themselves:

▶ **Lemma 8.** *Let ε be any positive constant. Assume that $\phi \geq \log^3 t$, $\alpha \leq \sqrt{\phi}$, $\alpha(p-\delta) < \beta_i \leq \alpha p + \frac{\alpha}{2\log t_i}$ for some $\delta \in [0,p)$.*

It holds asymptotically as $t \to \infty$ that $(1-\varepsilon)t_k \leq t \leq (1+\varepsilon)t_{k-1}$ and $(1-\varepsilon)X_{t_k} \leq X_{t_k} \leq (1+\varepsilon)t_{X_{k-1}}$.

Proof. Clearly from the previous lemmas we know that for any constant $\varepsilon > 0$ it is true that

$$\frac{t_k}{t_{k-1}} = 1 + \frac{\alpha \log t_{k-1}}{X_{t_{k-1}}} \leq 1 + \frac{\alpha \log t_{k-1}}{\sqrt{\phi t_{k-1}^{p-\delta}}} \in (1, 1+\varepsilon).$$

The first claim follows from this and from the fact that $t_{k-1} < t \leq t_k$.

Similarly, for any constant $\varepsilon > 0$ the second claim follows from the fact that $X_{t_{k-1}} < X_t \leq X_{t_k}$ and that

$$\frac{X_{t_k}}{X_{t_{k-1}}} = 1 + \frac{\beta_k \log t_k}{X_{t_{k-1}}} \leq 1 + \frac{\alpha \log t_{k-1}(p+\varepsilon)}{\sqrt{\phi t_{k-1}^{p-\delta}}} \in (1, 1+\varepsilon).$$

◀

Let us denote by $\mathcal{A}_i(s)$ the event that $\deg_{t_i}(s) \leq X_{t_i}$ for a fixed $s \leq t_i$. Now we proceed with the main theorem:

▶ **Theorem 9.** *For $G_t \sim DD(t, p, r)$ with $\frac{1}{2} < p < 1$ and $s \in [1, \log^4 t]$ it holds asymptotically that*

$$\Pr\left[\deg_t(s) > (1+\varepsilon)t^p \log^{5-4p} t\right] = O(t^{-A})$$

for any constants $\varepsilon > 0$, $A > 0$.

Proof. Throughout the proof we will use sequences $(t_i)_{i=0}^k$ and $(X_{t_i})_{i=0}^k$ with $\alpha = 273p^3(A+1)\log^2 t$, $\beta_i = \alpha p + \frac{\alpha}{2\log t_i}$ and $\phi = \log^4 t$.

Observe that all the assumptions of Lemma 5, Lemma 6 and Corollary 7 are met so we know that $\max\{\log^4 t, t_i^p\} \leq X_{t_i} \leq t_i^p \log^{5-4p} t$ for all $i = 0, 1, \ldots, k$ and also $k < \frac{t}{\log^2 t}$. Moreover, if $\mathcal{A}_i(s)$ holds, then the assumptions of Corollary 3 also are true for $\tau = t_i$ and $h = \frac{\alpha t_i \log t_i}{X_{t_i}}$ as $t_i \to \infty$ since for any constant $A > 0$ and $\varepsilon = \frac{1}{9p \log t_i}$ it holds that

$$\frac{3At_i \log t}{\varepsilon^2(pX_{t_i} + r)} < \frac{\alpha t_i \log t_i}{X_{t_i}} < \varepsilon X_{t_i}.$$

Moreover, since $\beta_i > \alpha p$, we know that for $\varepsilon = \frac{1}{9p \log t_i}$ asymptotically it is true that

$$X_{t_{i+1}} - X_{t_i} = \beta_i \log t_i \geq \beta_i \log t_i \frac{1 + \frac{1}{3p \log t_i}}{1 + \frac{1}{2p \log t_i}} \frac{pX_{t_i} + r}{pX_{t_i}} = (1+3\varepsilon)\frac{h(pX_{t_i} + r)}{t_i}.$$

Therefore, Corollary 3 implies that for any constant $A > 0$ and $\varepsilon = \frac{1}{6 \log t_i}$ it is true that $\Pr[\neg\mathcal{A}_{i+1}(s) | \mathcal{A}_i(s)] = O(t^{-A})$.

Clearly, for any $1 \leq s \leq t_0$ we know that $\mathcal{A}_0(s)$ always holds so $\Pr[\neg\mathcal{A}_0(s)] = 0$. Finally, we obtain using Lemma 8 and Corollary 3 that

$$\begin{aligned}\Pr[\deg_t(s) > X_{t_k}] &\leq \Pr[\deg_{t_k}(s) > X_{t_k}] = \Pr[\neg\mathcal{A}_k(s)] \\ &\leq \sum_{i=0}^{k-1} \Pr[\neg\mathcal{A}_{i+1}(s) | \mathcal{A}_i(s)] + \Pr[\neg\mathcal{A}_0(s)] = \sum_{i=0}^{k-1} O(t^{-A}) = O(t^{-A+1}).\end{aligned}$$

◀

3.2 Upper bound, late vertices

In the second part of the proof we also use the sequences $(t_i)_{i=0}^k$ and $(X_{t_i})_{i=0}^k$ as defined in Definition 4. Moreover, in their definition throughout this section we use the same constants as in the proof of Theorem 9: $\alpha = 273p^3(A+1)\log^2 t$, $\beta_i = \alpha p + \frac{\alpha}{2\log t_i}$ and $\phi = \log^4 t$.

The proof consist of showing that for $s \in [t_i, t_{i+1})$ for some $i = 0, 1, \ldots, k-1$ the degree of the vertex when it appears in the graph (i.e. $\deg_s(s)$) is with high probability significantly smaller than its respective $X_{t_{i+1}}$. Furthermore, we show that the increase of the degree between $\deg_s(s)$ and $\deg_{t_{i+1}}(s)$ with high probability also cannot compensate this difference. Thus, X_t (or, to be more precise, X_{t_k}) gives us a good upper bound on $\deg_t(s)$ for all s – and therefore also we obtain an upper bound for $\Delta(G_t)$.

Let us introduce an auxiliary event $\mathcal{B}_l(s) = \bigcup_{\tau=1}^{s} \mathcal{A}_l(\tau) = [\max\{\deg_{t_l}(\tau) \colon 1 \le \tau \le s\} \le X_{t_l}]$ for any s and l such that $s \le t_l$.

▶ **Lemma 10.** *Let $s \in (t_l, t_{l+1}]$ for some $l = 0, 1, \ldots, k-1$. Then, for any $\varepsilon \in (0,1)$*

$$\Pr\left[\deg_s(s) \ge (1+\varepsilon)(pX_{t_{l+1}} + r) | \mathcal{B}_l(t_l) \wedge \mathcal{B}_{l+1}(s-1)\right] \le \exp\left(-\frac{\varepsilon^2}{3}(pX_{t_{l+1}} + r)\right).$$

Proof. First, we notice the fact that $\max\{\deg_{t_{l+1}}(\tau) \colon 1 \le \tau \le s-1\} \le X_{t_{l+1}}$ guarantees that $\max\{\deg_s(\tau) \colon 1 \le \tau \le s-1\} \le X_{t_{l+1}}$. Therefore, $\deg_s(s)$ is stochastically dominated by $A_s \sim Bin\left(s, \frac{pX_{t_{l+1}}+r}{s}\right)$ so for any $\varepsilon \in (0,1)$ we obtain the result directly using the simple Chernoff bound with $\mathbb{E}[A_s] = pX_{t_{l+1}} + r$. ◀

Note that the result implies that with high probability at most slightly more than p fraction of maximum allowed degree was already used at time s. Therefore, we are interested in bounding the remaining part of the degree, i.e. $\deg_{t_{l+1}}(s) - \deg_s(s)$, by something smaller than the $(1-p)$ fraction of maximum allowed degree.

▶ **Lemma 11.** *Let $\frac{1}{2} < p < 1$ and $s \in (t_l, t_{l+1}]$ for some $l = 0, 1, \ldots, k-1$. Then asymptotically as $t \to \infty$, for any constant $A > 0$ it holds that*

$$\Pr\left[\deg_{t_{l+1}}(s) \ge X_{t_{l+1}} | \mathcal{B}_l(t_l) \wedge \mathcal{B}_{l+1}(s)\right] = O(t^{-A}).$$

Proof. Let us denote $d = \frac{1-p}{2}X_{t_{l+1}} - \frac{(1+p)r}{2p}$.

If $s \in [t_{l+1} - d, t_{l+1}]$, then the result is a direct implication from Lemma 10 with $\varepsilon = \frac{1-p}{2p}$, as the degree of the vertex during an interval of length d cannot grow more than d. Therefore, it is sufficient to use the bound from Lemma 5.

Otherwise $s \in (t_l, t_{l+1} - d)$. But if such s exists, then it is the case that $d \le t_{l+1} - t_l \le \frac{t_l \log t_l \log^2 t}{X_{t_l}}$ so from Lemma 5 with $\delta = 0$ and by the fact that $X_{t_i} \ge \phi$ we get that asymptotically $X_{t_l} \ge t_l^{\gamma p} \log^{4(1-\gamma)} t$ for any $\gamma \in [0,1]$ and therefore

$$t_l \log t_l \log^2 t \ge \left(\frac{1-p}{2}X_{t_{l+1}} - \frac{(1+p)r}{2p}\right) X_{t_l} \ge \frac{1-p}{4}X_{t_l}^2 \ge \frac{1-p}{4} t_l^{2\gamma p} \log^{8(1-\gamma)} t.$$

However, if we set e.g. $\gamma = \frac{3}{5}$, then we can bound the right side from below by $\frac{1-p}{4} t_l^{6/5} \log^{16/5} t$ – and for sufficiently large t we obtain a contradiction, as each term dominates the respective one on the left side. ◀

▶ **Lemma 12.** *Let $\frac{1}{2} < p < 1$ and $s \in (t_l, t_{l+1}]$ for some $l = 0, 1, \ldots, k-1$. Then asymptotically as $t \to \infty$, for any constant $A > 0$ it holds that*

$$\Pr\left[\neg\mathcal{B}_{l+1}(t_{l+1}) | \mathcal{B}_l(t_l)\right] = O(t^{-A}).$$

Proof. Let l be the first value for which the theorem does not hold. Then, from Lemma 11 we get that for any constant $A > 0$ it holds that

$$\Pr\left[\neg\mathcal{B}_{l+1}(t_{l+1})|\mathcal{B}_l(t_l) \wedge \mathcal{B}_{l+1}(t_l)\right] = \sum_{s=t_l}^{t_{l+1}-1} \Pr\left[\neg\mathcal{B}_{l+1}(s+1)|\mathcal{B}_l(t_l) \wedge \mathcal{B}_{l+1}(s)\right]$$
$$= \sum_{s=t_l}^{t_{l+1}-1} \Pr\left[\neg\mathcal{A}_{l+1}(s+1)|\mathcal{B}_l(t_l) \wedge \mathcal{B}_{l+1}(s)\right] = O(t^{-A}).$$

From Theorem 16 we know that $\Pr\left[\mathcal{B}_0(t_0)\right] = 1 - O(t^{-A})$. Therefore, by our assumption, $\Pr\left[\mathcal{B}_i(t_i)\right] = 1 - O(t^{-A})$ for all $i = 0, 1, \ldots, l$. We use this fact, the observation that $\mathcal{A}_l(s) \subseteq \mathcal{B}_l(t_l)$ and Theorem 9 to get

$$\Pr\left[\neg\mathcal{B}_{l+1}(t_l)|\mathcal{B}_l(t_l)\right] \le \sum_{s=1}^{t_l} \Pr\left[\neg\mathcal{A}_{l+1}(s)|\mathcal{B}_l(t_l)\right] \le \sum_{s=1}^{t_l} \frac{\Pr\left[\neg\mathcal{A}_{l+1}(s) \wedge \mathcal{B}_l(t_l)\right]}{\Pr\left[\mathcal{B}_l(t_l)\right]}$$
$$\le \sum_{s=1}^{t_l} \frac{\Pr\left[\neg\mathcal{A}_{l+1}(s) \wedge \mathcal{A}_l(s)\right]}{\Pr\left[\mathcal{B}_l(t_l)\right]} \le \sum_{s=1}^{t_l} \frac{\Pr\left[\neg\mathcal{A}_{l+1}(s)|\mathcal{A}_l(s)\right]}{\Pr\left[\mathcal{B}_l(t_l)\right]} = \sum_{s=1}^{t_l} \frac{O(t^{-A})}{1 - O(t^{-A})} = O(t^{-A}).$$

Finally, from the fact that for any events E_1, E_2, E_3 it follows that

$$\Pr[\neg E_1|E_2] = \Pr[\neg E_1 \wedge E_3|E_2] + \Pr[\neg E_1 \wedge \neg E_3|E_2]$$
$$\le \Pr[\neg E_1|E_3 \wedge E_2] + \Pr[\neg E_3|E_2],$$

and we substitute $E_1 = \mathcal{B}_{l+1}(t_{l+1})$, $E_2 = \mathcal{B}_l(t_l)$ and $E_3 = \mathcal{B}_{l+1}(t_l)$ to obtain the final result. ◀

▶ **Theorem 13.** *Let $\frac{1}{2} < p < 1$. Then asymptotically as $t \to \infty$, for any constant $A > 0$ it holds that*

$$\Pr\left[\Delta(G_t) \ge (1+\varepsilon)t^p \log^{5-4p} t\right] = O(t^{-A}).$$

Proof. We observe that

$$\Pr\left[\Delta(G_t) \ge (1+\varepsilon)t^p \log^{5-4p} t\right] \le \Pr\left[\neg\mathcal{B}_k(t_k)\right]$$
$$\le \sum_{l=0}^{k-1} \Pr\left[\neg\mathcal{B}_{l+1}(t_{l+1})|\mathcal{B}_l(t_l)\right] + \Pr\left[\neg\mathcal{B}_0(t_0)\right].$$

Now, from Theorem 16 and Lemma 12 we know that both $\Pr\left[\mathcal{B}_0(t_0)\right] = O(t^{-A})$ and $\Pr[\neg\mathcal{B}_{l+1}(t_l)|\mathcal{B}_l(t_l)] = O(t^{-A})$ for any $A > 0$, respectively. Putting this all together with Lemma 8 we obtain the result. ◀

3.3 Lower bound

Here we proceed analogously as in the case of upper bound for early vertices. First, we provide an appropriate Chernoff-type bound for the degree of a given vertex with respect to some deterministic sequence. Then we again use a special sequence, which has the desired rate of growth and serves as a lower bound on $\deg_t(s)$. Note that we don't need to extend our analysis for the late vertices since a lower bound for the degree of any vertex s at time t is also a lower bound for the minimum degree of G_t.

First, we note that if either we start from non-empty graph, then there exists $s \in [1, t_0]$ such that $\deg_{t_0}(s) \geq 1$. Moreover, even if the starting graph is empty, but $r > 0$, then with high probability there exist a vertex with positive degree, as the probability of adding another isolated vertex to an empty graph on t vertices is at most $(1 - \frac{r}{t})^t \leq \exp(-r)$, so within first $\frac{A}{r} \log t$ vertices for any $A > 0$ we have a non-isolated vertex with probability at least $1 - O(t^{-A})$. Of course, if we both start from an empty graph and $r = 0$, then there cannot arise any edge in the duplication process – yet in this case we have trivially $\Delta(G_t) = 0$, so we omit this case in further analysis.

Let us now return to the aforementioned Chernoff-type lower bound:

▶ **Lemma 14.** *Let $1 \leq s \leq \tau \leq t$. Let X_τ be any value such that $\deg_\tau(s) \geq X_\tau$. Then for any $h \leq \varepsilon\tau$ with $\varepsilon \in (0, 1/3)$ it is true that*

$$\Pr\left[\deg_{\tau+h}(s) \leq \deg_\tau(s) + (1 - 2\varepsilon)\,\frac{hpX_\tau}{\tau}\right] \leq \exp\left(-\frac{h\varepsilon^2(1-\varepsilon)pX_\tau}{2\tau}\right).$$

Proof. As in the proof of the previous Chernoff-type bound, let us recall that for $i = 0, 1, \ldots, h-1$ we have $\deg_{\tau+i+1}(s) = \deg_{\tau+i}(s) + I_{\tau+i}$ where $I_{\tau+i} \sim Be\left(\frac{p\deg_{\tau+i}(s)+r}{\tau+i}\right)$. Also clearly $\deg_{\tau+i}(s) \geq \deg_\tau(s)$ for any $i = 0, 1, \ldots, h$, so we have

$$\frac{\deg_{\tau+i}(s)}{\tau+i} \geq \frac{\deg_\tau(s)}{\tau+h} \geq \frac{X_\tau}{\tau(1+\varepsilon)} \geq (1-\varepsilon)\frac{X_\tau}{\tau}.$$

Therefore for any $i = 0, 1, \ldots, h-1$ we know that $I_{\tau+i}$ stochastically dominates $I^*_{\tau+i} \sim Be\left((1-\varepsilon)\frac{pX_\tau}{\tau}\right)$.

Now, from the well known Chernoff bound formula we know that for any $\varepsilon \in (0, 1)$

$$\Pr\left[\deg_{\tau+h}(s) - \deg_\tau(s) \leq (1-\varepsilon)\,\mathbb{E}\left[\sum_{i=0}^{h-1} I^*_{\tau+i}\right]\right] \leq \exp\left(-\frac{\varepsilon^2}{2}\mathbb{E}\left[\sum_{i=0}^{h-1} I^*_{\tau+i}\right]\right)$$

and therefore

$$\Pr\left[\deg_{\tau+h}(s) \leq \deg_\tau(s) + (1-2\varepsilon)\,\frac{hpX_\tau}{\tau}\right]$$
$$\leq \Pr\left[\deg_{\tau+h}(s) \leq \deg_\tau(s) + (1-\varepsilon)^2\,\frac{hpX_\tau}{\tau}\right] \leq \exp\left(-\frac{h\varepsilon^2(1-\varepsilon)pX_\tau}{2\tau}\right).$$

Finally, is is sufficient to see that if $\varepsilon < \frac{1}{3}$, then we can replace $\frac{1-\varepsilon}{2}$ by $\frac{1}{3}$ in the last formula, which completes the proof. ◀

▶ **Corollary 15.** *Let $1 \leq s \leq \tau \leq t$. Let $X_\tau \geq 0$, $A > 0$, $\varepsilon \in (0, 1/3)$ be values such that $\deg_\tau(s) \leq \tau$ and $3A\log t \leq \varepsilon^3 pX_\tau$. Then for any $h \in \left[\frac{3A\log t}{\varepsilon^2 pX_\tau}, \varepsilon\tau\right]$ it is true that*

$$\Pr\left[\deg_{\tau+h}(s) \leq \deg_\tau(s) + (1-2\varepsilon)\,\frac{hpX_\tau}{\tau}\right] = O(t^{-A}).$$

Next, we again use sequences $(t_i)_{i=1}^k$ and $(X_{t_i})_{i=1}^k$ from Definition 4. Let us also define $\mathcal{C}_i(s)$ as the event that $\deg_{t_i}(s) \geq X_{t_i} - \phi + 1$ for a fixed $s \leq t_i$. This allows us to proceed with the main theorem of this section:

▶ **Theorem 16.** *For $G_t \sim DD(t, p, r)$ with $\frac{1}{2} < p < 1$ there exists s such that it holds asymptotically that*

$$\Pr\left[\deg_t(s) < (1-\varepsilon)t^p\right] = O(t^{-A})$$

for any constants $\varepsilon > 0$ and $A > 0$.

Proof. Again let us use sequences $(t_i)_{i=0}^k$ and $(X_{t_i})_{i=0}^k$ with $\alpha = 12p^3(A+1)\log^2 t$, $\beta_i = \alpha p - \frac{\alpha}{\log t_i}$ and $\phi = \log^4 t$. These parameters satisfy the assumptions of Lemma 6 and Corollary 7.

Moreover, if $\mathcal{C}_i(s)$ holds, then the assumptions of Corollary 15 also are true for $\tau = t_i$ and $h = \frac{\alpha t_i \log t_i}{X_{t_i}}$ as $t_i \to \infty$ since for any constant $A > 0$ and $\varepsilon = \frac{1}{2p\log t_i}$ it holds that

$$\frac{3A\tau\log t}{\varepsilon^2(pX_{t_i}+r)} < \frac{\alpha t_i \log t_i}{X_{t_i}} < \varepsilon t_i,$$

and

$$X_{t_{i+1}} - X_{t_i} = \beta_i \log t_i \le \beta_i \log t_i \frac{pX_{t_i}}{pX_{t_i}} \frac{1 - \frac{2}{2p\log t_i}}{1 - \frac{1}{p\log t_i}} = (1-2\varepsilon)\frac{hpX_{t_i}}{t_i}.$$

Therefore, Corollary 15 implies that for any constant $A > 0$ it is true that $\Pr[\neg\mathcal{C}_{i+1}(s)|\mathcal{C}_i(s)] = O(t^{-A})$. Note that we apply this with a sequence $X_{t_i} - \phi + 1$, not with X_{t_i} itself this time.

Since $X_{t_0} = \log^4 t$ we know that $\mathcal{C}_0(s)$ holds with high probability: either the starting graph is nonempty or $r > 0$ and for first t_0 vertices at least one edge appears. Finally, we obtain using Lemma 8 and Corollary 15 that for any $\varepsilon > 0$

$$\Pr[\deg_t(s) < (1-\varepsilon)t^p] \le \Pr[\deg_t(s) < X_{t_{k-1}} - \phi + 1] \le \Pr[\deg_{t_{k-1}}(s) < X_{t_{k-1}} - \phi + 1]$$
$$= \Pr[\neg\mathcal{C}_{k-1}(s)] \le \sum_{i=0}^{k-2} \Pr[\neg\mathcal{C}_{i+1}(s)|\mathcal{C}_i(s)] + \Pr[\neg\mathcal{C}_0(s)] = \sum_{i=0}^{k-1} O(t^{-A}) = O(t^{-A+1}).$$

◀

▶ **Corollary 17.** *For $G_t \sim DD(t,p,r)$ with $\frac{1}{2} < p < 1$ it holds asymptotically that*

$$\Pr[\Delta(G_t) \le (1-\varepsilon)t^p] = O(t^{-A})$$

for any constants $\varepsilon > 0$ and $A > 0$.

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