Rainbow Thresholds

Alan Frieze*
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh PA 15213

Trent G. Marbach
Department of Mathematics
Ryerson University
Toronto ON M5B 2K3

October 14, 2021

Abstract

We extend a recent breakthrough result relating expectation thresholds and actual thresholds to include rainbow versions.

1 Introduction

It has been observed for a long time that the threshold for the existence of various combinatorial objects in random graphs and hypergraphs occurs close to where the expected number of such objects tends to infinity. This informal observation has been given rigorous validation in two recent breakthrough papers. First of all, Frankston, Kahn, Narayanan and Park [14] showed that under fairly general circumstances, the threshold for the existence of combinatorial objects is within a factor $O(\log n)$ of the point where the expected number begins to take off. In a follow up paper, Kahn, Narayanan and Park [13] tightened their analysis for the case of the square of a Hamilton cycle and solved the existence problem up to a constant factor. A remarkable achievement, given the complexity of proofs of earlier weaker results. A key notion in this analysis is that of *spread*, see (1), first used in the paper of Alweiss, Lovett, Wu and Zhang [1] that made significant progress in the resolution of the *Sunflower Conjecture* of Erdős.

Spiro [17] describes a refinement of the notion of spread. Espuny Diaz and Person [5] generalised the approach of [13] to handle some questions on spanning sructures from Frieze [8].

There has been considerable research on random graphs where the edges have been randomly colored. Most notably several authors have considered the existence of rainbow colored combinatorial objects. A set of colored edges will be called *rainbow* if each edge has a different color. Improving on earlier results of Cooper and Frieze [3] and Frieze and Loh [10], Ferber and Krivelevich [6] showed that w.h.p. at the threshold for Hamiltonicity, randomly coloring the edges of $G_{n,p}$ with $n+o(n)$ colors yields a rainbow Hamilton cycle. Our aim in this short paper is to show that the proof in [13] can be modified to incorporate rainbow questions. We begin by summarising the results of the papers [13] and [14].

*Research supported in part by NSF grant DMS1952285

A hypergraph $\mathcal{H}$ (thought of as a set of edges) is r-bounded if $e \in \mathcal{H}$ implies that $|e| \leq r$. The most important notion comes next. For a set $S \subseteq X = V(\mathcal{H})$ we let $\langle S \rangle = \{T : S \subseteq T \subseteq X\}$ denote the subsets of X that contain S. Let $\langle \mathcal{H} \rangle = \bigcup_{H \in \mathcal{H}} \langle H \rangle$ be the collection of subsets of X that contain an edge of $\mathcal{H}$. We say that $\mathcal{H}$ is κ-spread if we have the following bound on the number of edges of $\mathcal{H}$ that contain a particular set S: there is an absolute constant A such that

$$|\mathcal{H} \cap \langle S \rangle| \leq \frac{|\mathcal{H}|}{\kappa^{|S|}}, \quad \forall S \subseteq X. \tag{1}$$

Let X_m denote a random m-subset of X and X_p denote a subset of X where each $x \in X$ is included independently in X_p with probability p. The following theorem is from [14]:

Theorem 1. *Let $\mathcal{H}$ be an r-bounded, κ-spread hypergraph and let $X = V(\mathcal{H})$. There is an absolute constant $C > 0$ such that if*

$$m \geq \frac{(C \log r)|X|}{\kappa} \tag{2}$$

then w.h.p. X_m contains an edge of $\mathcal{H}$. Here w.h.p. assumes that $r \to \infty$.

Remark 1. *Let $p = 1/\kappa$ and Z denote the number of edges of $\mathcal{H}$ that are contained in X_p. Then, assuming that $\mathcal{H}$ is r-uniform, we have from (1) with $H \in \mathcal{H}$ that $|\mathcal{H}| \geq \kappa^r$ and then $\mathbb{E}(Z) = |\mathcal{H}|p^r \geq 1$. This gives the connection between spread and the expected value of Z. Theorem 1 inflates p by a factor of order $\log r$.*

Suppose now that the elements of X are randomly colored from a set $Q = [q]$. A set $S \subseteq X$ is rainbow colored if no two elements of S have the same color. We modify the proof of Theorem 1 to prove

Theorem 2. *Let $\mathcal{H}$ be an r-bounded, κ-spread hypergraph and let $X = V(\mathcal{H})$ be randomly colored from $Q = [q]$ where $q \geq r$. Suppose that (i) $r \leq |X|/2$, (ii) $r \leq C_1\kappa$ where C_1 is an absolute constant and (iii) $\kappa \gg \log |X|$[1]. Then there is an absolute constant $C > 0$ such that if*

$$m \geq \frac{(C \log r)|X|}{\kappa} \tag{3}$$

then w.h.p. X_m contains a rainbow colored edge of $\mathcal{H}$. Here w.h.p. assumes that $r \to \infty$.

(The additional constraints (i), (ii) are needed for our proof. We will assume that $\kappa \gg \log r$. If $\kappa = O(\log r)$ then $m \geq |X|$ for sufficiently large C_1.)

Applications: Taking $X = \binom{[n]}{2}$ (the edges of K_n), this shows for example that if $q = n$ and $m = Kn \log n$ then w.h.p. a randomly edge colored copy of $G_{n,m}$ contains a rainbow Hamilton cycle, see Bal and Frieze [2]. Here $\mathcal{H}$ is the n-uniform hypergraph with $(n-1)!/2$ edges, one for each Hamilton cycle of K_n, $r = n$ and $\kappa \geq n/e$.

Dudek, English and Frieze [4] studied rainbow Hamilton cycles in random hypergraphs. Theorem 2 strengthens Theorem 6 of that paper in that [4] requires $m \geq \omega n \log n$ where $\omega \to \infty$, whereas $m = C_1 n \log n$ follows from Theorem 2. Here $X = \binom{[n]}{k}$ and $\mathcal{H}$ is the $n/(k-1)$-uniform hypergraph with $\frac{(k-1)n!}{2n(k-2)!^{n/(k-1)}}$ edges, one for each loose Hamilton cycle. We can take $\kappa = \Omega(n^{k-1})$.

[1] We mean by this that as $n \to \infty$, $(\log |X|)/\kappa \to 0$.

Similarly, if $q = n-1$ and $m = C_1 n \log n$ and T is an n-vertex tree with bounded maximum degree $\Delta = O(1)$ then w.h.p. $G_{n,m}$ contains a rainbow copy of T. It suffices to take $\kappa = n/\Delta$. The uncolored version is due to Montgomery [16].

We also obtain the rainbow version of Shamir's problem, Corollary 1.2 of Bal and Frieze [2]. Here $X = \binom{[n]}{k}$ and $\mathcal{H}$ is the $n/(k-1)$-uniform hypergraph with $\frac{n!}{(n/k)!k!^{n/k}}$ edges, one for each perfect matching. It suffices to take $\kappa = n^{k-1}/k!$.

The paper [13] focusses exclusively on the square of Hamilton cycles. It removed a $\log n$ factor to get within a constant factor of a sharp threshold. The proof is similar to that of Theorem 1. Here $\kappa = O(n^{1/2})$ and the constraint that $r = O(\kappa)$ prevents us from generalising this result in the same way.

Remark 2. *An obvious line of attack here is to (i) replace each edge* $e = \{v_1, v_2, \ldots, v_k\} \in \mathcal{H}$ *by a set of edges* $\{v_1, v_2, \ldots, v_k, c\}$, $c \in [q] \subseteq X^* = X \times [q]$, *(ii) verify that the new hypergraph is* κ'*-spread for some value* κ' *and then apply [13] or [14]. The appropriate value of* κ' *seems to be* $q\kappa/e$, *see* (5)*. The only problem with this approach is that a random* m*-subset of* X^* *may not correspond to a randomly colored subset of* X*. Let* $\mathcal{B}$ *denote the subsets of* X^*_m *that contain a pair* $(x, c_1), (x, c_2)$ *i.e. where* $x \in X$ *has been given two colors. If* $N = |X|$ *then the expected number of such pairs in a random subset* X^*_m *is* $\approx \frac{Nq^2}{2} \cdot \left(\frac{m}{qN}\right)^2 = O\left(\frac{N \log^2 r}{\kappa^2}\right)$*. So we see immediately that Theorem 1 holds if* $\frac{\kappa}{N^{1/2}\log r} \to \infty$*, otherwise we have to work round the problem.*

2 Proof of Theorem 2

We closely follow [14] changing the proof to account for the coloring. The idea is to choose a small randomly colored random set W^* and argue that w.h.p. there exists a rainbow colored edge $H^* \in \mathcal{H}^*$ such that $|H^* \setminus W^*|$ is significantly smaller than $|H^*|$. We then repeat the argument with respect to the hypergraph $\mathcal{H}^* \setminus W^* = \{H^* \setminus W^* : e \in \mathcal{H}^*\}$. In this way, we build up a member of $\mathcal{H}^*$ piece by piece. After $O(\log r)$ iterations we can prove the existence of a final small piece by using a small modification to Janson's inequality [12].

It is important to realise that the edge sets of the hypergraphs encountered are multi-sets i.e. the same edge can be repeated many times. It will be helpful if from now on we use a superscript $*$ to indicate colored objects.

Let γ be a moderately small constant (e.g. $\gamma = 0.1$ suffices), and let C_0 be a constant large enough to support the estimates that follow. Let $\mathcal{H}$ be an r-bounded, κ-spread hypergraph on a Q-colored set X of size N, with $r, \kappa \geq C_0^2$. Let $X^* = X \times [q]$ and for $x \in X^*$, let $c(x)$ denote the color of x. We say that a set $S^* \subseteq X^*$ is rainbow if $c(x) \neq c(y)$ for $x, y \in S^*$. Then let

$$\mathcal{H}^* = \{(x_i, c_i) \in X^*, 1 \leq i \leq r : \{x_1, x_2, \ldots, x_r\} \in \mathcal{H},\ c_i \neq c_j \text{ for } i \neq j\}, \tag{4}$$

be the set of rainbow edges of $\mathcal{H}$.

For $x^* = (x, c) \in X^*$ we define $\xi(x^*) = x, c(x^*) = c$ and for $S^* \subseteq X^*$, we let $\xi(S^*) = \{\xi(x), x \in S^*\}$. We say that $A^*, B^* \subseteq X^*$ are *compatible* and write $A^* \sim B^*$ if $z = \xi(a^*) = \xi(b^*)$ where $a^* \in A^*, b^* \in B^*$ then $c(a^*) = c(b^*)$.

If $S^* \subseteq X^*$ is rainbow and $|S^*| = s$, then we have

$$|\mathcal{H}^* \cap \langle S^* \rangle| = \sum_{\substack{H^* \in \mathcal{H}^* \\ \xi(H^*) \supseteq \xi(S^*) \\ H^* \sim S^*}} (q-s)_{|\xi(H^*|)-s} \leq (q-s)_{r-s} \frac{|\mathcal{H}|}{\kappa^s} = \frac{(q-s)_{r-s}}{(q)_r} \frac{|\mathcal{H}^*|}{\kappa^s} \leq \frac{e^s |\mathcal{H}^*|}{(q\kappa)^s}. \tag{5}$$

If S^* is not rainbow or $|\xi(S^*)| \neq |S^*|$ then $\mathcal{H}^* \cap \langle S^* \rangle = \emptyset$ and so $\mathcal{H}^*$ is $q\kappa/e$ spread. It follows from Theorem 1 that w.h.p. X_m^* contains an edge of $\mathcal{H}^*$ w.h.p. for $m \geq \frac{(K \log r)|X^*|}{q\kappa} = \frac{KN \log r}{\kappa}$, but this is not the end of the story as discussed in Remark 2. Let

$$\widehat{E}^* = \{W^* \subseteq X^* : W^* \notin \mathcal{B}\},$$

where $\mathcal{B}$ is defined in Remark 2. Set $p = C_2/\kappa$ and $m = Np$, $N = |X|$ with $C_0 \leq C_2 \leq \kappa/C_0$ (so $p \leq 1/C_0$), $r' = (1-\gamma)r$. Let W_m^* be chosen randomly from $\binom{\widehat{E}^*}{m}$. We remark that choosing W_m^* in this way, is the same as choosing m randomly colored vertices. Also let W_p^* be chosen from $\widehat{E}^*$ by selecting each vertex of X with probability p and then randomly coloring the selected vertices. (We distinguish this from X_p^* which is chosen from $X \times [q]$ by choosing each colored vertex independently with probability p/q.)

Let

$$\mathcal{H}_{W^*}^* = \left\{H^* \in \mathcal{H}^* : H^* \cup W^* \in \widehat{E}^*\right\}. \tag{6}$$

Lemma 3. *If $\mathcal{H}$ is r-uniform and κ-spread and $\mu = \mathbb{E}(|\mathcal{H}_{W_p^*}^*|)$ then*

$$\mu = |\mathcal{H}^*| \left(1 - p + \frac{p}{q}\right)^r \text{ and } \mathbb{P}\left(|\mathcal{H}_{W_p^*}^*| \leq \frac{\mu}{2}\right) \leq \exp\left\{-\frac{\kappa}{32}\right\}.$$

Proof. To establish (9) we set $\zeta_{H^*} = 1_{\{H^* \cup W_p^* \in \widehat{E}^*\}}$ for $H^* \in \mathcal{H}^*$. Then

$$\mu = \sum_{H^* \in \mathcal{H}^*} \mathbb{E}(\zeta_{H^*}) = \sum_{H^* \in \mathcal{H}^*} \sum_{t=1}^{r} \binom{r}{t} q^{-t} p^t (1-p)^{r-t} = |\mathcal{H}^*| \left(1 - p + \frac{p}{q}\right)^r.$$

Let

$$\begin{aligned}
\Delta = \sum_{H^* \cap J^* \neq \emptyset} \mathbb{E}(\zeta_{H^*} \zeta_{J^*}) = \\
\sum_{H^* \cap J^* \neq \emptyset} \left(1 - p + \frac{p}{q}\right)^{2r - |H^* \cap J^*|} &= \sum_{H^* \cap J^* \neq \emptyset} \left(1 - p + \frac{p}{q}\right)^{2r} \sum_{t=1}^{r} \sum_{\substack{T^* \subseteq H^* \\ |T^*| = t}} \sum_{\substack{J^* \\ H^* \cap J^* = T^*}} \left(1 - p + \frac{p}{q}\right)^{-t} \\
&\leq \mu \left(1 - p + \frac{p}{q}\right)^r \max_{H^* \in \mathcal{H}^*} \left\{\sum_{t=1}^{r} \left(1 - p + \frac{p}{q}\right)^{-t} \sum_{\substack{T^* \subseteq H^* \\ |T^*| = t}} |\{J^* : H^* \cap J^* = T^*\}|\right\} \\
&\leq \mu \left(1 - p + \frac{p}{q}\right)^r \sum_{t=1}^{r} \binom{r}{t} \left(1 - p + \frac{p}{q}\right)^{-t} \left(\frac{e}{q\kappa}\right)^t |\mathcal{H}^*| \\
&= \mu^2 \sum_{t=1}^{r} \binom{r}{t} \left(1 - p + \frac{p}{q}\right)^{-t} \left(\frac{e}{q\kappa}\right)^t = \mu^2 \left(\left(1 + \frac{e}{q\kappa\left(1 - p + \frac{p}{q}\right)}\right)^r - 1\right) \leq \\
&\mu^2 \left(\exp\left\{\frac{3r}{q\kappa}\right\} - 1\right) \leq \frac{4\mu^2}{\kappa}. \qquad (7)
\end{aligned}$$

If applicable, Janson's inequality would bound the probability in (6) by $\exp\{-\mu^2/8\Delta\}$. We cannot do this. As explained previously, randomly coloring a random subset of X is not the same as choosing a random subset of X^*. We claim however, that a straightforward modification of Janson's inequality allows us to prove $\mathbb{P}(|\mathcal{H}^*_{W^*}| \leq \mu/2) \leq \exp\left\{-\frac{\mu^2}{8\Delta}\right\}$. We give details of our modification in Section 2.3. $\square$

Corollary 4. *If $\mathcal{H}$ is r-bounded and κ-spread then*

$$\mathbb{P}\left(|\mathcal{H}^*_{W^*_p}| \leq \frac{1}{2}|\mathcal{H}^*|\left(1-p+\frac{p}{q}\right)^r\right) \leq \exp\left\{-\frac{\kappa}{32}\right\}.$$

Proof. For each $H \in \mathcal{H}$ we add $r - |H|$ distinct new elements X_H to create the edge $\widehat{H}$. This creates a hypergraphs on the set $\widehat{X} = X \cup \bigcup_{H\in\mathcal{H}} X_H$ and the set $\widehat{X}^* = \widehat{X} \times [q]$. The hypergraph $\widehat{\mathcal{H}}$ is κ-spread. Indeed, if $S \cap X_H \neq \emptyset$ then $\widehat{H}$ is the unique edge of $\widehat{\mathcal{H}}$ that contains S. Otherwise, $S \cap X_H = \emptyset$ for all $H \in \mathcal{H}$ and $|\widehat{\mathcal{H}} \cap \langle S\rangle| = |\mathcal{H} \cap \langle S\rangle| \leq |\mathcal{H}|/\kappa^{|S|} = |\widehat{\mathcal{H}}|/\kappa^{|S|}$.

Finally observe that if we denote the random subset of $\widehat{X}^*$ chosen by $\widehat{W}^*$ and let $W^* = \widehat{W}^* \cap X^*$ then $\widehat{H}^* \in \widehat{\mathcal{H}}^*_{\widehat{W}^*}$ implies $H^* \in \mathcal{H}^*_{W^*}$. ($\widehat{\mathcal{H}}^*_{\widehat{W}^*}$ is defined analogously to $\mathcal{H}^*_{W^*}$ in (6).) Also, in this case the distribution of W^* is uniform from $\binom{\widehat{E}^*}{m}$. The corollary then follows by applying Lemma 3 to $\widehat{\mathcal{H}}^*$. $\square$

Our assumption that $r \leq C_1\kappa$ implies that

$$\left(1-p+\frac{p}{q}\right)^r \geq e^{-2C_1C_2}. \tag{8}$$

Corollary 5. *If $\mathcal{H}$ is r-bounded and κ-spread and $\kappa \gg \log N$ and $m = Np$ then*

$$\mathbb{P}\left(|\mathcal{H}^*_{W^*_m}| \leq \frac{1}{2}|\mathcal{H}^*|\left(1-p+\frac{p}{q}\right)^r\right) \leq \exp\left\{-\frac{\kappa}{33}\right\}.$$

Proof. We have

$$\begin{aligned}\mathbb{P}\left(|\mathcal{H}^*_{W^*_m}| \leq \frac{1}{2}|\mathcal{H}^*|\left(1-p+\frac{p}{q}\right)^r\right) &= \mathbb{P}\left(|\mathcal{H}^*_{W^*_p}| \leq \frac{1}{2}|\mathcal{H}^*|\left(1-p+\frac{p}{q}\right)^r \,\middle|\, |W^*_p| = m\right)\\ &\leq \frac{\mathbb{P}\left(|\mathcal{H}^*_{W^*_p}| \leq \frac{1}{2}|\mathcal{H}^*|\left(1-p+\frac{p}{q}\right)^r\right)}{\mathbb{P}(|W^*_p| = m)} \leq N^{1/2}\mathbb{P}\left(|\mathcal{H}^*_{W^*_p}| \leq \frac{1}{2}|\mathcal{H}^*|\left(1-p+\frac{p}{q}\right)^r\right) \leq N^{1/2}\exp\left\{-\frac{\kappa}{32}\right\}.\end{aligned}$$

The lemma follows after we use our assumption that $\kappa \gg \log N$. $\square$

Given $Z^* \in \widehat{E}^*_m$ we let $\psi(Z^*) \in \mathcal{H}^*$ be the rainbow edge $H^*_{Z^*}$ that minimises $|H^* \setminus Z^*|$ and let $\chi(Z^*) = \psi(Z^*) \setminus Z^*$. For $H^* \in \mathcal{H}^*$ and $W^* \in \widehat{E}^*$ such that $Z^* = H^* \cup W^* \in \widehat{E}^*$ set $\chi(H^*, W^*) = \psi(Z^*) \setminus W^*$. (So if $H^* \cup W^* = \tilde{H}^* \cup W^*$ then $\xi(H^*, W^*) = \chi(\tilde{H}^*, W^*)$.)

For $H^* \in \mathcal{H}^*$ and $W^* \in \widehat{E}^*$ we say that the pair (H^*, W^*) is *bad* if $|\chi(Z^*) \setminus W^*| > r'$ and *good* otherwise. Now comes the main lemma driving the proof.

Lemma 6. *For $\mathcal{H}$ as above, and W^* chosen uniformly from $\widehat{E}^*_m = \left\{E \in \widehat{E}^* : |E| = m\right\}$:*

$$\mathbb{E}(|\{H^* \in \mathcal{H}^*_{W^*} : (H^*, W^*) \text{ is bad}\}|) \leq |\mathcal{H}^*_{W^*}|C^{-r/3} \tag{9}$$

Proof. Let $W^* = \{(x_i, c_i) : i = 1, 2, \ldots, m\}$ be distributed as $\widehat{X}^*_m$. It is enough to show that if $s \in (r', r]$ and $\mathcal{H}^*_{s,W^*} = \{H^* \in \mathcal{H}^*_{W^*} : |H^*| = s\}$ then,

$$\mathbb{E}(|\left\{H^* \in \mathcal{H}^*_{s,W^*} : (H^*, W^*) \text{ is bad}\right\}|) \leq (\gamma r)^{-1} |\mathcal{H}^*_{W^*}| C^{-r/3}. \tag{10}$$

(Note that $\gamma r = r - r'$ bounds the number of s for which the set in question can be nonempty, whence the factor $(\gamma r)^{-1}$.)

Now,

$$\mathbb{E}\left(\left|\left\{H^* \in \mathcal{H}^*_{s,W^*} : (H^*, W^*) \text{ is bad}\right\}\right|\right) = \\ = \frac{1}{\binom{N}{Np} q^{Np}} \left|\left\{\left(H^* \in \mathcal{H}^*_{s,W^*}, W^* \in \widehat{E}^*_m\right) : \ (H^*, W^*) \text{ is bad and } H^* \cup W^* \in \widehat{E}^*\right\}\right|.$$

So, we instead concentrate on showing,

$$\left|\left\{\left(H^* \in \mathcal{H}^*_{s,W^*}, W^*_1 \in \binom{X}{Np}, \mathbf{c} \in [q]^{Np}\right) : \ (H^*, W^*) \text{ is bad and } H^* \cup W^* \in \widehat{E}^*\right\}\right| \\ \leq (\gamma r)^{-1} \binom{N}{Np} q^{Np} |\mathcal{H}^*_{W^*}| C^{-r/3}. \tag{11}$$

For $Z^* \supseteq H^* \in \mathcal{H}^*_{s,W^*}$, we say that (H^*, Z^*) is *pathological* if there is $T^* \subseteq H^*$ with $t := |T^*| > r'$ and

$$|\{\tilde{H}^* \in \mathcal{H}^*_{s,W^*} : \tilde{H}^* \in [T^*, Z^*]\}| > C^{r/2} |\mathcal{H}^*_{W^*}| (q\kappa)^{-t} \left(\frac{p}{q}\right)^{s-t}. \tag{12}$$

Here $[T^*, Z^*] = \left\{\tilde{H} : T \subseteq \tilde{H}^* \subseteq Z^*\right\}$.

We bound the non-pathological and pathological parts of (11) separately.

Non-pathological contributions. We first bound the number of pairs (H^*, W^*) in the left hand side of (11) with (H^*, Z^*) non-pathological, $Z^* = H^* \cup W^*$.

Step 1: There are at most

$$\sum_{i=0}^{s} \binom{N}{Np+i} q^{Np+i} \leq \binom{N+s}{Np+s} q^{Np+s} \leq \binom{N}{Np} \frac{q^{Np+s}}{p^s} \tag{13}$$

choices for $Z^* = W^* \cup H^*$.

Step 2: Given Z^*, let $\tilde{H}^* = \psi(Z^*)$. Choose $T^* := H^* \cap \tilde{H}^*$, for which there are at most $2^{|\tilde{H}^*|} \leq 2^r$ possibilities, and set $t = |T^*| > r'$. (If $t \leq r'$ then (H^*, W^*) cannot be bad, as $\chi(H^*, W^*) = \tilde{H}^* \setminus W^* \subseteq T^*$.)

Step 3: Since we are only interested in non-pathological choices, and we choose $H^* \setminus \tilde{H}^*$ by choosing $H^* \in [T^*, Z^*]$, the number of possibilities for H^* is now at most

$$C^{r/2} |\mathcal{H}^*_{W^*}| (q\kappa)^{-t} \left(\frac{p}{q}\right)^{s-t}.$$

Step 4: Complete the specification of (H^*,W^*) by choosing $W^*\cap H^*$, the number of possibilities for which is at most 2^s.

Because $s\le r$ and $t>r'=(1-\gamma)r$, the number of nonpathological possibilities is at most

$$\binom{N}{Np}\frac{q^{Np+s}}{p^s}\cdot 2^r\cdot C^{r/2}|\mathcal{H}^*_{W^*}|(q\kappa)^{-t}\left(\frac{p}{q}\right)^{s-t}\cdot 2^s = 2^{r+s}\binom{N}{Np}q^{Np}|\mathcal{H}^*_{W^*}|C^{r/2}(p\kappa)^{-t}$$
$$\le \binom{N}{Np}q^{Np}|\mathcal{H}^*_{W^*}|(4C^{1/2})^rC^{-t} < \binom{N}{Np}q^{Np}|\mathcal{H}^*_{W^*}|\,[4C^{-(1/2-\gamma)}]^r. \quad (14)$$

Pathological contributions. We next bound the number of (H^*,W^*) in the left hand side of (11) with (H^*,Z^*) pathological, $Z^*=H^*\cup W^*$.

Step 1: There are at most $|\mathcal{H}^*_{W^*}|$ possibilities for H^*.

Step 2: Choose $T^*\subseteq H^*$ witnessing the pathology of (H^*,Z^*) (i.e. for which (12) holds); there are at most 2^s possibilities for T^*.

Step 3: Choose $U^*\in[T^*,H^*]$ for which

$$|\mathcal{H}^*_{s,W^*}\cap[U^*,(Z^*\setminus H^*)\cup U^*]|>2^{-(s-t)}C^{r/2}|\mathcal{H}^*_{W^*}|(q\kappa)^{-t}\left(\frac{p}{q}\right)^{s-t}. \quad (15)$$

(Here the left hand side counts the members of $\mathcal{H}^*_{s,W}$ in Z^* whose intersection with H^* is precisely U^*. Of course, the existence of U^* as in (15) follows from (12).) The number of possibilities for this choice is at most 2^{s-t}.

Step 4: Choose $Y^*=Z^*\setminus H^*$. Write Φ for the R.H.S. of (15). Noting that $Z^*\setminus H^*$ must belong to the set $\bigcup_{i=0}^s\left(\binom{X\setminus\xi(H^*)}{Np-i}\times[q]^{Np-i}\right)$, we consider, for Y^* drawn uniformly from this set,

$$\mathbb{P}(|\mathcal{H}^*_{s,W^*}\cap[U^*,Y^*\cup U^*]|>\Phi). \quad (16)$$

We can then bound the number of choices for $Z^*\setminus H^*$ by $\binom{N}{Np}q^{Np}$ times this probability. Set $|U^*|=u$. We have

$$\mathbb{E}(|\mathcal{H}^*_{s,W^*}\cap\langle U^*\rangle|)=\sum_{\substack{H^*\in\mathcal{H}^*\\ H^*\supseteq U^*}}\mathbb{P}(H^*\cup W^*\in\widehat{E}^*)\le|\mathcal{H}^*|\left(\frac{e}{q\kappa}\right)^u\left(1-p+\frac{p}{q}\right)^r\le 2|\mathcal{H}^*_{W^*}|\left(\frac{e}{q\kappa}\right)^u,$$

while, for any $\tilde{H}^*\in\mathcal{H}^*_{s,W^*}\cap\langle U^*\rangle$,

$$\mathbb{P}(Y^*\supseteq\tilde{H}^*\setminus U^*)\le\left(\frac{Np}{q(N-s)}\right)^{s-u}$$

(of course if $\tilde{H}^*\cap H^*\ne U^*$ the probability is zero); so

$$\theta:=\mathbb{E}\left(|\mathcal{H}^*_{s,W^*}\cap[U^*,Y^*\cup U^*]|\right)\le 2|\mathcal{H}^*_{W^*}|\left(\frac{e}{q\kappa}\right)^u\left(\frac{Np}{q(N-s)}\right)^{s-u}\le 2|\mathcal{H}^*_{W^*}|\left(\frac{e}{q\kappa}\right)^u\left(\frac{2p}{q}\right)^{s-u}$$

(since $N-s\ge N/2$). Markov's inequality then bounds the probability in (16) by θ/Φ, and this bounds the number of possibilities for $Z^*\setminus H^*$ by $\binom{N}{Np}q^{Np}(\theta/\Phi)$.

Step 5: Complete the specification of (H^*, W^*) by choosing $H^* \cap W^*$, which can be done in at most 2^s ways.

Combining we find that the number of pathological possibilities is at most

$$|\mathcal{H}^*_{W^*}| \cdot 2^s \cdot 2^{s-t} \cdot \binom{N}{Np} q^{Np} \cdot \frac{2|\mathcal{H}^*_{W^*}| \left(\frac{e}{q\kappa}\right)^u \left(\frac{2p}{q}\right)^{s-u}}{2^{-(s-t)} C^{r/2} |\mathcal{H}^*_{W^*}| (q\kappa)^{-t} \left(\frac{p}{q}\right)^{s-t}} \cdot 2^s =$$

$$\binom{N}{Np} q^{Np} |\mathcal{H}^*|\, e^u 2^{5s-2t-u+1} \kappa^{t-u} p^{t-u} C^{-r/2}$$

$$= \binom{N}{Np} q^{Np} |\mathcal{H}^*_{W^*}| e^u 2^{5s-2t-u+1} C^{t-u-r/2} \leq \binom{N}{Np} q^{Np} |\mathcal{H}^*_{W^*}| C^{-2r/5}. \quad (17)$$

Finally, the sum of the bounds in (14) and (17) is less than the $(\gamma r)^{-1}\binom{N}{Np} q^{Np} |\mathcal{H}^*_{W^*}| C^{-r/3}$ of (11), for r large. □

2.1 Small uniformities

Small set sizes are handled by the same modification of Janson's inequality used in Lemma 3. For $Y = X \times [q]$ and $\alpha \in (0, 1)$ we define the random subset Y^*_α as follows: we start with the random set X_α and then each $x \in X_\alpha$ we give x a random color $c(x) \in [q]$. The resulting set $Y^*_\alpha = \{(x, c(x)), x \in X_\alpha\}$.

Lemma 7. *For an r-bounded, κ-spread $\mathcal{G}$ on Y, and $\alpha \in (0, 1)$,*

$$\mathbb{P}(Y^*_\alpha \notin \langle \mathcal{G}^* \rangle) \leq \exp\left\{-2\left(\sum_{t=1}^{r} \binom{r}{t} \left(\frac{e}{\alpha\kappa}\right)^t\right)^{-1}\right\}, \quad (18)$$

where $\mathcal{G}^*$ is the set of rainbow edges of $\mathcal{G}$.

Proof. Denote the members of $\mathcal{G}$ by $\{G\}$ and the members of $\mathcal{G}^*$ by $\{G^*\}$ where G^* is the randomly colored G. Then set $\zeta_{G^*} = 1_{\{Y^*_\alpha \supseteq G^*\}}$. We add $r - |G|$ new elements and colors (x, c) to each G to create an r-uniform hypergraph $\widehat{\mathcal{G}}^*$ on a set of vertices $\widehat{Y}^*$. Note that $\widehat{Y}^*_\alpha \in \langle \widehat{\mathcal{G}}^* \rangle$ implies $Y^*_\alpha \in \langle \mathcal{G}^* \rangle$. This addition of vertices and we let $\mathcal{G}^* = \widehat{\mathcal{G}}^*$, assuming that $\mathcal{G}$ is r-uniform. Then

$$\mu := \sum_{G^* \in \mathcal{G}^*} \mathbb{E}(\zeta_G) = \left(\frac{\alpha}{q}\right)^r |\mathcal{G}^*| = \frac{\alpha^r (q)_r}{q^r} |\mathcal{G}|.$$

Let

$$\Delta = \sum_{G^*, H^* : G^* \cap H^* \neq \emptyset} \mathbb{E}(\zeta_{G^*} \zeta_{H^*}) = \sum_{G \in \mathcal{G}} \frac{\alpha^r (q)_r}{q^r} \sum_{t=1}^{r} \sum_{\substack{T \subseteq G \\ |T| = t}} \sum_{G \cap H = T} \frac{\alpha^{r-t} (q-t)_{r-t}}{q^{r-t}}$$

$$\leq \mu \max_{G \in \mathcal{G}} \left\{\sum_{t=1}^{r} \alpha^{-t} \sum_{\substack{T \subseteq G \\ |T| = t}} \sum_{G \cap H = T} \frac{\alpha^r (q)_r}{q^r} \cdot \frac{(q-t)_{r-t} q^t}{(q)_r}\right\} \leq \mu \sum_{t=1}^{r} \left(\frac{e}{\alpha}\right)^t \max_{G \in \mathcal{G}} \left\{\sum_{\substack{T \subseteq G \\ |T| = t}} \sum_{H \supseteq T} \frac{\alpha^r (q)_r}{q^r}\right\}$$

$$\leq \mu \sum_{t=1}^{r} \binom{r}{t} \left(\frac{e}{\alpha\kappa}\right)^t \frac{\alpha^r (q)_r}{q^r} |\mathcal{G}| = \mu^2 \sum_{t=1}^{r} \binom{r}{t} \left(\frac{e}{\alpha\kappa}\right)^t. \quad (19)$$

If applicable, Janson's inequality would bound the probability in (18) by $\exp\{-\mu^2/2\Delta\}$. We claim however, that the modification of Janson's inequality in Section 2.3 allows us to prove

$$\mathbb{P}(Y_\alpha^* \notin \langle \mathcal{G}^* \rangle) \leq \exp\left\{-\frac{\mu^2}{2\Delta}\right\}. \tag{20}$$

□

Corollary 8. *Let $\mathcal{G}$ be as in Lemma 7, let $m = \alpha|Y|$ be an integer with $\alpha\kappa \geq 2er$, and let W^* be distributed $\widehat{X}_m^*$ i.e. choose a random member of $\binom{X}{m}$ and then give each element $x \in W^*$ a random color $c(x) \in [q]$. Then*

$$\mathbb{P}(W^* \notin \langle \mathcal{G}^* \rangle) \leq 2\exp\left\{-\frac{\alpha\kappa}{4er}\right\}.$$

Proof. Note that

$$\sum_{t=1}^{r} \binom{r}{t} \left(\frac{e}{\alpha\kappa}\right)^t = \left(1 + \frac{e}{\alpha\kappa}\right)^r - 1 \leq \frac{2er}{\alpha\kappa}.$$

Thus Lemma 7 gives

$$\exp\left\{-\frac{\alpha\kappa}{4er}\right\} \geq \mathbb{P}(Y_\alpha^* \notin \langle \mathcal{G}^* \rangle) \geq \mathbb{P}(|Y_\alpha^*| \leq m)\mathbb{P}(W^* \notin \langle \mathcal{G}^* \rangle) \geq \mathbb{P}(W^* \notin \langle \mathcal{G}^* \rangle)/2,$$

where we use the fact that any binomial ξ with $\mathbb{E}[\xi] \in \mathbb{Z}$ satisfies $\mathbb{P}(\xi \leq \mathbb{E}[\xi]) \geq 1/2$, see e..g. Lord [15]. □

At this point the reader might worry that if $r = \kappa$ (as allowed in Theorem 2) then $e^{-\alpha\kappa/4er}$ will not be small. We will see however in the next section that we apply this inequality to a hypergraph where κ is unchanged and where the edges are much smaller than the edges of $\mathcal{H}$.

2.2 Completing the proof

Let γ and C_0 be as in Section 2 and $\mathcal{H}$ be as in the statement of Theorem 1, and recall that asymptotics refer to r.

In what follows we will have a sequence $\mathcal{H}_i^*$, with $\mathcal{H}_0^* = \mathcal{H}^*$ and

$$\mathcal{H}_i^* \subseteq \{\chi_i(H^*, W_i^*) : H^* \in \mathcal{H}_{i-1,W_i^*}^*\}, \text{ where } \mathcal{H}_{i-1,W_i^*}^* = \left\{H^* \in \mathcal{H}_{i-1}^* : H^* \cup W_i^* \in \widehat{E}^*\right\},$$

where W_i^* and χ_i will be defined below (with χ_i a version of the χ of Section 2).

Set $C = C_0$ and $p = C/\kappa$, define ℓ to be the smallest positive integer such that $(1-\gamma)^\ell \leq \sqrt{\log r}/r$, and set $\rho = \log r/\kappa$. Then $\ell \leq \gamma^{-1}\log r$ because $(1-\gamma)^{-1+\gamma^{-1}\log r} \leq (1-\gamma)^{-1}\exp\{-\log r\} \leq \sqrt{\log r}/r$ and Theorem 2 will follow from the next assertion.

Claim 1. *If W^* is a random element of $\widehat{E}^*$ of size $((\ell p + \rho)N)$, then $W^* \in \langle \mathcal{H}^* \rangle = \bigcup_{H^* \in \mathcal{H}^*} \langle H^* \rangle$ w.h.p.*

Proof. Set $\delta = 1/(2\ell)$. Let $r_0 = r$ and $r_i = (1-\gamma)r_{i-1} = (1-\gamma)^i r_0$ for $i \in [\ell]$. Let $X_0 = X$ and $X_0^* = X \times [q]$ and, for $i = 1 \ldots, \ell$, let W_i^* be chosen with the distribution Y_p (defined in Section 2.1 with α in place of p)

and set $X_i = X_{i-1} \setminus \xi(W_i^*)$ and $X_i^* = X_i \times [q]$. The sequence $X_i^*, W_i^*, i = 1, 2, \ldots, \ell+1$ is defined below. Note the assumption $\kappa \gg \log r$ ensures that

$$|X_\ell| = N - \ell Np \geq N\left(1 - \frac{C_0\gamma^{-1}\log r}{\kappa}\right) \geq \frac{N}{2}.$$

For $H^* \in \mathcal{H}^*_{i-1,W_i^*}$ let $\chi_i(H^*, W_i^*) = \psi_i(H^* \cup W_i^*) \setminus W_i^*$, where $\psi_i(Z^*)$ selects a member H^* of $\mathcal{H}^*_{i-1,W_i^*}$ that minimises $|H^* \setminus W_i^*|$. Say that H^* is *good* if $|\chi_i(H^*, W_i^*)| \leq r_i$ (and *bad* otherwise), and set

$$\mathcal{H}_i^* = \{\chi_i(H^*, W_i^*) : H^* \in \mathcal{H}^*_{i-1} \text{ is good}\}.$$

Thus $\mathcal{H}_i^*$ is an r_i-bounded collection of subsets of X_i. Finally, choose $W^*_{\ell+1}$ uniformly from $\binom{X_\ell}{N\rho}$ and randomly color each vertex. Then $W^* = W_1^* \cup \cdots \cup W^*_{\ell+1}$ is distributed as required for Claim 1. Note also that $W^* \in \langle \mathcal{H}^* \rangle$ whenever $W^*_{\ell+1} \in \langle \mathcal{H}^*_\ell \rangle$. (More generally, $W_1^* \cup \cdots \cup W_i^* \cup Y \in \langle \mathcal{H}^* \rangle$ whenever $Y \subseteq X_i$ lies in $\langle \mathcal{H}_i^* \rangle$.)

So to prove the claim, we just need to show

$$\mathbb{P}(W^*_{\ell+1} \in \langle \mathcal{H}^*_\ell \rangle) = 1 - o(1) \tag{21}$$

(where the $\mathbb{P}$ refers to the entire sequence $W_1^* \ldots W^*_{\ell+1}$).

For $i \in [\ell]$ call W_i^* *successful* if $|\mathcal{H}_i^*| \geq (1-\delta)|\mathcal{H}^*_{i-1}|$, call $W^*_{\ell+1}$ successful if it lies in $\langle \mathcal{H}^*_\ell \rangle$, and say a sequence of W_i^*'s is successful if each of its entries is. We show that

$$\mathbb{P}(W_1^* \ldots W^*_{\ell+1} \text{ is successful}) = 1 - \exp\left[-\Omega(\sqrt{\log r})\right]. \tag{22}$$

Now $W_1^* \ldots W^*_{i-1}$ successful implies that $|\mathcal{H}^*_{i-1}| > (1-\delta)^\ell |\mathcal{H}^*| > |\mathcal{H}^*|/2$. Let $A = 2e^{2C_1C_2}$ and $\kappa' = \kappa/A$. So for $I \subseteq X_{i-1}$ we have that with probability at least $e^{-\kappa'/33}$,

$$|\mathcal{H}^*_{i-1} \cap \langle I \rangle| \leq |\mathcal{H}^* \cap \langle I \rangle| \leq \left(\frac{e}{q\kappa}\right)^{|I|} |\mathcal{H}^*| \leq A\left(\frac{e}{q\kappa}\right)^{|I|} |\mathcal{H}^*_{i-1}| \leq \left(\frac{e}{q\kappa'}\right)^{|I|} |\mathcal{H}^*_{i-1}|.$$

We therefore have the κ'-spread condition (5) for $\mathcal{H}^*_{i-1}$. For $i \in [\ell]$, according to Corollary 5 and Lemma 6 (and the Markov inequality),

$$\mathbb{P}(W_i^* \text{ is } \textit{not} \text{ successful} \,|\, W_1^* \ldots W^*_{i-1} \text{ is successful}) < \delta^{-1} C^{-r_{i-1}/3} + e^{-\kappa'/33}.$$

Thus

$$\mathbb{P}(W_1^* \ldots W_\ell^* \text{ is successful}) > 1 - \delta^{-1}\sum_{i=1}^{\ell} C^{-r_{i-1}/3} - \ell e^{-\kappa'/33} > 1 - \exp\left\{-\sqrt{\log r}/4e\right\} \tag{23}$$

(using $r_\ell = \sqrt{\log r}$). (Replacing κ by κ' leads to employing a larger C in Lemma 6.)

Finally, if $W_1^* \ldots W_\ell^*$ is successful, then Corollary 8 applied with $\mathcal{G} = \mathcal{H}_\ell = \{\xi(H^*) : H^* \in \mathcal{H}^*_\ell\}$, $Y = X_\ell$, $\alpha = N\rho/|Y| \geq \rho$, $r = r_\ell$, and $W^* = W^*_{\ell+1}$ gives

$$\mathbb{P}(W^*_{\ell+1} \notin \langle \mathcal{H}^*_\ell \rangle) \leq 2\exp\left\{-\sqrt{\log r}/4e\right\}, \tag{24}$$

and we have (22) and the claim.

We used the following to obtain (24):

$$\frac{\alpha\kappa}{er_\ell} \geq \frac{\rho\kappa}{e\sqrt{\log r}} = \frac{\sqrt{\log r}}{4e}.$$

□

This completes the proof of Theorem 2, modulo the verification of inequality (20).

2.3 Proof of inequality (20)

We begin with a modification of Harris's inequality [11].

2.3.1 A modification of Harris's inequality

We have a partition $P_1, P_2, \ldots, P_N$ of the set $[n]$ and $\Omega = \left\{ \mathbf{x} \in \{0,1\}^n : \sum_{t \in P_j} x_t \leq 1, \text{for } j \in [N] \right\}$. Let f, g be real monotone increasing functions defined on Ω in the sense that if $\mathbf{x}, \mathbf{y} \in \Omega$ and $\mathbf{x} \leq \mathbf{y}$ then $f(\mathbf{x}) \leq f(\mathbf{y})$, and similarly for g. Turn Ω into a probability space by setting $\xi_j = 1, j \in [N]$ independently with probability p_j for $j \in [N]$. Then for each $j \in [N]$ for which $\xi_j = 1$, choose t uniformly at random from P_j and set $x_t = 1$ and $x_s = 0$ for $s \in P_j \setminus \{t\}$. Harris's inequality comprises the following lemma when $N = n$ and $P_j = \{j\}$, $j \in [N]$.

Lemma 9. *With f, g, Ω as above, $\mathbb{E}(fg) \geq \mathbb{E}(f)\mathbb{E}(g)$.*

Proof. We just repeat the proof of Harris's inequality. We will prove the lemma by induction on N. If $N = 0$ then $\mathbb{E}(f) = a, \mathbb{E}(g) = b$ and $\mathbb{E}(fg) = ab$ for some constants a, b.

So assume the truth for $N - 1$. Suppose that $\mathbb{E}(f \mid \xi_N = 0) = a$ and $\mathbb{E}(g \mid \xi_N = 0) = b$ then

$$\mathbb{E}((f-a)(g-b)) - \mathbb{E}(f-a)\mathbb{E}(g-b) = \mathbb{E}(fg) - \mathbb{E}(f)\mathbb{E}(g).$$

By replacing f by $f - a$ and g by $g - b$ we may therefore assume that $\mathbb{E}(f \mid \xi_N = 0) = \mathbb{E}(g \mid \xi_N = 0) = 0$. By monotonicity, we see that $\mathbb{E}(f \mid \xi_N = 1), \mathbb{E}(g \mid \xi_N = 1) \geq 0$.

We observe that by the induction hypothesis that

$$\begin{aligned}
\mathbb{E}(fg \mid \xi_N = 0) &\geq \mathbb{E}(f \mid \xi_N = 0)\mathbb{E}(g \mid \xi_N = 0) = 0 \\
\mathbb{E}(fg \mid \xi_N = 1) &\geq \mathbb{E}(f \mid \xi_N = 1)\mathbb{E}(g \mid \xi_N = 1) \geq 0
\end{aligned}$$

Now, by the above inequalities,

$$\begin{aligned}
\mathbb{E}(fg) &= \mathbb{E}(fg \mid \xi_N = 0)(1 - p_N) + \mathbb{E}(fg \mid \xi_N = 1)p_N \\
&\geq \mathbb{E}(f \mid \xi_N = 1)\mathbb{E}(g \mid \xi_N = 1)p_N. \qquad (25)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}(f)\mathbb{E}(g) = (\mathbb{E}(f \mid \xi_N = 0)(1 - p_N) + \mathbb{E}(f \mid \xi_N = 1)p_N) \times \\
(\mathbb{E}(g \mid \xi_N = 0)(1 - p_N) + \mathbb{E}(g \mid \xi_N = 1)p_N) \\
= \mathbb{E}(f \mid \xi_N = 1)\mathbb{E}(g \mid \xi_N = 1)p_N^2. \qquad (26)
\end{aligned}$$

The result follows by comparing (25) and (26) and using the fact that $\mathbb{E}(f \mid \xi_N = 1), \mathbb{E}(g \mid \xi_N = 1) \geq 0$ and $0 \leq p_N \leq 1$. ☐

Of course, if f, g are both decreasing then we also have

$$\mathbb{E}(fg) = \mathbb{E}((-f)(-g)) \geq \mathbb{E}(-f)\mathbb{E}(-g) = \mathbb{E}(f)\mathbb{E}(g). \qquad (27)$$

2.3.2 A modification of Janson's inequality

For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ define $D(\mathbf{x}) \subseteq [n]$, by $i \in D(\mathbf{x})$ iff $x_i = 1$. On the other hand, given $D' \subseteq [n]$, define $\mathbf{x}(D')$ by $D(\mathbf{x}(D')) = D'$. Fix a family of m subsets $D_i \subseteq [n], i \in [m]$ where $\mathbf{x}(D_i) \in \Omega$ for $i \in [m]$, where Ω is as in Section 2.3.1. For each i, let $E_i = \{j : D_i \cap P_j \neq \emptyset\}$.

To obtain (20) we partition $Y = X \times [q]$ into $P_i = X \times \{q\}$ for $i = 1, 2, \ldots, N$ and let the D_i correspond to the edges of $\mathcal{G}^*$. We then apply Lemma 10 with $t = \mu$.

We repeat the proof of Janson's inequality. Let $R = D(\mathbf{x})$ where $\mathbf{x}$ is chosen randomly from Ω as in Section 2.3.1. Let $\mathcal{A}_i$ be the event that D_i is a subset of R. Moreover, let I_i be the indicator of the event $\mathcal{A}_i$. Note that, I_i and I_j are independent if $E_i \cap E_j = \emptyset$ and if $t \in E_i \cap E_j$ then $D_i \cap P_t = D_j \cap P_t$ or $I_i I_j = 0$. One can easily see that the I_i's are increasing. We let

$$S_m = I_1 + I_2 + \cdots + I_m,$$

and

$$\mu = \mathbb{E} S_m = \sum_{i=1}^{m} \mathbb{E}(I_i).$$

We write $i \sim j$ if $E_i \cap E_j \neq \emptyset$. Then, let

$$\Delta = \sum_{\{(i,j): i \sim j\}} \mathbb{E}(I_i I_j). \tag{28}$$

Let $\phi(x) = (1+x)\log(1+x) - x$. Now, with S_m, Δ, ϕ given above one can establish the following upper bound on the lower tail of the distribution of S_m.

Lemma 10. *For any real t, $0 \leq t \leq \mu$,*

$$\mathbb{P}(S_m \leq \mu - t) \leq \exp\left\{-\frac{\phi(-t/\mu)\mu^2}{\Delta}\right\} \leq \exp\left\{-\frac{t^2}{2\Delta}\right\}. \tag{29}$$

Proof. Put $\psi(\lambda) = \mathbb{E}(e^{-\lambda S_m}), \lambda \geq 0$. By Markov's inequality we have

$$\mathbb{P}(S_m \leq \mu - t) \leq e^{\lambda(\mu - t)} \mathbb{E}(e^{-\lambda S_m}).$$

Therefore,

$$\log \mathbb{P}(S_m \leq \mu - t) \leq \log \psi(\lambda) + \lambda(\mu - t). \tag{30}$$

Now let us estimate $\log \psi(\lambda)$ and minimise the right-hand-side of (30) with respect to λ.

Note that

$$-\psi'(\lambda) = \mathbb{E}(S_m e^{-\lambda S_m}) = \sum_{i=1}^{m} \mathbb{E}(I_i e^{-\lambda S_m}). \tag{31}$$

Now for every $i \in [n]$, split S_m into Y_i and Z_i^*, where

$$Y_i = \sum_{j: j \sim i} I_j, \quad Z_i = \sum_{j: j \not\sim i} I_j, \quad S_m = Y_i + Z_i.$$

We note that Lemma 9 remains true if we condition on $I_i = 1$ for some $i \in [m]$. The conditioning basically just restricts Ω to a smaller set. Then by Lemma 9 with $f = -e^{-\lambda Y_i}, g = -e^{-\lambda Z_i}$ and R with I_i fixed at one, we get, setting $p_i = \mathbb{E}(I_i)$,

$$\mathbb{E}(I_i e^{-\lambda S_m}) = p_i \mathbb{E}(e^{-\lambda Y_i} e^{-\lambda Z_i} \mid I_i = 1) \geq p_i \mathbb{E}(e^{-\lambda Y_i} \mid I_i = 1) \mathbb{E}(e^{-\lambda Z_i} \mid I_i = 1).$$

Since Z_i and I_i are independent we get

$$\mathbb{E}(I_i e^{-\lambda S_m}) \geq p_i \mathbb{E}(e^{-\lambda Y_i} \mid I_i = 1) \mathbb{E}(e^{-\lambda Z_i}) \geq p_i \mathbb{E}(e^{-\lambda Y_i} \mid I_i = 1) \psi(\lambda). \tag{32}$$

From (31) and (32), applying Jensen's inequality (twice) and remembering that $\mu = \mathbb{E} S_m = \sum_{i=1}^m p_i$, we get

$$\begin{aligned} -(\log \psi(\lambda))' = -\frac{\psi'(\lambda)}{\psi(\lambda)} &\geq \sum_{i=1}^m p_i \mathbb{E}(e^{-\lambda Y_i} \mid I_i = 1) \\ &\geq \mu \sum_{i=1}^m \frac{p_i}{\mu} \exp\left\{-\mathbb{E}(\lambda Y_i \mid I_i = 1)\right\} \geq \mu \exp\left\{-\frac{1}{\mu} \sum_{i=1}^m p_i \mathbb{E}(\lambda Y_i \mid I_i = 1)\right\} \\ &= \mu \exp\left\{-\frac{\lambda}{\mu} \sum_{i=1}^m \mathbb{E}(Y_i I_i)\right\} = \mu e^{-\lambda \Delta/\mu}. \end{aligned} \tag{33}$$

So

$$-(\log \psi(\lambda))' \geq \mu e^{-\lambda \Delta/\mu} \tag{34}$$

which implies that

$$-\log \psi(\lambda) \geq \int_0^\lambda \mu e^{-z\Delta/\mu} dz = \frac{\mu^2}{\Delta}(1 - e^{-\lambda \Delta/\mu}). \tag{35}$$

Hence by (35) and (30)

$$\log \mathbb{P}(S_m \leq \mu - t) \leq -\frac{\mu^2}{\Delta}(1 - e^{-\lambda \Delta/\mu}) + \lambda(\mu - t), \tag{36}$$

which is minimized by choosing $\lambda = -\log(1 - \frac{t}{\mu})\mu/\Delta$. It yields the first bound in (29), while the final bound in (29) follows from the fact that $\phi(x) \geq \frac{x^2}{2}$ for $x \leq 0$. □

3 Final Remarks

An earlier version claimed to have full generalisations of the threshold results of [13], [14]. Excellent reviewing pointed out significant errors and we can only claim a (partial) generalisation of [14]. We nevertheless are confident that the threshold results of these papers can be generalised to rainbow versions.

Acknowledgement We thank Sam Spiro for pointing out an issue in a previous version.

References

[1] R. Alweiss, S. Lovett, K. Wu, and J. Zhang, Improved bounds for the sunflower lemma, *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing* (2020) 624-630.

[2] D. Bal and A.M. Frieze, Rainbow Matchings and Hamilton Cycles in Random Graphs, *Random Structures and Algorithms* 48 (2016) 503-523.

[3] C. Cooper and A.M. Frieze, Multi-coloured Hamilton cycles in randomly coloured random graphs, *Combinatorics, Probability and Computing* 11 (2002) 129-134.

[4] A. Dudek, S, English and A.M. Frieze, On rainbow Hamilton cycles in random hypergraphs, *Electronic Journal of Combinatorics* 25 (2018).

[5] A Espuny Diaz and Y. Person, Spanning F-cycles in random graphs.

[6] A. Ferber and M. Krivelevich, Rainbow Hamilton cycles in random graphs and hypergraphs, in *Recent trends in combinatorics, IMA Volumes in Mathematics and its applications, A. Beveridge, J. R. Griggs, L. Hogben, G. Musiker and P. Tetali, Eds.*, Springer 2016, 167-189.

[7] E. Friedgut, Hunting for Sharp Thresholds, *Random Structures Algorithms* 26 (2005) 37-51.

[8] A.M. Frieze, A note on spanning K_r-cycles in random graphs, *AIMS Mathematics* 5, 4849-4852

[9] A.M. Frieze and M. Karoński, Introduction to Random Graphs, Cambridge University Press, 2015.

[10] A.M. Frieze and P. Loh, Rainbow Hamilton cycles in random graphs, *Random Structures and Algorithms* 44 (2014) 328-354.

[11] T. Harris, A lower bound for the critical probability in a certain percolation, *Proceedings of the Cambridge Philosophical Society* 56 (1960) 13-20.

[12] S. Janson, Poisson approximation for large deviations, *Random Structures and Algorithms* 1 (1990) 221-230.

[13] J. Kahn, B. Narayanan and J. Park, The threshold for the square of a Hamilton cycle.

[14] K. Frankston, J. Kahn, B. Narayanan and J. Park, Thresholds versus fractional expectation thresholds.

[15] N. Lord, Binomial averages when the mean is an integer, *Mathematical Gazette* 94 (2010) 331-332.

[16] R. Montgomery, Spanning trees in random graphs, *Advances in Mathematics* 356 (2019).

[17] S. Spiro, A Smoother Notion of Spread Hypergraphs.