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ON THE STRENGTH OF CONNECTIVITY OF RANDOM SUBGRAPHS OF THE n-CUBE

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Suppose V_n is randomly sampled from the vertex set C^n of the n-cube so that $\Pr(x \in V_n)=p_v$ independently for each $x \in C^n$. Let $E(V_n)$ denote the edges of the subgraph of C^n induced by V_n under the usual adjacency relation in C^n. Suppose that A_n is now randomly sampled from $E(V_n)$ so that $\Pr(a \in A_n)=p_e$ independently for each $a \in E(V_n)$. Let $\Gamma_n=(V_n, A_n)$ be the random graph so produced. We show that for $s \geqslant 0$ integer, c constant and

$$p_e p_v=\tfrac{1}{2}+(\tfrac{1}{2}s\ln n+c)/n$$

that

$$\lim_{n\to\infty} \Pr(\Gamma_n \text{ is } s+1\text{-connected})$$

$$=1-\lim_{n\to\infty} \Pr(\Gamma_n \text{ is } s\text{-connected})$$

$$=e^{-\lambda}$$

where $\lambda=(\lim_{n\to\infty} p_v)\times e^{-2c}/s!$

1. Introduction

We consider the set $C^n=\{0, 1\}^n$, the vertex set of the unit hypercube. For any two points $x, y \in C^n$ we call x, y *adjacent* (or neighbours) if they differ in precisely one coordinate. This relation endows C^n with a graph structure. For any set

$V \subseteq C^n$, let

$$E(V)=\{\{x,y\}: x,y \in V \text{ and } x,y \text{ are adjacent}\}.$$

Let p_e, p_v, $0 \leqslant p_e, p_v \leqslant 1$ satisfy

$$p=p_e p_v=\tfrac{1}{2}+(\tfrac{1}{2}s\ln n+c)/n$$

where $s \geqslant 0$ is an integer.

We construct a random subgraph $\Gamma_n=(V_n, A_n)$ of C^n in the following manner:

(a) V_n is randomly sampled from C^n so that $\Pr(x \in V_n)=p_v$ independently for each $x \in C^n$;

(b) A_n is randomly sampled from $E(V_n)$ so that $\Pr(a \in A_n)=p_e$ independently for each $a \in A_n$.

The following theorem is the main result of this paper. It establishes the asymptotic probable connectivity of Γ_n.

Theorem 1.1.

$$\lim_{n\to\infty} \Pr(\Gamma_n \text{ is } s\text{-connected})=1-e^{-\lambda}$$

$$\lim_{n\to\infty} \Pr(\Gamma_n \text{ is } s+1\text{-connected})=e^{-\lambda}$$

where

$$\lambda=\bar{p}_v e^{-2c}/s! \tag{1.1}$$

with $\bar{p}_v=\lim_{n\to\infty} p_v$. *(Note that* $\frac{1}{2} \leqslant \bar{p}_v \leqslant 1$.*)*

The first results on this problem are due to Saposhenko [10] and Burtin [3]. They considered the case $s=0$, $p_v=1$ and p_e constant. Burtin considered the case $p_e \neq \frac{1}{2}$, and showed that if $p_e<\frac{1}{2}$ then Γ_n is almost surely not connected, but is almost surely connected when $p_e>\frac{1}{2}$. Saposhenko considered the case $p_e=\frac{1}{2}$, and gives various properties of Γ_n. (Note that our use of "almost surely" (a.s.) simply means with probability tending to 1 as $n \to \infty$.)

Erdös and Spencer [5] tightened this result and considered the case $s=0$, $p_v=1$ and $p_e=\frac{1}{2}+c/n$ for constant c. See also Bollobás [2]. The case $s=0$, $p_e=1$ and $p_v=\frac{1}{2}$ was also considered by Saposhenko [9], where the random subset

V_n is considered to define a random boolean function of n variables. (See also the paper [11] by Weber for a short review.) Theorem 1.1 generalises the above results in two ways. First we consider a model in which both random edge and vertex deletions occur, and secondly we examine the *strength* of connectivity of the resulting graph.

2. Notation and preliminaries

We will write $N=2^n$, ln denotes natural logarithms and lg logs to base 2.

For $x, y \in C^n$, the (Hamming) distance between x and y is the minimal number of edges in a path connecting x and y in $G(C^n)=G_n$. The graph G_n has N vertices and $\frac{1}{2}nN$ edges. More generally we may consider the number $H_k(n)$ of connected subgraphs of size k which are vertex-induced subgraphs of G_n. Since any component of size $(k-1)$ has at most $(k-1)n$ incident edges, we have

$$H_k(n) \leqslant N.n.2n. \dots (k-1)n=(k-1)!\, n^{k-1} N. \tag{2.1}$$

We will have recourse to the following simple inequalities:

For any $0<r\leqslant n$,

$$\left(\frac{ne}{r}\right)^r \geqslant \binom{n}{r} \geqslant \left(\frac{n}{r}\right)^r. \tag{2.2}$$

For any $S\subseteq C^n$, let $B(S)=\{y \in C^n - S : y$ is adjacent to some $x \in S\}$. Thus $B(S)$ is the set of neighbours of S. More generally we can consider the set $B^i(S)$ of points which are within a (shortest) distance i from S, defined by

$$B^0(S)=S, \quad B^i(S)=B\Big(\bigcup_{j=0}^{i-1} B^j(S)\Big).$$

Clearly $B(S)=B^1(S)$. Now, for $1\leqslant k\leqslant 2^n$, let $b(k)=\min\{|B(S)| : |S|=k\}$.

We first establish the following facts about $b(k)$, which will be required in the analysis of the main problem. The following theorem describes sets which attain $b(k)$.

Theorem 2.1. (Katona [8]). *For each $i=0, 1, \dots, n$ let $Z_i=\{x \in C^n : d(x, 0)=i\}$. Order the $x \in C^n$ so that all $x \in Z_i$ occur before all $x \in Z_j$ if $i<j$, and within each Z_i order the vertices lexicographically. Then $b(k)$ is attained by the set $S=M(k)$ which consists of the first k vertices in this ordering.*

Note that $|Z_i|=\binom{n}{i}$ for $i=0, 1, \ldots, n$. Write

$$T(i)=\sum_{j=0}^{i}\binom{n}{j}=\Big|\bigcup_{j=0}^{i} Z_j\Big|.$$

Then, for any k, let $q(k)$ be defined by

$$T(q-1)<k\leqslant T(q). \tag{2.3}$$

Thus $M(k)$ consists of $\bigcup_{j=0}^{q-1} Z_j$ together with the (lexicographic) first $(k-T(q-1))$ vertices in Z_q.

We deduce the following facts. The first deals with "small" sets.

Lemma 2.2. *For* $1\leqslant k\leqslant n+1$, $b(k)=kn-\frac{1}{2}(k-1)(k+2)$.

Proof. If $k=1$, the formula gives $b(k)=n$. Thus we may assume $k\geqslant 2$ and hence $q(k)=1$. Hence, letting e_j be the jth unit vector, and $e_0=0$, $M(k)=\{e_0, e_1, e_2, \ldots, e_{k-1}\}$. Now $M(k)$ has precisely the following neighbours:

(i) All vertices of the form e_i+e_j $(0\leqslant i\leqslant k-1, k\leqslant j\leqslant n)$

(ii) All vertices of the form e_i+e_j $(1\leqslant i\leqslant j\leqslant k-1)$

and all these are distinct. There are $k(n-k+1)$ of type (i) and $\binom{k-1}{2}$ of type (ii). Thus

$$b(k)=k(n-k+1)+\binom{k-1}{2}=kn-\tfrac{1}{2}(k-1)(k+2). \qquad \square$$

Lemma 2.3. *For all* k, $b(k)>\left(\dfrac{n-2q-1}{q+1}\right)k$ *where* $q=q(k)$.

Proof. Let $t=k-T(q-1)$. Now $M(k)$ has the following neighbours:

(i) All vertices of Z_q except for the first t.

(ii) All vertices in Z_{q+1} adjacent to the first t in Z_q.

There are clearly $\binom{n}{q}-t$ of type (i). To count those of type (ii), consider $G(Z_q \cup Z_{q+1})$. This is bipartite with each vertex in Z_q having degree $(n-q)$, and each vertex in Z_{q+1} degree $(q+1)$. Thus by a simple count of edges, any t vertices in Z_q must have at least $(n-q)t/(q+1)$ neighbours in Z_{q+1}. Thus there is at least this number of type (ii) vertices. Therefore

$$b(k) \geqslant \binom{n}{q} - t + (n-q)t/(q+1), \quad \text{i.e.}$$

$$b(k) \geqslant \binom{n}{q} + (n-2q-1)t/(q+1) \tag{2.4}$$

$$= (n-2q-1)k/(q+1) + \left\{ \binom{n}{q} - \left(\frac{n-2q-1}{q+1} \right) T(q-1) \right\}.$$

Thus we need only show that $T(q-1) < \dfrac{q+1}{n-2q-1}\dbinom{n}{q}$ to complete the proof. This follows since

$$T(q-1) = \sum_{i=1}^{q} \binom{n}{q-i}$$

$$= \binom{n}{q} \sum_{i=1}^{q} \frac{q(q-1)\ldots(q-i+1)}{(n-q+1)\ldots(n-q+i)}$$

$$\leqslant \binom{n}{q} \sum_{i=1}^{q} \left(\frac{q}{n-q+1} \right)^i$$

$$< \binom{n}{q} \frac{q/(n-q+1)}{1-q/(n-q+1)} \quad \text{provided } q < \tfrac{1}{2}(n+1)$$

$$= \binom{n}{q} \frac{q}{n-2q+1}$$

$$< \frac{q+1}{n-2q-1} \binom{n}{q} \quad \text{as required.} \quad \square$$

Note that for $q > \frac{1}{2}(n-1)$ the right hand side of the inequality in Lemma 2.3 is negative, and hence of no use. We will therefore need the following result, which states roughly that sets which are large, but not too large, have many neighbours. Its exact statement conforms with later requirements.

Lemma 2.4. *Let* $K = K(n) = [2^{\frac{1}{4}\sqrt{n \lg n}}]$. *Then, for large* n, *if* $K \leqslant k \leqslant N-K$ *then* $b(k) > 2^{\frac{1}{4}\sqrt{n \lg n}}$.

Proof. First note that $0 \leqslant t \leqslant \binom{n}{q}$ in (2.4) implies

$$b(k) \geqslant \min\left\{\binom{n}{q}, \binom{n}{q+1}\right\}. \tag{2.5}$$

Let $m = \lfloor \frac{1}{2}\sqrt{n/\lg n} \rfloor$. A simple calculation yields

$$\begin{aligned} T(m) &= (1+\mathrm{o}(1))\binom{n}{m} \\ &\leqslant (1+\mathrm{o}(1))(ne/m)^m < K \end{aligned}$$

for large n.

Using $T(m-1)+T(n-m)=N$, we obtain $T(n-m)>N-K$ also. Thus $K \leqslant k \leqslant N-K$ implies $m<q(k)<n-m$. The result now follows from (2.5) and (2.2). □

3. Minimum degree of Γ_n

As one might expect from previous work on the strength of connectivity of random graphs, e.g. Bollobás [1], Erdös and Rényi [4] or Fenner and Frieze [6], the connectivity of Γ_n is essentially determined by its minimum degree.

Let v_t denote the number of vertices of degree t in Γ_n. We have the following result.

Theorem 3.1.

(a) *For* $s \geqslant 1$, $c_n \nrightarrow -\infty$, $\lim_{n\to\infty} \Pr(\sum_{t=0}^{s-1} v_t > 0) = 0$;

(b) *For* $s \geqslant 0$, $c_n \to c$, $\lim_{n\to\infty} \Pr(v_s = i) = \mathrm{e}^{-\lambda}\lambda^i/i!$ $(i \geqslant 0)$;

(c) *For* $c_n \to +\infty$, $\lim_{n\to\infty} \Pr(v_s > 0) = 0$;

(d) *For* $c_n \to -\infty$, $\lim_{n\to\infty} \Pr(\sum_{t=0}^{s} v_t > 0) = 1$;

where λ *is given by* (1.1). *Thus* v_s *is asymptotically Poisson with mean* λ.

Proof.

(a) For $x \in C^n$ and $t<s$, we have that

$$\begin{aligned}
&\Pr(x \in V_n \text{ and degree of } x=t)\\
&=p_v\binom{n}{t}p^t(1-p)^{n-t}\\
&<n^t 2^{-n}\left(1-\frac{s\ln n+2c}{n}\right)^n(1+o(1))\\
&<n^t 2^{-n} n^{-s} e^{-2c}(1+o(1)).
\end{aligned}$$

Thus, for $t<s$, the expected number of vertices of degree t is $O(n^{-1})$, and hence part (a) of the theorem follows.

(b) We use inclusion-exclusion. Let E_x be the event that $x \in C^n$ is a vertex of degree s in Γ_n. For any $S \subseteq C^n$, write $E_S=\bigcap_{x\in S} E_x$ and

$$\theta_t(n)=\sum_{|S|=t}\Pr(E_S).$$

Then the inclusion-exclusion formula gives

$$\Pr(v_s=i)=\sum_{t=i}^{N}(-1)^{t-1}\binom{t}{i}\theta_t(n) \tag{3.1}$$

and the sum on the right hand side of (3.1) alternates in value about the left hand side.

Let $\alpha_t=\lim_{n\to\infty}\theta_t(n)$, then using the alternation property of the sum in (3.1) it may be shown that

$$\lim_{n\to\infty}\Pr(v_s=i)=\sum_{t=i}^{\infty}(-1)^{t-1}\binom{t}{i}\alpha_t \tag{3.2}$$

provided the sum converges. Thus it remains to estimate α_t. We may obviously assume $n>t$. Let S be any set with $|S|=t$. Note that

$$\begin{aligned}
\Pr(E_S)&=\left(p_v\binom{n}{s}p^s(1-p)^{n-s}\right)^t\\
&=\mu_t(n), \quad \text{say},
\end{aligned}$$

unless two vertices in S are either adjacent or within a distance 2 of each other (i.e. they have a common neighbour). We show that the contribution to $\sum_{|S|=t} \Pr(E_S)$ from such sets S is asymptotically negligible. Let this contribution be $\beta_t(n)$. Now there are at most

$$N\left(n+\binom{n}{2}\right)\binom{N-2}{t-2}<n^2N^{t-1}$$

sets S which either have two adjacent vertices or two vertices with a common neighbour. For such sets S we have

$$nt-\tfrac{1}{2}(t-1)(t+2)\leqslant M=|B(S)|\leqslant nt.$$

For each $x \in B(S)$, let y_x be a particular neighbour of x in S. Let $T=\{x \in B(S) : x \in V_n \text{ and } \{x, y_x\} \in A_n\}$. If E_S occurs then $|T|\leqslant ts$. But

$$\begin{aligned}\Pr(|T|\leqslant ts)&\leqslant \sum_{r=0}^{ts}\binom{M}{r}p^r(1-p)^{M-r}\\ &=(1+o(1))\binom{M}{ts}p^{ts}(1-p)^{M-ts}\\ &\leqslant(1+o(1))M^{ts}2^{-M}.\end{aligned}$$

Thus

$$0\leqslant\beta(n)\leqslant(1+o(1))(tn)^{ts}2^{t^2}N^{-t}n^2N^{t-1}.$$

Hence, as $n\to\infty$, $\beta_t(n)\to 0$, as claimed.

It remains to bound the number of sets S which have no common neighbour. There are clearly at most $\binom{N}{t}\leqslant N^t/t!$ such sets, and at least

$$\begin{aligned}&N\left(N-\left(1+n+\binom{n}{2}\right)\right)\ldots\left(N-(t-1)\left(1+n+\binom{n}{2}\right)\right)/t!\\ &>(N-tn^2)/t!\end{aligned}$$

Therefore

$$\frac{(N-tn^2)^t}{t!}\mu_t(n)\leqslant\theta_t(n)-\beta_t(n)\leqslant\frac{N^t}{t!}\mu_t(n).$$

Letting $n\to\infty$, we obtain

$$\lambda^t/t! \leqslant \alpha_t \leqslant \lambda^t/t!$$

Hence the right hand side of (3.2) is

$$\sum_{t=i}^{\infty} (-1)^{t-1}\binom{t}{i}\lambda^t/t! = \lambda^i e^{-\lambda}/i! \tag{3.3}$$

(c) Proceeding as in (a), we find that the expected number of vertices of degree s tends to zero, since in this case $e^{-2c_n}\to 0$.

(d) In this case, the expected number of vertices of degree at most s tends to infinity. A routine use of the Chebyshev inequality yields the result. One, of course, must compute the variance, but this is a simple exercise whose details are left to the reader. □

Corollary 3.2. *If* $c_n\to c$, *then*

$$\lim_{n\to\infty} \Pr(\delta(\Gamma_n)=s)=1-e^{-\lambda} \quad \textit{and}$$

$$\lim_{n\to\infty} \Pr(\delta(\Gamma_n)=s+1)=e^{-\lambda}$$

where δ *denotes the minimum degree of the graph, as usual.*

For $c_n\to+\infty$, *we have*

$$\lim_{n\to\infty} \Pr(\delta(\Gamma_n)\leqslant s)=0,$$

and for $c_n\to-\infty$,

$$\lim_{n\to\infty} \Pr(\delta(\Gamma_n)\leqslant s)=1.$$

Corollary 3.3. *If* $p_v p_e = 1 + c_n/n$ *where* $c_n\to-\infty$, *then*

$$\lim_{n\to\infty} \Pr(\Gamma_n \text{ is connected})=0.$$

4. Threshold for connectivity

We start our proof of Theorem 1.1 with the case $s=0$. In this case we have to establish the limiting probability that Γ_n is connected. In view of Corollary 3.3 we may assume that $c_n\not\to-\infty$. We shall only treat the case $c_n\to c$ in detail. All the calculations go through *a fortiori* for $c_n\to\infty$.

Since c_n plays only a minor role in the subsequent analysis, we shall assume for convenience that $c_n = c$.

Let

$$\Pi(n, k) = \Pr\,(\Gamma_n \text{ has a component of size } k)$$

and

$$\Pi(n, k_1, k_2) = \Pr\,(\Gamma_n \text{ has a component of size } k,\ k_1 \leqslant k \leqslant k_2).$$

Clearly

$$\Pi(n, k_1, k_2) \leqslant \sum_{k=k_1}^{k_2} \Pi(n, k).$$

Corollary 3.2 can be re-expressed here as

$$\lim_{n\to\infty} \Pi(n, 1) = 1 - e^{-\lambda} \tag{4.1}$$

where $\lambda = \bar{p}_v e^{-2c}$.

We can therefore prove our theorem for the case $s=0$ by showing

$$\lim_{n\to\infty} \Pi(n, 2, \tfrac{1}{2}N) = 0. \tag{4.2}$$

5. Small components

We show here that the probability that Γ_n has a component of size k, $2 \leqslant k \leqslant 2^{\frac{1}{4}\sqrt{n \lg n}}$, is very small. Our estimates of $\Pi(n, k)$ in this section are based on the bound

$$\Pi(n, k) \leqslant H_k(n)\, p_v^k (1-p)^{b(k)}. \tag{5.1}$$

The right hand side of (5.1) is an upper bound to the expected number of components of size k in Γ_n. To see this, let $S \subseteq C^n$ and $|S| = k$. For each $y \in B(S)$, choose an edge $e(y)$ from y to a vertex in S. If S is a component of Γ_n, then $V_n \supseteq S$ and $e(y) \notin A_n$ for $y \in B(S)$. Thus

$$\Pr\,(S \text{ is a component of } \Gamma_n) \leqslant p_v^{|S|}(1-p)^{|B(S)|}$$

and (5.1) follows.

We shall actually use a weakening of (5.1) to

$$\Pi(n,k) \leqslant H_k(n) 2^{-b(k)}\left(1-\frac{2c}{n}\right)^{b(k)}$$

$$\leqslant H_k(n) 2^{-b(k)} e^{2k|c|}. \quad \text{(For } c_n \to \infty, \tag{5.2}$$

we would obviously omit the modulus).

Lemma 5.1. *For $2 \leqslant k \leqslant n^2/2$ and large enough n, $\Pi(n,k) \leqslant 2^{-\frac{1}{4}kn}$.*

Proof.

(i) First consider $k=2$, i.e. an isolated edge. Now C^n has only $n2^{n-1}$ edges altogether. Thus, using (5.2) and Lemma 2.2, we obtain

$$\Pi(n,2) \leqslant n 2^{n-1} 2^{-2(n-1)} e^{4|c|} \leqslant 2^{-\frac{1}{2}n}$$

for large n.

(ii) $3 \leqslant k \leqslant \frac{1}{2}n$.

$$H_k(n) \leqslant 2^n n^{k-1}(k-1)! \leqslant 2^{n+2k\lg n}. \tag{5.3}$$

Thus, using (5.2) and Lemma 2.2,

$$\Pi(n,k) \leqslant 2^{n+2k\lg n-(kn-\frac{1}{2}(k-1)(k+2))} e^{2k|c|}$$

$$\leqslant 2^{-(1-o(1))kn} \text{ for } k \geqslant 3.$$

(iii) $\frac{1}{2}n \leqslant k \leqslant \frac{1}{2}n^2$.

In this range of k, $q(k) \leqslant 2$ and so $b(k) \geqslant k(n-5)/3$, using Lemma 2.3. Using (5.2) and (5.3) we obtain

$$\Pi(n,k) \leqslant 2^{n+2k\lg n-k(n-5)/3} e^{2k|c|}$$

$$\leqslant 2^{-\frac{1}{4}kn} \text{ for large enough } n. \quad \square$$

Lemma 5.2. *For $\frac{1}{2}n^2 \leqslant k \leqslant K(n) = [2^{\frac{1}{3}\sqrt{n\lg n}}]$ and large n,*

$$\Pi(n,k) \leqslant 2^{-k\sqrt{n\lg n}/3}.$$

Proof. Now $H_k(n) \leqslant (2kn)^k$ and $b(k) > k(n-2q-1)/(q+1)$ by Lemma 2.3, where $q = q(k)$. Thus, using (5.2),

$$\Pi(n,k) \leqslant (2kn/2^{(n-2q-1)/(q+1)})^k e^{2k|c|}. \tag{5.4}$$

Let $m=\lfloor\sqrt{n/\lg n}\rfloor$. It is easy to see from (2.2) that $T(m-1)>\binom{n}{m-1}>K$. Thus, within our range of $k, q(k)\leqslant m$. Consequently, for large n, $(n-2q-1)/(q+1)>9\sqrt{n\lg n}/10$. Hence, from (5.4),

$$\Pi(n,k)\leqslant(2n2^{\frac{1}{4}\sqrt{n\lg n}}/2^{9\sqrt{n\lg n}/10})^k e^{2k|c|}$$

$$\leqslant 2^{-k\sqrt{n\lg n}/3} \text{ for large } n. \quad \square$$

Corollary 5.3.

$$\Pi(n,2,2^{\frac{1}{4}\sqrt{n\lg n}})\leqslant 2^{-n/3} \quad \textit{and}$$

$$\Pi(n,n,2^{\frac{1}{4}\sqrt{n\lg n}})\leqslant 2^{-n^2/5},$$

in both cases for large enough n. $\square$

We conclude this section by showing that there are few vertices in "very small" components. We prove only a fairly crude bound.

Lemma 5.4. *The probability that Γ_n has a total of more than n^5 vertices in components of size at most n is less than $2^{-n^2/5}$, for large enough n.*

Proof. The condition of the Lemma implies that for some $1\leqslant k\leqslant n$, there must be at least n^4 vertices in components of size k. Call two components S, S' of size k *independent* if $B(S)\cap B(S')=\emptyset$. Consider an associated graph with vertices S, and an edge SS' whenever S and S' are not independent. Now, for any vertex S, $|B(S)|\leqslant kn$ and each vertex in $B(S)$ is adjacent to at most $(n-1)$ other components S'. Hence the associated graph has degree bound $k\binom{n}{2}$. It follows easily that it possesses an independent set of size $r=(n^4/k)/(kn(n-1)+1)$, since it must have at least n^4/k vertices. Thus $r\geqslant n^2/k^2$. Consider first $k=1$. The probability that Γ_n contains n^2 independent isolated vertices is no more than $\binom{N}{n^2}(1-p)^{n^3}=o(2^{-n^2})$. For $k\geqslant 2$, following the lines of the proof of Lemma 5.1, the probability of at least r independent components of size k is less than $(H_k(n)2^{-b(k)}e^{2k|c|})^r\leqslant 2^{-n^2/4}$. Thus the probability that for any $1\leqslant k\leqslant n$, there are at least n^4 vertices in components of this size is less than $n2^{-n^2/4}<2^{-n^2/5}$ for large enough n. $\square$

Remark 5.5. We say that $x \in C^n$ is an *isolated non-vertex* if $(\{x\} \cup B(\{x\})) \cap V_n = \emptyset$. Arguing as in Lemma 5.4, we may show that

$$\Pr(C^n \text{ has at least } n^4 \text{ isolated non-vertices}) = o(2^{-n^2}).$$

6. Large components

We show that there are no disconnected large components by "bootstrapping" the results of §5. The approach is to dissect C^{n+1} into two copies of C^n in an obvious way. We require the following result, which follows immediately from results of Hoeffding [7].

Lemma 6.1. *For large n,*

$$\Pr((21\bar{p}_v - 2)N/20 \leqslant |V_n| \leqslant (1+\bar{p}_v)N/2)$$
$$\geqslant 1 - e^{-2^{\frac{1}{2}n}}.$$

Proof. From Hoeffding's results, noting that $|V_n|$ is a sum of N independent zero-one random variables, each with expectation p_v.

$$\Pr(|V_n| \leqslant (1-\varepsilon)Np_v) \leqslant e^{-\frac{1}{2}\varepsilon^2 Np_v} \tag{6.1}$$

$$\Pr(|V_n| \geqslant (1+\varepsilon)Np_v) \leqslant e^{-\frac{1}{3}\varepsilon^2 Np_v} \tag{6.2}$$

for any ε, $0 \leqslant \varepsilon \leqslant 1$.

Putting $\varepsilon = 1/n$ and using some obvious estimates in (6.1) and (6.2) gives the conclusion. □

In view of Lemma 6.1, it is sufficient to show that, with high probability, Γ_n possesses no connected component of size at most $(1+\bar{p}_v)N/4$, i.e. that

$$\lim_{n\to\infty} \Pi(n, 2, (1+\bar{p}_v)N/4) = 0.$$

Definition. For any $0 \leqslant f \leqslant n$, an f-face of C^n is a maximal subset for which the values of $(n-f)$ coordinates are constant.

We now consider C^{n+1} in order to derive a relationship between $\Pi(n+1, n+1, 4k)$ and $\Pi(n, n, k)$ for large (but not too large) values of k.

We will denote the n-face of C^{n+1} on which $x_j = i$ $(i = 0, 1)$ by C_{ij}^{n+1}. Let $X_{ij} = V_{n+1} \cap C_{ij}^{n+1}$. Then we observe that X_{ij} is a random subset of the n-cube

C_{ij}^{n+1}, and that for $i=0, 1$ these are independent of each other. This simple observation provides the main tool.

Consider a fixed value of j, and for convenience drop this subscript. We will say a subset S_0 of X_0 is *incident* with a subset S_1 of X_1 if there exist $x_0 \in X_0$, $x_1 \in X_1$ with $\{x_0, x_1\} \in A_{n+1}$. Clearly if S_0, S_1 are incident they lie in the same connected component of Γ_{n+1}.

We define a component of X_{ij} to be a component of the subgraph of Γ_{n+1} induced by X_{ij}. Let us call such a component of any X_{ij} *trivial* if it has at most n vertices, and *large* if it has at least $K=\lceil 2^{\frac{1}{2}\sqrt{n \lg n}}\rceil$ vertices.

Lemma 6.2. *The probability that, for any i, j there exists a large component in X_{ij} which is not incident with any non-trivial component of $X_{1-i,j}$ is less than $2^{-n^2/6}$.*

Proof. We condition on X_{ij}, i.e. we consider it fixed and use the inequality

$$\begin{aligned}\Pr(A) &= \sum_X \Pr(A|X_{ij}=X)\Pr(X_{ij}=X)\\ &\leqslant \max_X \Pr(A|X_{ij}=X)\end{aligned}$$

for any event A. Again we will drop the suffix j temporarily and assume without loss that $i=0$. Suppose that X_0 has a component S of size at least K. Since X_1 is independent of X_0, it is unconditioned. If S is incident with s vertices of X_1, then it follows that

$$\Pr(s \leqslant n^5) \leqslant \binom{K}{n^5}(1-p)^{K-n^5}$$

which, substituting for K and noting that $\binom{K}{n^5} \leqslant K^{n^5}$, clearly implies that $s > n^5$ with probability $1 - o(2^{-\frac{1}{2}K})$, say. However, from Lemma 5.4, with probability at least $1-2^{-n^2/5}$, this implies that S is incident with a non-trivial component of X_1. Thus the probability that the Lemma fails is at most

$$2^{-n^2/5} + o(2^{-\frac{1}{2}K}) \text{ for this } S, i, j. \tag{6.3}$$

There are obviously less than 2^n large components in X_0, and C^{n+1} has only $2(n+1)$ n-faces. Thus (6.3) needs to be inflated by a factor of less than $2^n \times 2(n+1)$. For large enough n this gives the conclusion. □

Now, if S_0, S'_0 are both components of X_0, we will say S_0 is *bridged* to S'_0 if they are both incident to a connected component S_1 of X_1, for any such S_1. Clearly if S_0, S'_0 are bridged, they lie in the same component of Γ_{n+1}.

Lemma 6.3. *The probability that any large component (of any X_{ij}) with less than $(1+\bar{p}_v)N/4$ vertices is not bridged to a distinct non-trivial component is less than $2^{-n^2/6}$ for large n.*

Proof. Let S be any such component of X_0, as in Lemma 6.2. Then $K<|S|<(1+\bar{p}_v)N/4$. Thus, from Lemma 6.1, with probability at least $1-e^{-2^{\frac{1}{2}n}}$,

$$|X_0-S|>(21\bar{p}_v-2)N/20-(1+\bar{p}_v)N/4\geq N/20.$$

Hence $|S\cup B_0(S)|\leq N-|X_0-S|<N-N/20<N-K$, where $B_0(S)$ is the set of neighbours of S in $C_{0j_t}^{n+1}$. Thus, from Lemma 2.4,

$$|B_0^2(S)|=|B_0(S\cup B_0(S))|>2^{\frac{1}{2}\sqrt{n\lg n}}=K_1\,,\ \text{say}.$$

Now every vertex of $B_0^2(S)$ is clearly either

(i) a vertex of some other component of X_0, or

(ii) adjacent to a vertex of some other component of X_0, or

(iii) an isolated non-vertex.

However, from Remark 5.5, with probability at least $1-2^{-n^2}$, there are less than n^4 vertices of type (iii). Thus $B_0^2(S)$ has at least K_1-n^4 vertices of types (i) and (ii). Now, since there are only $1+n+\binom{n}{2}<n^2$ vertices within a distance at most 2 from any given vertex, there are at least $(K_1-n^4)/n^2$ vertex-disjoint paths of length 2 edges (or 3 vertices) to vertices of types (i) or (ii) in $B_0^2(S)$ from S. Since a vertex of another component can be reached in at most one further step from the terminal vertex of each such path, and each vertex has degree n in C^n, there are thus at least $(K_1-n^4)/n^3=K_2$ vertex-disjoint paths of length at most 4 vertices into a distinct component of X_0. Now the corresponding path exist in X_1 with probability at least $p_v^4p_e^3>1/20$, say, for large n. Thus the probability that no more than n^5 of these paths exist in X_1 is at most

$$\binom{K_2}{n^5}(19/20)^{K_2-n^5}=o(2^{-n^3}).$$

Thus, using Lemma 5.4, the probability that the Lemma fails for S is at most

$$e^{-2^{\frac{1}{5}n}}+2^{-n^2}+o(2^{-n^3})+2^{-n^2/5}. \tag{6.4}$$

Again, as in Lemma 6.2, (6.4) needs to be inflated by a factor of at most $(n+1)2^{n+1}$ to give the conclusion. □

Before proving the main result of this section, we need one further Lemma.

Lemma 6.4. *Any component of Γ_{n+1} with at most $n+1$ vertices is contained entirely in some C_{ij}^{n+1}.*

Proof. We prove a stronger statement which clearly implies the Lemma.

Claim. *Any connected subgraph Y of Γ_{n+1} with at most $k \leqslant n+1$ vertices is contained entirely in some $(k-1)$-face of C^{n+1}.*

Proof of Claim. By induction on k. It is obvious for $k=1$. For $k>1$, fix any connected subgraph Y' of Y with $(k-1)$ vertices. By hypothesis this lies entirely in some $(k-2)$-face F. All vertices of F agree on $(n-k+3)$ "constant" coordinates. We now add the excluded vertex, which is connected by an edge to Y', and hence is either in F or is joined to it by an edge of C_{n+1}. Thus its coordinates agree with those of F on all but at most one of the constant coordinates. Thus the coordinates of all vertices in Y agree on at least $(n-k+2)$ coordinates, which gives the result. □

We now prove the main result of this section. In order to avoid confusing the flow of the argument with too many probability statements, we will write σ_1 implies $(\varepsilon)\,\sigma_2$ for two events σ_1, σ_2 if the implication holds on an event of probability at least $1-\varepsilon$. The whole argument then holds on an event of probability at least $(1-\Sigma\varepsilon)$, where the sum is over the values of ε in all such implications used. We will have to amend our notation slightly so that

$$\Pi(n, k_1, k_2, p)$$

$$=\Pr(\Gamma_n \text{ contains a connected component of size } k,$$

$$k_1 \leqslant k \leqslant k_2, \text{ assuming } p_v p_e = p).$$

The need for this amendment comes from

Lemma 6.5. *If* $K(n)\leqslant k\leqslant(1+\bar{p}_v)N/8$, *then*

$$\Pi(n+1,n+1,4k,\tfrac{1}{2}+c/(n+1))$$
$$\leqslant 2(n+1)\Pi(n,n,k,\tfrac{1}{2}+c/(n+1))+2^{-n^2/7}.$$

Proof. Suppose Γ_{n+1} has a component S of size $n+1\leqslant|S|\leqslant 4k$. Choose now any sub-component (i.e. connected subset) of S with $n+1$ vertices, then by Lemma 6.4 there exists i, j such that C_{ij}^{n+1} contains a connected sub-component of S with at least $n+1$ vertices. We will assume that $i=0$. We wish to show that, with high probability, either X_0 or X_1 contains a sub-component of S with size in the range n to k inclusive. Let us call such a component of X_0 or X_1 a *bad* component. Let $S_0=S\cap X_0$. We have $(n+1)\leqslant|S_0|$. If $|S_0|\leqslant k$, then S_0 is a bad component. Therefore assume $|S_0|>k$. If $|S_0|\leqslant 2k\leqslant(1+\bar{p}_v)N/4$, then by Lemma 6.3 this implies $(2^{-n^2/6})$ that S_0 is bridged to a non-trivial component $S_0'\subseteq S$ of X_0. Thus $|S_0'|\geqslant n+1$. If $|S_0'|\leqslant k$, then it is a bad component. Otherwise $|S_0'|>k$, so $|S_0|+|S_0'|>2k$. Thus $|X_1\cap S|<2k$. However, Lemma 6.2 implies $(2^{-n^2/6})$ that S_0 is incident with a non-trivial component $S_1\subseteq S$ of X_1. Thus $|S_1|\geqslant(n+1)$. If $|S_1|\leqslant k$, then it is a bad component. Otherwise $|S_1|>k$, and Lemma 6.3 implies $(2^{-n^2/6})$ that it is bridged to a component $S_1'\subseteq S$ with $|S_1'|\geqslant(n+1)$. Clearly $|S_1'|<2k-|S_1|<k$, and hence S_1' is a bad component.

Otherwise, we must assume $|S_0|>2k$, then this again implies $(2^{-n^2/6})$ that either S_1 or S_1' is a bad component.

The above proof clearly holds on an event of probability at least $1-5.2^{-n^2/6}\geqslant 1-2^{-n^2/7}$ for large n. On this event the existence of S implies the existence of a bad component in some C_{ij}^{n+1}, which has probability at most $2(n+1)\Pi(n,n,k,\tfrac{1}{2}+c/(n+1))$, since there are $2(n+1)$ such n-faces. □

We can iterate the formula of Lemma 6.5. There is a technical point to check, that for large n, $k\geqslant K(n)=2^{\frac{1}{2}\sqrt{n\lg n}}$ implies $4k\geqslant K(n+1)$. This follows easily from the fact that

$$\sqrt{(n+1)\lg(n+1)}-\sqrt{n\lg n}\to 0 \text{ as } n\to\infty.$$

We only iterate as long as $4k\leqslant(1+\bar{p}_v)2^{n-1}$, which is the maximum component size we wish to consider in C^{n+1}.

Thus, iterating r times (provided r is not too large), it follows that, using fairly crude estimates,

$$\Pi(n+r, n+r, 4^r k, c/(n+r)) \leqslant 2^r(n+r)^r \Pi(n, n, k, c/(n+r)) + 2^{-n^2/8}. \tag{6.5}$$

Putting $k=K(n)$ in (6.5) and noting from Corollary 5.3 that for large n we have $\Pi(n, n, K(n), c/(n+r)) \leqslant 2^{-n^2/5}$, it follows that $\Pi(n, n, n, c/(n+r)) + 2^{-n^2/8} < 2^{-n^2/9}$, say. Also we may assume $r<n$, since we must have $4^r K(n) \leqslant 2^{n+r} = |C^{n+r}|$.

Thus the right side of (6.5) is bounded by $(4n)^n 2^{-n^2/9} < 2^{-n^2/10}$, say. Therefore we have

$$\Pi(n+r, n+r, 4^r K(n), \tfrac{1}{2}+c/(n+r)) < 2^{-n^2/10}. \tag{6.6}$$

Now put $r=r(n)=\lfloor n-2+\lg(1+\bar{p}_v)-\lg K(n)\rfloor$ to show that

$$\Pi(n+r, n+r, (1+\bar{p}_v)2^{n+r-3}, \tfrac{1}{2}+c/(n+r)) < 2^{-n^2/10}. \tag{6.7}$$

We would like to replace $(n+r)$ by m in (6.7) to obtain an inequality which is valid for all large m. We are hampered by the fact that, for some m, there may be no n such that $m=n+r(n)$. Therefore let

$$\varphi(m) = \max\{n+r(n) : n+r(n) \leqslant m\}.$$

Since $(n+1)+r(n+1)-(n+r(n)) \leqslant 2$, we deduce that

$$\varphi(m) = m - \alpha(m) \text{ where } \alpha(m) = 0 \text{ or } 1.$$

Define $\psi(m)$ by $\varphi(m)=\psi(m)+r(\psi(m))$, and note that $\psi(m) \geqslant \varphi(m)/2$. We thus have from (6.7) that

$$\Pi(\varphi(m), \varphi(m), (1+\bar{p}_v)2^{\varphi(m)-3\alpha(m)}, \tfrac{1}{2}+c/\varphi(m)) \leqslant 2^{-m^2/50}$$

for large m.

Applying Lemma 6.5 $\alpha(m)+1$ times yields

$$\Pi(m+1, m+1, (1+\bar{p}_v)2^{m-1}, \tfrac{1}{2}+c/(m+1)) \leqslant 2^{-m^2/60}$$

for large m.

Finally, putting $n=m+1$ and dropping the fourth parameter

$$\Pi(n, n, (1+\bar{p}_v)2^{n-2}) \leqslant 2^{-n^2/70} \text{ for large } n.$$

This, combined with Corollary 5.3 and Lemma 6.1, gives the theorem for the case $s=0$. □

7. General case: $s>0$

We shall use an induction on s based on the partition of C^{n+1} into two copies of C^n, somewhat similarly to §6. We first note, however, that

Lemma 7.1. *If* $s \geqslant 1$, *then*

$$\Pr(\Gamma_n \text{ has a vertex of degree at most } s-1) = O(1/n) \quad (7.1a)$$

$$\Pr(\Gamma_n \text{ is not connected}) = O(1/n). \quad (7.1b)$$

Proof. The calculation in Theorem 3.1(a) gives (7.1a) – see the paragraph following (3.1).

$$\Pi(n, 2, N/2) \leqslant 2^{-n/3} + 2^{-n^2/70}$$

follows from the calculations done for the case $s=0$, and so (7.1b) follows also. □

Definition. A set S is a *proper disconnector* (PD) of a connected graph G if the subgraph of G induced by $V(G)-S$ is (a) not connected, and (b) contains no isolated vertices.

Let now $\Delta(n, s) = \Pr(\Gamma_n$ has a PD S with $|S|=s)$. In view of Theorem 3.1(b) we have only to prove

$$\lim_{n\to\infty} \Delta(n, s) = 0 \quad (7.2)$$

in order to prove our theorem.

For a set $S \subseteq V_n$, let Γ_n/S denote the subgraph of Γ_n induced by $V_n - S$.

Lemma 7.2.

Pr (There exists a PD S, $|S|=s$, of Γ_n such that Γ_n/S has a component Z, $2 \leqslant |Z| \leqslant K(n)$) $\leqslant 2^{-\frac{1}{4}n}$.

Proof. If we put

$$\alpha(s,k)=\Pr(\text{There exists a pair } S, Z \text{ as above, } |Z|=k)$$

$$\leqslant H_k(n)\binom{N}{s}(1-p)^{b(k)-s}$$

$$\leqslant H_k(n)2^{-b(k)+s(n+1)}e^{2k|c|}. \tag{7.3}$$

The right hand side of (7.3) bounds the expected number of such pairs – $H_k(n)$ counts the number of Z's, $\binom{N}{s}$ bounds the number of S's and $(1-p)^{b(k)-s}$ bounds the probability that Z is a component of Γ_n/S.

For $k\geqslant s+2$, we proceed exactly as in Lemmas 5.1 and 5.2 to estimate $H_k(n)2^{-b(k)}$. This yields

$$\alpha(s,k)=O(2^{-2n/3}) \text{ for } k\geqslant s+2.$$

For $k\leqslant s+1$, we see that if Z, S exist, then Z contains a pair of adjacent vertices x, y for which

$$|N(\{x,y\})|\leqslant s+k-2<2s$$

where for $T\subseteq V_n$, $N(T)=\{v\in V_n-T : v \text{ is adjacent in } C^n \text{ to some } w\in T\}$.

But

$$\Pr(\text{There exist adjacent } x, y \text{ such that } |N(\{x,y\})|<2s)$$

$$\leqslant Nn\binom{2n-2}{2s-1}(1-p)^{2n-2s}$$

$$=O(n^{2s}2^{-n}).$$

We shall now use the relationship between C^{n+1} and C^n similarly to the proof of Lemma 6.4. Let now

$$\Delta(n,s,p)$$

$$=\Pr(\Gamma_n \text{ has a PD } S \text{ with } |S|=s, \text{ and } p_e p_v=p).$$

Lemma 7.3.

$$\Delta(n+1, s, p) \leqslant \sum_{t=1}^{s-1} \Delta(n, t, p) + O(1/n) \tag{7.4}$$

where $p = \frac{1}{2} + (\frac{1}{2}s \ln n + c)/n$ *and* $s \geqslant 1$.

Proof. We shall, somewhat loosely, refer to the subgraph induced by a subset Y of V_n by Y itself. This should not lead to any confusion,. given the context.

Let $X_i = X_{i1}$ for $i=0$ or 1 as in §6.

Let D denote the event that Γ_{n+1} contains a PD S with $|S| = s$. We note that

$$D \subseteq E_\alpha \cup E_{\beta 0} \cup E_{\beta 1} \cup E_\gamma \cup E_\delta \tag{7.5}$$

where

E_α is the event that X_i has minimum degree at most $s-1$ for some $i=0$ or 1.

$E_{\beta i}$ is the event: $S \subseteq X_i$ but not E_α, for $i=0, 1$, and D occurs.

E_γ is the event $S_i = S \cap X_i$ is a PD of X_i, $0 < |S_i| < s$, for some $i=0$ or 1, then the event D occurs but not E_α or $E_{\beta 0}$ or $E_{\beta 1}$. (Note that $|S_i| < s$ because of $\bar{E}_{\beta 0}$ and $\bar{E}_{\beta 1}$. Because of $\bar{E}_\alpha$, $S_i \neq N(x, i) = \{y \in X_i : \{x, y\} \in A_n\}$.)

E_δ is the event that D occurs and $X_i - S$ is connected for $i=0$, 1 and there are no more than s vertices $x \in X_0$ such that $\nu(x) \in X_1$ and $\{x, \nu(x)\} \in A_{n+1}$. Here $\nu(x)$ is obtained from x by changing its first coordinate.

We note first that Lemma 3.1(a) shows

$$\Pr(E_\alpha) = O(1/n) \tag{7.6}$$

and that

$$\Pr(E_\gamma) \leqslant 2 \sum_{t=1}^{s-1} \Delta(n, t, p), \quad (E_\gamma = \emptyset \text{ for } s=1). \tag{7.7}$$

Furthermore

$$\Pr(E_\delta) \leqslant \Pr(|\{x \in X_0 : \nu(x) \in X_1, \{x, \nu(x)\} \in A_{n+1}\}| \leqslant s)$$

$$\leqslant e^{-2^{\frac{1}{2}n}} + \binom{N}{s}(1-p)^{(21\bar{p}_\nu - 2)N/20 - s}, \tag{7.8}$$

using Lemma 6.1 and the fact that, given X_0, X_1, the edges joining X_0, X_1 are unconditioned.

Let us now consider the event $E_{\beta 0}$ (and hence by symmetry $E_{\beta 1}$). We have

$$E_{\beta 0} \subseteq E_1^* \cup E_2^* \cup E_3^* \cup E_4^* \tag{7.9}$$

where

E_1^* is the event that X_1 is not connected.

E_2^* is the event that $E_{\beta 0}$ occurs, X_1 is connected and $S = N(x, 0)$ and $\{x, \nu(x)\} \in A_{n+1}$, some $x \in X_0$. (If $S = N(x, 0)$ and $\{x, \nu(x)\} \notin A_{n+1}$, then $S = N(\{x\})$ and hence S is not a PD.)

E_3^* is the event that $E_{\beta 0}$ occurs, X_1 is connected and $X_0 - S$ contains a component of size k, $2 \leqslant k \leqslant K(n)$.

E_4^* is the event that $E_{\beta 0}$ occurs, X_1 is connected and $X_0 - S$ contains a component T of size $k > K(n)$ such that $\{x, \nu(x)\} \notin A_{n+1}$ for all $x \in T$.

Now Lemma 7.1 gives

$$\Pr(E_1^*) = O(1/n). \tag{7.10}$$

Lemma 7.2 gives

$$\Pr(E_3^*) \leqslant 2^{-\frac{1}{4}n} \tag{7.11}$$

and clearly

$$\Pr(E_4^*) \leqslant (N/K)(1-p)^K \leqslant 2^{-2^{\sqrt{n \lg n}/3}} \tag{7.12}$$

for n large.

Let us now consider the event E_2^*. We note that

$$E_2^* \subseteq E_3^* \cup E_4^* \cup E_5^* \tag{7.13}$$

where

E_5^* is the event $\bar{E}_\alpha$, D and $X_0 - S$ contains an isolated vertex x' different from the x defining E_2^*.

Now $|B(\{x\}) \cap B(\{x'\})| \leqslant 2$ for any $x, x' \in C^n$, and thus, given $\bar{E}_\alpha$, we have

$$|N(x, 0) \cup N(x', 0)| \geqslant 2s - 2 \text{ for } x, x' \in X_0.$$

Thus

$$E_5^* = \emptyset \text{ for all } s \geqslant 3\,. \tag{7.14}$$

We have now only to consider

Case 1: $s=1$

Now E_5^* implies that X_0 contains two vertices x, x' of degree 1 with $d(x, x')=2$. Thus

$$\Pr(E_5^*) \leqslant Nn^4(1-p)^{2n-4} < 2^{-\frac{1}{2}n} \text{ for large } n\,. \tag{7.15}$$

Case 2: $s=2$

In this case, E_5^* implies E_b^*, the event that X_0 contains 2 vertices x, x' of degree 2 with $d(x, x')=2$.

Now,

$$\Pr(E_b^*) \leqslant Nn^6(1-p)^{2n-6} < 2^{-\frac{1}{2}n} \tag{7.16}$$

for n large. The Lemma now follows from (7.5) to (7.16). □

Theorem 1.1 now follows easily from Lemma 7.3, the case $s=0$ and Theorem 3.1.

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