Some online Maker-Breaker games

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Abstract

We consider some Maker-Breaker games of the following flavor. We have some set V of items for purchase. Maker's goal is to purchase some member of a given family $\mathcal{H}$ of subsets of V as cheaply as possible and Breaker's goal is to make the purchase as expensive as possible. Each player has a pointer and during a player's turn their pointer moves through the items in the order of the permutation until the player decides to take one. We mostly focus on the case where the permutation is random and unknown to the players (it is revealed by the players as their pointers move).

1 Introduction

In this note, we consider a few related games with two players, Maker and Breaker. In general, we have some set V of items for purchase, and Maker's goal is to purchase some member of a given family $\mathcal{H}$ of subsets of V (as a subset of the items that Maker purchases). We will focus on cases where Maker can guarantee (or "almost" guarantee) that they are able to get some member of $\mathcal{H}$. But each item has a cost, and Maker would also like to minimize the total cost of the items they purchase. Breaker's goal will be to maximize the cost Maker needs to pay. In this sense Breaker does not play the same role as the type of player usually called "Breaker," whose goal is to entirely prevent Maker from collecting some set of items. A typical Maker-Breaker game is played on a board, which is often the complete graph K_n. Maker and Breaker take turns choosing items. Maker is trying to acquire a set of edges containing some structure e.g. a spanning tree; a Hamilton cycle etc., and Breaker is trying to prevent this. Breaker takes b items for every one of Maker's. There is some threshold b^* such that if $b \leq b^*$ then Maker will succeed and if $b > b^*$ then Breaker will succeed. (See Hefetz, Krivelevich, Stojaković and Szabó [4] for nice monograph on positional games.) Instead, in our games $b \leq b^*$ (sometimes we make the weaker assumption that b is small enough that Maker can obtain the structure they want with high probability) and Breaker is resigned to allowing Maker to succeed in collecting a desired set of items, but wants to make the total cost of Maker's items as expensive as possible.

The items in V will be ordered in a uniform random permutation. Each item has a uniform random cost in the interval $[0, 1]$. The permutation of V and the costs are unknown to the players in the beginning. Each player has a pointer which in the beginning points at the first item of V in the permutation. As the game progresses the pointers will move forward through the permutation. Once either player's pointer moves past an item of V, both players know the item's cost and what position it is in the random permutation of V. Of course, our analysis of these games will often involve "revealing" the random costs and the permutation as the pointers move forward, leaving the rest of the permutation and random costs "unrevealed." The players

alternate turns and Breaker's turn is first. Breaker gets to take b items per term and Maker gets to take one item. Breaker's pointer moves forward one item at a time, and as Breaker's pointer moves over an item they see its cost and decides to either take it or reject it. Once Breaker has taken a total of b items (or reached the end of V), it is Maker's turn. Maker's pointer is still at the first item, but Maker knows everything Breaker has revealed about the permutation and costs up to wherever Breaker's pointer is now. Maker examines the items in order and for each one that hasn't already been taken by Breaker, Maker can purchase it or reject it. Thus Maker can either take an item rejected by Breaker, or Maker could wait until Maker's pointer moves past Breaker's to start revealing new items until Maker decides to purchase one (or reaches the end of V). Then Breaker gets another turn, starting wherever Breaker's pointer was when Breaker's last turn ended. And so on.

In the first game we consider, $\mathcal{H}$ is just the set of all singletons, i.e. Maker's goal is to purchase one item. We call this game ITEM since Maker just wants one item. We prove the following.

Theorem 1. *For the game* ITEM *with $b = 1$ there exists a strategy for Maker which always purchases an item and where the expected cost paid by Maker is at most $(4 + o(1))/n$. For all $2 \le b \le n/\log^4 n$ there exists such a strategy for Maker where the expected cost is at most $O\left(\frac{b \log^2 b}{n}\right)$. Furthermore, there exists a strategy for Breaker for which Maker's expected cost is $\Omega\left(\frac{b}{n}\right)$.*

Sometimes we will make an additional restriction on Breaker to prevent them from getting too far ahead of Maker. Maker's strategy will often involve using the early items in the permutation to build a collection that, for example, contains all but one item of e for many sets $e \in \mathcal{H}$. Maker will then look in the later items in the permutation to find a single low-cost item which would complete such a set e. But if Breaker knows that Maker plans to follow such a strategy then Breaker can pass over the early part of the permutation, observing the structure that Maker will build from it, and spoil Maker's strategy in the later parts of the permutation by removing the items Maker will want. In some situations it seems Maker is powerless to stop Breaker from doing this, since Maker will have to spend a significant amount of turns in the earlier part of the permutation in order to purchase the necessary items. While Maker is stuck in the earlier part of the permutation, Breaker is free to cause this kind of trouble in the later part. Thus, we sometimes add a restriction to prevent Breaker from doing this. We say the game is k-phase restricted if the permutation of items is divided into k contiguous pieces we call the *phases*, each consisting of $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ steps, such that Breaker is not allowed to start a phase until Maker has reached the end of the previous phase.

We now consider the case of trying to purchase a subgraph. More precisely, our set of items V will be the edge set of K_n, and $\mathcal{H}$ some collection of sets of edges in K_n. Our next result is when $\mathcal{H}$ is the collection of triples of edges that form a triangle in K_n (more generally, $\binom{k}{2}$-tuples of edges forming a k-clique). The game where Maker tries to build a k-clique will be called k-CLIQUE.

Theorem 2. *For the game 3-CLIQUE with $b \le n^{2/3}/10$ and no phase restriction, there is a strategy for Maker which w.h.p. succeeds in purchasing a 3-clique and spends at most $O(bn^{-1/3} \log n)$. For each fixed $k \ge 3$ and for all b with $1 \le b = o(n^{\alpha_k/4})$, $\alpha_k = \frac{1}{11 \cdot 2^{k-5}-1}$ there exists a k-phase restricted strategy for Maker which w.h.p. succeeds in purchasing a k-clique and spends at most $O\left(bn^{-\alpha_k} \log^2(b+1)\right)$.*

Next we consider the problem where Maker wants a path from u to v for two given vertices u, v. For the case $b = 0$, Frieze and Pegden [3] proved that this can be done with an expected cost of $n^{-2/3+o(1)}$. We say an event holds with high probability (w.h.p.) if the probability tends to 1 as $n \to \infty$.

Theorem 3. *For $b \le n^{1-1/\log \log n}$ there is an $O(\log \log n)$-phase restricted strategy for Maker to purchase a path between two given vertices which succeeds with high probability with a total cost of at most $bn^{-2/3+o(1)}$*

Finally, we consider an "order-restricted" version of the box game studied by Chvátal and Erdős [1]. The original box game from [1] is also a Maker-Breaker game (although in our description we will swap the names of Maker and Breaker compared to the description in [1]). We start with n boxes, each having m balls. The players take turns, Maker going first. Maker takes one ball each turn, and Breaker takes b balls. Maker's goal is to take one ball from every box. Chvátal and Erdős [1] gave the exact value $m_0 = m_0(n, b)$ such that Maker has a winning strategy if and only if $m \geq m_0$. For large n, this m_0 is about $b \log n$.

In the "order-restricted" version we consider here, the balls have some ordering which is initially unknown to the players (the ordering is revealed as the game is played). On each player's turn they are offered the balls according to the order, picking up from wherever they left off at their last turn (and of course they cannot take a ball that was already taken by the other player). We call our version BoxGame. We consider the case where the balls are randomly ordered as well as the case of general orderings.

Theorem 4. *If the ordering of balls is arbitrary (possibly adversarial favoring Breaker, and adaptive to the moves made so far), then for all $b \geq 1$, Maker has a winning strategy for* BoxGame *if and only if $m \geq bn+1$. If the ordering is random and $b \geq b_0 = 100\varepsilon^{-2} \log n$, then for any fixed $\varepsilon > 0$ and $m \leq (1 - \varepsilon)bn$, there is a strategy for Breaker to win with high probability.*

2 Purchasing one item

In this section we prove Theorem 1. The lower bound is easy so we do it first.

2.1 Lower bound

Breaker's strategy is to take any item that costs at most $b/2n$. The expected number of such items is $b/2$, so by Markov's inequality there are at most b of them with probability at least $1/2$. In that event Breaker will get all of those items and then Maker has to pay at least $b/2n$. Thus Maker's expected cost is at least $b/4n$. This completes the proof of the lower bound.

2.2 Upper bound for $2 \leq b \leq n/\log^4 n$

We handle the case where $2 \leq b \leq n/\log^4 n$. Let

$$\alpha := 10 + 10\lceil \log b \rceil \tag{1}$$

Set

$$N := \left\lceil \frac{n}{b+1} \right\rceil, \qquad m_i := \frac{\alpha}{N + \alpha - i}. \tag{2}$$

Maker's strategy is as follows. First we break the n steps into $b + 1$ *phases* of N consecutive steps. In each phase, Maker will try to take the first item i such that its cost c_i satisfies $c_i \leq m_i$ (where the value of i restarts at each phase so each phase has $i = 1, \ldots, N$). Since $m_N = 1$ it is guaranteed that each phase has such a value i. Of course when we say Maker "tries" to take the item, we mean that Maker takes it unless Breaker already has. But, since there are $b + 1$ phases, Maker will always be able to obtain one of them.

For $1 \leq i \leq N$, let $\mathcal{E}_i$ be the event that out of all $b + 1$ phases, i is the largest indexed item that Maker has tried to take. In other words, in one of the phases Maker has tried to take item i and in all the other phases

Maker has tried to take some item at most i. In the event $\mathcal{E}_i$ Maker gets some item indexed at most i which could cost at most m_i. The expected cost paid by Maker is at most

$$\sum_{i=1}^{N} \mathbb{P}\left(\mathcal{E}_i\right) m_i. \tag{3}$$

We now start trying to bound (3) from above. We have

$$\mathbb{P}\left(\mathcal{E}_i\right) \leq (b+1) \left[\left(\prod_{j=1}^{i-1}(1 - m_j) \right) m_i \right] \left[1 - \left(\prod_{j=1}^{i}(1 - m_j) \right) \right]^{b}. \tag{4}$$

We address the large values of i as follows. Set $x := \lceil \alpha^2 \log N \rceil$, and note that since we have $2 \leq b \leq n/\log^4 n$, (1) and (2), it follows that

$$\Omega(1) \leq \alpha \leq O(\log n), \qquad N = \Omega(\log^4 n), \qquad \alpha = o(N), \qquad \alpha^2 = o(x). \tag{5}$$

For $i > N - x$ we have

$$\mathbb{P}\left(\mathcal{E}_i\right) \leq (b+1) \prod_{j=1}^{N-x}(1 - m_j) \leq (b+1) \exp\left\{ -\sum_{j=1}^{N-x} m_j \right\}. \tag{6}$$

Now by the approximation of the harmonic series, with γ equal to the Euler constant,

$$\sum_{\ell=1}^{m} \frac{1}{\ell} = \log(m) + \gamma + O\left(\frac{1}{m}\right). \tag{7}$$

Thus we have

$$\begin{aligned} \sum_{j=1}^{N-x} m_j &= \sum_{j=1}^{N-x} \frac{\alpha}{N + \alpha - j} \\ &= \alpha\left(\log(N + \alpha - 1) - \log(x + \alpha - 1) + O\left(\frac{1}{x + \alpha - 1}\right) \right) \\ &= -\log\left(\frac{x + \alpha - 1}{N + \alpha - 1}\right)^{\alpha} + o(1), \end{aligned} \tag{8}$$

where on the last line we have used $\frac{\alpha}{x+\alpha-1} = o(1)$ which follows from (5). Thus the total contribution to (3) by all terms $i > N - x$ is at most x times (6), which in light of (8) is at most

$$O\left(x(b+1) \left(\frac{x + \alpha - 1}{N + \alpha - 1}\right)^{\alpha} \right) = O\left(x(b+1) \left(\frac{2x}{N}\right)^{\alpha} \right), \tag{9}$$

where we have used that $x \to \infty$ and $\alpha = o(\sqrt{x})$ (see (5)). Now we address the rest of the values i. For $i \leq N - x$ we have for all $j \leq i$ that $m_j \leq m_{N-x} = \alpha/(x + \alpha) = o(1)$. Thus, using the Taylor series for $\log(1 - x)$ gives

$$\prod_{j=1}^{i}(1 - m_j) = \exp\left\{ \sum_{j=1}^{i} \log(1 - m_j) \right\} = \exp\left\{ -\sum_{j=1}^{i}(m_j + O(m_j^2)) \right\}. \tag{10}$$

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Using (7) again we have

$$\sum_{j=1}^{i} m_j = \sum_{j=1}^{i} \frac{\alpha}{N + \alpha - j}$$

$$= \alpha \left(\log(N + \alpha - 1) - \log(N + \alpha - i - 1) + O \left(\frac{1}{N + \alpha - i - 1} \right) \right)$$

$$= -\log \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} + o(1), \tag{11}$$

where on the last line we used $\frac{\alpha}{N+\alpha-i-1} \le \frac{\alpha}{x+\alpha-1} = o(1)$ by (5). Using

$$\sum_{\ell=1}^{m} \frac{1}{\ell^2} = \frac{\pi^2}{6} + O \left(\frac{1}{m} \right),$$

we have

$$\sum_{j=1}^{i} m_j^2 = \sum_{j=1}^{i} \frac{\alpha^2}{(N + \alpha - j)^2} = O \left(\frac{\alpha^2}{N + \alpha - i} \right) = o(1), \tag{12}$$

since $\frac{\alpha^2}{N+\alpha-i-1} \le \frac{\alpha^2}{x+\alpha-1} = o(1)$ by (5). Now by (11) and (12), (10) is

$$\exp \left\{ \log \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} + o(1) \right\} = (1 + o(1)) \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha}.$$

Therefore for $i \le N - x$, the right hand side of (4) is

$$(1 + o(1))(b + 1)m_i \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} \left[1 - (1 + o(1)) \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} \right]^{b}$$

$$\le 2(b + 1)m_i \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} \left[1 - \frac{1}{2} \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} \right]^{b}$$

$$\le 2(b + 1)m_i \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} \exp \left\{ -\frac{1}{2}b \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} \right\}.$$

Thus, the contribution to (3) by terms $i \le N - x$ is at most

$$2(b + 1) \sum_{i=1}^{N-x} m_i^2 \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} \exp \left\{ -\frac{1}{2}b \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} \right\}$$

$$= 2(b + 1) \sum_{i=1}^{N-x} \left(\frac{\alpha}{N + \alpha - i} \right)^{2} \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} \exp \left\{ -\frac{1}{2}b \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} \right\}$$

$$\le \frac{2\alpha^2(b + 1)}{(N + \alpha - 1)^2} \sum_{i=1}^{N-x} \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha - 2} \exp \left\{ -\frac{1}{2}b \left(\frac{N + \alpha - i - 1}{N + \alpha - 1} \right)^{\alpha} \right\}. \tag{13}$$

Letting

$$f(t) := t^{\alpha - 2} e^{-\frac{1}{2}bt^{\alpha}},$$

elementary calculus shows that the maximum of f over $0 \le t \le 1$ occurs when $t = \left(\frac{2(\alpha-2)}{\alpha b} \right)^{1/\alpha}$. Thus each term on the right hand side of (13) is at most

$$f \left(\left(\frac{2(\alpha - 2)}{\alpha b} \right)^{1/\alpha} \right) = \left(\frac{2(\alpha - 2)}{\alpha b} \right)^{1 - 2/\alpha} e^{-1 + 2/\alpha} = O \left(b^{-1 + 2/\alpha} \right) = O(b^{-1}),$$

where we have used (1). Thus the right hand side of (13) is at most

$$\frac{2\alpha^2(b+1)}{(N+\alpha-1)^2} \cdot (N-x) \cdot O(b^{-1}) = O\left(\frac{\alpha^2}{N}\right) = O\left(\frac{b\log^2 b}{n}\right), \tag{14}$$

where we have used (5), (2) and (1). It remains to show that (9) is negligible compared to (14). Suppose first that $2 \le b \le \sqrt{n}$. Then we have $N = \Omega(\sqrt{n})$, $x = O(\log^3 n)$ and $\alpha \ge 10$. In this case (9) is

$$O\left(x(b+1)\left(\frac{2x}{N}\right)^\alpha\right) = O\left(\log^3 n \cdot \sqrt{n}\left(\frac{\log^3 n}{\sqrt{n}}\right)^{10}\right),$$

which is negligible compared to (14). Now suppose $\sqrt{n} \le b \le n/\log^4 n$. Then $N = \Omega(\log^4 n)$, $x = O(\log^3 n)$ and $\alpha \ge 5\log n$. In this case (9) is

$$O\left(x(b+1)\left(\frac{2x}{N}\right)^\alpha\right) = O\left(\log^3 n \cdot \frac{n}{\log^4 n}\left(\frac{C\log^3 n}{\log^4 n}\right)^{5\log n}\right),$$

where C is some constant. The last expression above is also negligible compared to (14). Thus the expected cost paid by Maker is $O\left(\frac{b\log^2 b}{n}\right)$, completing the proof of Theorem 1 in the case where $2 \le b \le n/\log^4 n$.

2.3 Upper bound for $b = 1$

Finally we do the upper bound for $b = 1$. Recall that c_i is the cost of item i. Maker's strategy will be to take the first vertex i which is not already taken by Breaker and such that $c_i \le m_i$, where

$$m_i = \frac{2}{n-i+1} \quad \text{for } i \le n-1, \quad \text{and } m_n = 1.$$

Say Breaker's strategy is to remove the first vertex j such that $c_j \le b_j$ (for some deterministic $b_1, \ldots b_n$). We will show that for any choice of the b_j, Maker's expected cost is at most asymptotically $4/n$.

Let the random variables M, B be the names of the vertices taken my Maker and Breaker, respectively. The expected cost paid by Maker is precisely

$$\mathbb{E}[c_M] = \sum_{i=1}^{n} \mathbb{P}(M = i \ \wedge \ B > i)\mathbb{E}[c_i \mid M = i \ \wedge \ B > i] \tag{15}$$

$$+ \sum_{i=2}^{n} \sum_{j=1}^{i-1} \mathbb{P}(M = i \ \wedge \ B = j)\mathbb{E}[c_i \mid M = i \ \wedge \ B = j]. \tag{16}$$

For each j, let $a_j := \max\{b_j, m_j\}$. Then for all $i \ge 1$ we have

$$\mathbb{P}(M = i \ \wedge \ B > i) = \left(\prod_{k=1}^{i-1}(1 - a_k)\right) \cdot (a_i - b_i),$$

where we regard an empty sum as 0 and an empty product as 1. Now either $a_i = b_i$ in which case the above is 0, or else $b_i < m_i$ in which case we have

$$\mathbb{E}[c_i \mid M = i \ \wedge \ B > i] = \frac{m_i + b_i}{2} = \frac{a_i + b_i}{2}.$$

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Meanwhile we have for $1 \leq j < i$ that

$$\mathbb{P}(M = i \ \wedge \ B = j) = \left(\prod_{k=1}^{j-1}(1 - a_k)\right) \cdot b_j \cdot \left(\prod_{k=j+1}^{i-1}(1 - m_k)\right) \cdot m_i,$$

and

$$\mathbb{E}[c_i \mid M = i \ \wedge \ B = j] = \frac{m_i}{2}.$$

Thus, letting $\mathbf{b}$ be the vector whose entries are the b_i, we define the function

$$c(\mathbf{b}) := \mathbb{E}[c_M] = \sum_{i=1}^{n}\left(\prod_{k=1}^{i-1}(1 - a_k)\right) \cdot \frac{a_i^2 - b_i^2}{2} \tag{17}$$

$$+ \sum_{i=2}^{n}\sum_{j=1}^{i-1}\left(\prod_{k=1}^{j-1}(1 - a_k)\right) \cdot b_j \cdot \left(\prod_{k=j+1}^{i-1}(1 - m_k)\right) \cdot \frac{m_i^2}{2}, \tag{18}$$

where we have used (15) and (16).

We now maximize $c(\mathbf{b})$ over all possible $\mathbf{b}$.

Claim 1. *There is some choice for $\mathbf{b}$ maximizing $c(\mathbf{b})$ such that*

$$b_{i^*} = \tilde{b}_{i^*} := \frac{2}{n - i^* - 1} - \frac{2}{(n - i^*)(n - i^* - 1)}\sum_{\ell=1}^{n-i^*}\frac{1}{\ell} \quad \text{for all } 1 \leq i^* \leq n - 10. \tag{19}$$

Proof. First we claim that, for some $\psi \geq 0$ which does not depend on b_{i^*} (but which may depend on all the other b_i, m_i) that we have

$$\frac{\partial c(\mathbf{b})}{\partial b_{i^*}} = -\psi \cdot \mathbb{1}_{b_{i^*} \geq m_{i^*}} - \left(\prod_{k=1}^{i^*-1}(1 - a_k)\right) \cdot b_{i^*} + \sum_{i=i^*+1}^{n}\left(\prod_{k=1}^{i^*-1}(1 - a_k)\right) \cdot \left(\prod_{k=i^*+1}^{i-1}(1 - m_k)\right) \cdot \frac{m_i^2}{2}. \tag{20}$$

Indeed, the first term above is from the dependence on $a_{i^*} = \max\{b_{i^*}, m_{i^*}\}$, and takes care of all the terms on line (17) with $i > i^*$ as well as the terms on line (18) with $j > i^*$. The second term above is the derivative of the term in line (17) with $i = i^*$. The final sum above is the derivative of all terms in line (18) with $i = i^*$. The rest of the terms in (17) and (18) have no dependence on b_{i^*}.

Note that the product of $1 - m_k$ is telescoping. In particular, when $i^* + 1 \leq i \leq n$ we have

$$\prod_{k=i^{**}+1}^{i-1}(1 - m_k) = \prod_{k=i^{**}+1}^{i-1}\frac{n - k - 1}{n - k + 1} = \frac{(n - i + 1)(n - i)}{(n - i^{**})(n - i^{**} - 1)}.$$

Thus, after reindexing the sum in $\tilde{b}_{i^*}$ we have

$$\tilde{b}_{i^*} = \frac{2}{(n - i^*)(n - i^* - 1)}\sum_{i=i^*+1}^{n}\frac{n - i}{n - i + 1}$$

$$= \sum_{i=i^*+1}^{n}\frac{(n - i + 1)(n - i)}{(n - i^*)(n - i^* - 1)} \cdot \frac{2}{(n - i + 1)^2}$$

$$= \sum_{i=i^*+1}^{n}\left(\prod_{k=i^*+1}^{i-1}(1 - m_k)\right) \cdot \frac{m_i^2}{2}.$$

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Using the last expression for $\tilde{b}_{i^*}$, (20) becomes

$$\frac{\partial c(\mathbf{b})}{\partial b_{i^*}} = -\psi \cdot \mathbb{1}_{b_{i^*} \geq m_{i^*}} - \left(\prod_{k=1}^{i^*-1}(1-a_k)\right) \cdot (b_{i^*} - \tilde{b}_{i^*}). \tag{21}$$

Now, using (19) and bounding the harmonic series, we have

$$\tilde{b}_{i^*} \leq \frac{2}{n-i^*-1}\left(1 - \frac{1}{n-i^*}\log(n-i^*)\right) \leq \frac{2}{n-i^*-1}\left(1 - \frac{2}{n-i^*+1}\right) = m_{i^*}$$

where the second inequality follows since $\frac{\log(x)}{x} \geq \frac{2}{x+1}$ for all $x \geq 10$.

Now if $b_{i^*} < \tilde{b}_{i^*}$, (21) is nonnegative (here the first term is 0 since $b_{i^*} < m_{i^*}$). If $b_{i^*} > \tilde{b}_{i^*}$, (21) is nonpositive. This completes the proof of Claim 1. $\qquad\square$

Supposing Maker plays by strategy $\mathbf{m}$ then by Claim 1 the best Breaker can do is to play by the strategy given in (19). We now estimate (17)+(18) for these strategies. Actually since we only need a lower bound, we drop line (17) and all the terms of (18) with $i > n - 10$. Since we have $b_i \leq m_i = a_i$ for all such i, (18) is at least

$$\sum_{i=2}^{n-10}\sum_{j=1}^{i-1}\left(\prod_{k=1}^{j-1}(1-m_k)\right)\cdot b_j \cdot \left(\prod_{k=j+1}^{i-1}(1-m_k)\right)\cdot\frac{m_i^2}{2}$$

$$= \sum_{i=2}^{n-10}\sum_{j=1}^{i-1}\frac{(n-i+1)(n-i)(n-j+1)}{n(n-1)(n-j-1)}\cdot\frac{b_j m_i^2}{2}$$

$$= \frac{1}{2n(n-1)}\sum_{i=2}^{n-10}(n-i+1)(n-i)m_i^2\sum_{j=1}^{i-1}\left(1+\frac{2}{n-j-1}\right)b_j$$

$$= \frac{1}{2n(n-1)}\sum_{i=2}^{n-10}\left(4 - \frac{1}{n-i+1}\right)\sum_{j=1}^{i-1}\left(\frac{2}{n-j} + O\left(\frac{\log n}{(n-j)^2}\right)\right)$$

$$= \frac{1}{n(n-1)}\sum_{i=2}^{n-10}\left(4 - \frac{1}{n-i+1}\right)\left(-\log\left(\frac{n-i}{n}\right) + O\left(\frac{\log n}{n-i}\right)\right)$$

$$\sim \frac{4}{n},$$

since $\sum_{i=2}^{n-10}\log\left(\frac{n-i}{n}\right) \sim n\int_0^1 \log(1-x)\,dx = -n$. This completes the proof of Theorem 1 for $b = 1$.

3 Purchasing a k-clique

In this section we prove Theorem 2. We start with the game 3-CLIQUE without any phase restriction. Here and several times in the future we will use the Chernoff–Hoeffding bound (see, for example, Theorem 23.6 in [2])

Theorem 5. *Let X be distributed as $Bin(n, p)$ and $0 < \varepsilon < 1$. Then*

$$\mathbb{P}(|X - np| > \varepsilon np) \leq 2\exp(-\varepsilon^2 np/3)$$

We will now from now on define integer quantities without expressly rounding upper or down. The reader can be assured that either choice will lead to a satisfactory conclusion.

3.1 Unrestricted 3-clique

Maker will first build a $n^{1/3}$-star by the time Maker has seen half of the edges. Maker can do this by arbitrarily choosing a root v at the beginning, and then taking every offered edge incident with v of cost at most $8(b+1)n^{-2/3}$. Indeed, Theorem 5 implies that the number of such edges among the first $\binom{n}{2}/2$ edges is at least half its expectation, i.e. $2(b+1)n^{1/3}$, with probability at least $1 - \exp(-\Omega(n^{1/3}))$. Of course Breaker can take some, but Maker will get at least a $1/(b+1)$ fraction of them so Maker easily gets $n^{1/3}$ of them, completing the $n^{1/3}$-star rooted at v. The total cost paid by Maker for the star is at most $8(b+1)n^{-1/3}$. Maker still gets to look at the second half of the edges to complete the triangle.

When Maker has finished building the $n^{1/3}$-star, Breaker has only taken $bn^{1/3}$ edges total. Unfortunately we have to deal with the possibility that Breaker has taken many edges in the neighborhood of v. Let K be the set of leaves of our $n^{1/3}$-star. The expected number of edges in K among the second half of the edges in the random permutation of cost at most $40bn^{-1/3}\log n$ is $\frac{1}{2}\binom{n^{1/3}}{2} \cdot \frac{40b\log n}{n^{1/3}} \sim 10bn^{1/3}\log n$. By Theorem 5, the probability that this number of edges is at most $5bn^{1/3}\log n$ (i.e. half its expectation) is at most $\exp\left(-10bn^{1/3}\log n/12\right)$ which is small enough to beat a union bound over choices of K, the number of choices here being $\binom{n}{n^{1/3}} \leq \exp(3n^{1/3}\log n/4)$. Thus, with high probability Maker is able to close the triangle using an edge of cost at most $20bn^{-1/3}\log n$.

Altogether, Maker pays a cost at most

$$8(b+1)n^{-1/3} + 20bn^{-1/3}\log n \leq 30(b+1)n^{-1/3}\log n.$$

3.2 k-phase restricted k-Clique

In this version we consider the first $\frac{1}{k}\binom{n}{2}$ edges to be Phase 1, the next $\frac{1}{k}\binom{n}{2}$ edges to be Phase 2, and so on. If Breaker reaches the end of a phase before Maker, then Maker is allowed to take any edges Maker wants from what remains to be taken in. Breaker is not allowed to begin the next phase until Maker has done this.

In each of the first $k-3$ phases, Maker will build a $(k-3)$-clique with a large common neighborhood. Maker does this by building a star in each of these phases, each star contained inside the leaves of the last star. Finally in the last three phases Maker will build a triangle contained in the common neighborhood of the vertices in this $(k-3)$-clique, making a k-clique.

3.2.1 The first $k-3$ phases

We describe how to build our stars. Set

$$r := n^{-\alpha_k}, \qquad \ell_i := r \cdot \left(\frac{n}{r}\right)^{2^{-i}}, 0 \leq i \leq k-3, \tag{22}$$

and note that we have

$$\frac{\ell_i^2}{\ell_{i-1}} = r = \ell_{k-3}^{-4/7}, \qquad 1 \leq i \leq k-3. \tag{23}$$

$$\ell_{k-3} = \omega(b^7). \tag{24}$$

When we start Phase i for $1 \leq i \leq k-3$, assume we have chosen a set L_{i-1} of ℓ_{i-1} vertices (note that $\ell_0 = n$, so L_0 is all the vertices). Maker chooses an arbitrary vertex $v_i \in L_{i-1}$, and during Phase i Maker will take

any edge of cost at most $10k(b+1)\ell_i/\ell_{i-1}$ with one end at v_i and the other in L_{i-1}. The probability that any particular edge appears in Phase i is $1/k$. So the expected number of such edges in L_{i-1} incident with vertex v_i that appear in Phase i is $10(b+1)\ell_i \in [n^{\beta_k}, o(n)]$ for some $\beta_k > 0$. So with high probability there are at least $(b+1)\ell_i$ such edges and Maker can get ℓ_i of them at a total cost of at most $10k(b+1)\ell_i^2/\ell_{i-1} = 10k(b+1)r$ (using (23)). For $1 \le i \le k-3$, L_i will be the set of leaves of the star built at Phase i.

Altogether for Phases 1 through $k-3$, Maker pays at most $10(b+1)k^2r$. If Maker builds a triangle in L_{k-3} then Maker is done.

3.2.2   Phase $k-2$

Set $\ell := \ell_{k-3}$, $L := L_{k-3}$. Maker's strategy now is to take any edge with both ends in L whose cost is at most $10k(b+1)\ell^{-9/7}$ until Maker has $2\ell^{5/7}$ such edges. The expected number of Phase $k-2$ edges that have cost at most $10k(b+1)\ell^{-9/7}$ is

$$\frac{1}{k}\binom{\ell}{2} \cdot 10k(b+1)\ell^{-9/7} \sim 5(b+1)\ell^{5/7},$$

which grows like a power of n. Thus, an easy application of Theorem 5 gives us that there are at least $2(b+1)\ell^{5/7}$ such edges with exponentially small failure probability. Maker will get at least $2\ell^{5/7}$ of them regardless of what Breaker does. Now we would like to claim that this set of $2\ell^{5/7}$ edges contains a matching on $\ell^{5/7}$ edges. Indeed, the number of vertices that are adjacent to two edges of cost at most $10(b+1)\ell^{-9/7}$ has expectation at most

$$\ell^3 \cdot \left(10k(b+1)\ell^{-9/7}\right)^2 = O(b^2\ell^{3/7}) = o\left(\ell^{5/7}\right), \qquad \text{by (24)},$$

and so we can easily remove $o\left(\ell^{5/7}\right)$ of Maker's $2\ell^{5/7}$ edges to get our matching.

Maker ends Phase $k-2$ with a matching on $\ell^{5/7}$ edges. The total cost paid in Phase $k-2$ is at most

$$\sim 2\ell^{5/7} \cdot 10k(b+1)\ell^{-9/7} = 20(b+1)k\ell^{-4/7} = 20(b+1)kr,$$

where we have used (23).

3.2.3   Phase $k-1$

In Phase $k-1$, Maker's strategy is as follows. Let M be the matching on $m = \ell^{5/7}$ edges from Phase $k-2$. In Phase $k-1$ Maker will take any offered edge e with the following properties, until Maker has $\ell^{4/7}$ of them:

1. $e \subseteq L$ and is adjacent to some edge $e' \in M$;

2. e has cost at most $5(b+1)k^2\ell^{-8/7}$;

3. the edge e'' which would complete a triangle with e, e' has not been offered to either player yet.

By Theorem 5 it holds with exponentially small failure probability that for every edge $f \in M$ there are $\sim \frac{2}{k^2}\ell$ pairs of edges $f', f'' \subseteq L$ making a triangle with f such that f' is a Phase $k-1$ edge and f'' is a Phase k

edge. (Actually, this property holds w.h.p. for every edge f seen in the first $k - 3$ phases and so Breaker cannot prevent this strategy by influencing M.) There are then

$$\sim \ell^{5/7} \cdot \frac{2}{k^2} \ell \cdot 5(b+1)k^2 \ell^{-8/7} = 10(b+1)\ell^{4/7}$$

such triples f, f', f'' where the cost of f' is low enough that Maker would want it. In this scenario if Maker is offered f' during Phase $k - 1$ then Maker would take it (since it is impossible that f'' has been offered to anyone yet).

Technically, we will not condition on the event from the previous paragraph. We will condition on Phase $k-2$ completing as described, and then we will reveal the edges of Phase $k - 1$ one by one as they are revealed by the players (i.e. whenever a player is offered an edge which has not been offered to either player yet). Of all the steps during which an edge is revealed which Maker would want, Maker gets at least a $(b+1)^{-1}$ fraction of them. For each edge e' taken by Maker, conditional on having revealed all the edges up to e' there is probability at least $\frac{1}{2}$ that the edge e'' is a Phase k edge. Indeed, the remaining unrevealed edges are a uniform random permutation of edges consisting of some of the phase $k - 1$ edges and all of the phase k edges. As discussed in the above paragraph, with exponentially small failure probability there are at least $\sim 10(b+1)\ell^{4/7}$ steps when such an edge e' is revealed. Thus, w.h.p. Maker gets at least

$$\frac{1}{2} \cdot 10(b+1)\ell^{4/7} \cdot (b+1)^{-1} \cdot \frac{1}{2} > \ell^{4/7}$$

edges f' such that f'' is a Phase k edge. The total cost paid in Phase $k - 1$ is at most $\ell^{4/7} \cdot 5(b+1)k^2\ell^{-8/7}$.

3.2.4 Phase k and finishing the proof

Now Maker just wants to take one of the $\ell^{4/7}$ possible edges which would close a triangle, and by Theorem 1 and (23) Maker can do that with an expected cost of $O(b\log^2(b+1)/\ell^{4/7}) = O(br\log^2(b+1))$. Note that in our application of Theorem 1, we have replaced $\log^2 b$ with $\log^2(b+1)$ to handle the case where $b = 1$ (and $\log^2 b$=0). The cost of all previous phases is absorbed into the big-O. This completes the proof of Theorem 2.

4 Paths

In this section we prove Theorem 3

We describe how to adapt the strategy of the second author and Pegden [3]. They showed that for $b = 0$ Maker can build a path connecting two given vertices with a total cost $n^{-2/3+o(1)}$. Our strategy for Maker will essentially follow the strategy in [3], which still works even though Breaker will take some of the edges Maker wants.

Let $k := \log \log n$. Maker's strategy will be $3k$-phase restricted. In the first k phases, Maker will build a sequence of trees $T_1 \subseteq \ldots \subseteq T_k$ containing u. Specifically, we let T_0 consist of just the vertex u, and for $1 \leq i \leq k$, at phase i Maker will take any offered edge with one endpoint in T_{i-1} whose cost is at most $(b+1)p$ where $p := n^{-1+1/3k}$, until Maker has $((1-\varepsilon)np/3k)^i$ such edges, where $\varepsilon = n^{-1/9k}$. Note that since $b \leq n^{1-1/k}$ we have that $(b+1)p \leq 1$. Of course there is the possibility that Maker will not find this number of edges in which case phase i fails. However, we will now see that is unlikely. Indeed, assuming phases $1, \ldots, i - 1$ have

succeeded, the number of vertices in T_{i-1} satisfies

$$((1-\varepsilon)np/3k)^{i-1} \le |V(T_{i-1})| = \sum_{j=0}^{i-1}((1-\varepsilon)np/3k)^j \le (1+o(1))((1-\varepsilon)np/3k)^{i-1}.$$

The number of vertices w outside of T_{i-1} such that there is a phase i edge from w to T_{i-1} of cost at most p is distributed as $Bin(|V(T_{i-1})|(n-|V(T_{i-1})|),(b+1)p/3k)$. By Theorem 5, the probability that this random variable is less than $1-\varepsilon/2$ times its expectation μ where

$$\mu = |V(T_{i-1})|(n-|V(T_{i-1})|)(b+1)p/3k \ge ((1-\varepsilon)np/3k)^{i-1}(1-\varepsilon/3)(b+1)np/3k$$

is at most $2\exp(-\varepsilon^2\mu/12) = o(n^{-10})$. Thus with probability at least $1-o(n^{-10})$ there are at least $(1-\varepsilon/2)\mu \ge (b+1)((1-\varepsilon)np/3k)^i$ edges that Maker would want in phase i. Since Maker will get at least a $1/(b+1)$ proportion of these edges, in this case phase i is successful. Thus with probability at least $1-o(n^{-9})$ phases $1,\ldots,k$ are all successful. The amount paid by Maker during these phases is at most

$$|V(T_k)|(b+1)p = O\left(((1-\varepsilon)np/3k)^k(b+1)p\right) = (b+1)n^{-2/3+o(1)}$$

In phases $k+1,\ldots,2k$, Maker builds a sequence of trees $T_1' \subseteq \ldots \subseteq T_k'$ containing v. This works similarly the first k phases with v in place of u. Maker is able to build T_k' containing $((1-\varepsilon)np/3k)^k = n^{1/3+o(1)}$ vertices including v, and paying a total cost of at most $(b+1)n^{-2/3+o(1)}$.

Now if T_k, T_k' do not already intersect, Maker can very likely find a cheap edge connecting them in one of the phases $2k+1,\ldots 3k$ (we really do not even need to treat these as separate phases). Indeed, the expected number of edges from T_k to T_k' that cost at most $(b+1)\log^2 n/|V(T_k)| \cdot |V(T_k)'| = (b+1)n^{-2/3+o(1)}$ in the last k phases is at least

$$|V(T_k)| \cdot |V(T_k)'| \cdot \frac{1}{3} \cdot \frac{(b+1)\log^2 n}{|V(T_k)| \cdot |V(T_k)'|} = \frac{1}{3}(b+1)\log^2 n.$$

By Theorem 5 there are easily $b+1$ such edges with high probability and so Maker gets one. Maker is done, and the total cost paid is $(b+1)n^{-2/3+o(1)}$. This completes the proof of Theorem 3.

5   Box game

Here we prove Theorem 4. First assume the ordering is arbitrary, possibly adversarial and adaptive. Then Maker has a winning strategy if and only if $m \ge bn+1$. Indeed, if m is that large then Maker can win by taking a ball if and only if it belongs to a box which has the fewest balls that do not yet belong to Breaker (here we say a ball belongs to Breaker if Breaker has taken it or if it has been rejected by Maker). After Breaker has taken i turns there is never a box with more than bi balls belonging to Breaker including those rejected by Maker by induction on i. Thus, Maker can always win if $m \ge bn+1$. Now suppose $m \le bn$. The ordering can be such that (at least towards the beginning) Breaker is always offered balls from the same box while Maker is only offered balls from the other $n-1$ boxes. Maker needs one ball from each of the other boxes so Maker is forced to use up at least $n-1$ turns. Meanwhile during these $n-1$ turns Breaker has accumulated $b(n-1)$ balls from the same box and wins the game on the next turn by taking b more.

Now assume the ordering of the balls is random, $b \ge b_0$, $\varepsilon > 0$ is fixed and $m \le (1-\varepsilon)bn$. By Theorem 5, the probability that we do not see box 1 b times over the course of the first $(1+\varepsilon/2)bn$ steps is at most n^{-3}. Likewise the probability that we see box 1 more than b times or less than $(1-\varepsilon)b$ times in the first $(1-\varepsilon/2)bn$

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steps has the same bound. This bound is small enough so that we can take the union bound over all boxes and all sequences of consecutive steps. Thus with high probability our sequence of balls is such that in any consecutive $(1 + \varepsilon/2)bn$ steps we see every ball b times, and in any consecutive $(1 - \varepsilon/2)bn$ steps we see every ball between $(1 - \varepsilon)b$ and b times.

On Breaker's first turn, Breaker will always take b balls from the first box. Breaker will only continue to take balls from the first box until Maker takes a ball from that box (which requires Maker to look ahead of Breaker in the ordering, since Breaker has taken all balls in box 1 before Breaker's pointer). If Maker does take a ball from the first box, then Breaker chooses any other box which Maker has no balls in yet and starts taking balls from that box until Maker gets one, and then Breaker chooses a new box to focus on. Every time Maker takes Breaker's box, this requires Maker's pointer to go ahead of Breaker's. Breaker only takes balls that are ahead of Maker's pointer so whenever Maker gets ahead, Breaker will skip over some balls to get ahead. Breaker needs to look at at least $(1 - \varepsilon/2)bn$ balls each turn in order to get b from the desired box. In the process of looking at those $(1 - \varepsilon/2)bn$ balls, Breaker must pass over at least $(1 - \varepsilon)b$ balls in every other box. If Maker allows Breaker to take balls from the same box for $k \geq 1$ turns, then Breaker has looked at least $k(1 - \varepsilon/2)bn$ balls during these k turns. Maker cannot get ahead of Breaker to take that box without passing over $k(1 - \varepsilon)b$ balls in every other box. Therefore when Breaker picks a new box to focus on, over these k turns Maker will have rejected $k(1 - \varepsilon)b$ balls from that box already. By induction, after a total of j turns there is always a box which has at least $j(1 - \varepsilon)b$ balls which are unavailable to Maker. This completes the proof of Theorem 4.

6   Closing remarks

Our results are probably not optimal. In Theorem 1 the $\log^2 b$ factor is likely not necessary. Indeed, it is natural to expect that the optimal strategy for Maker gives an expected cost of $\sim 2(b+1)/n$. This is because, as Frieze and Pegden [3] showed, when $b = 0$ the optimal expected cost is $\sim 2/n$. Heuristically, Breaker should be able to inflate that by a factor $b + 1$ since Maker can only get one out of every $b + 1$ cheap items. However, in Theorem 1 we were only able to prove it for $b = 1$.

Regarding Theorem 2, Frieze and Pegden [3] showed that for $b = 0$ the expected cost of a triangle is at least $n^{-4/7}$ and that the $-4/7$ power is optimal. Thus it is somewhat satisfying to know that our result gets the same expected cost inflated by not much more than a factor of b. However while it is natural to assume that this b factor is necessary, it would be nice to see a strategy for Breaker which comes close to our upper bound. There is also the analogous question about the b factor in Theorem 3. Also, for Theorems 2 and 3 it is unclear whether the phase restrictions are strictly necessary.

Finally, in Theorem 4 for the randomly ordered BoxGame, we believe the condition $b \geq b_0$ is unnecessary and the result should hold for all $b \geq 1$. We use $b \geq b_0$ to establish that long-enough intervals in the ordering of balls are all nicely concentrated in terms of the number of times each ball appears. For smaller b we would not have the necessary concentration for our argument to go through. It seems that Breaker should actually be able to take advantage of this lack of concentration, but we are unable to prove it at the moment.

References

[1] V. Chvátal and P. Erdős. Biased positional games. *Ann. Discrete Math.*, 2:221–229, 1978.

[2] A. Frieze and M. Karoński. *Introduction to random graphs.* Cambridge University Press, Cambridge, 2016.

[3] A. Frieze and W. Pegden. Online purchasing under uncertainty. *Random Structures Algorithms*, 53(2):327–351, 2018.

[4] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. *Positional games.* Oberwolfach Seminars. Birkhäuser, 2014.