Rainbow copies of spanning subgraphs

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Abstract

Let $\mathbb{G}_{n,p}^{[\kappa]}$ denote the space of n-vertex edge coloured graphs, where each edge occurs independently with probability p. The colour of each existing edge is chosen independently and uniformly at random from the set $[\kappa]$. We consider the threshold for the existence of rainbow colored copies of a spanning subgraph H. We provide lower bounds on p and κ sufficient to prove the existence of such copies w.h.p.

1 Introduction

Let $\mathbb{G}_{n,p}^{[\kappa]}$ denote the space of n-vertex edge coloured graphs, where each edge occurs independently with probability p. The colour of each existing edge is chosen independently and uniformly at random from the set $[\kappa]$. Alon and Furédi [1] and Riordan [7] gave upper bounds for the threshold for $G_{n,p}$ to contain a copy of a spanning subgraph H w.h.p. We give comparable estimates for $\mathbb{G}_{n,p}^{[\kappa]}$ to contain a *rainbow* copy of H w.h.p., i.e., a copy of a spanning subgraph H where each edge has a different color.

1.1 Background

There has been considerable research on the existence of rainbow spanning objects in randomly colored random graphs. One of the earliest results is in Frieze and McKay [6]. They considered the existence of rainbow spanning trees and proved the following result. Suppose we add randomly colored edges one by one, to an initially empty graph then w.h.p. there is a rainbow spanning tree once the graph is connected and at least $n-1$ colors have appeared. Cooper and Frieze [2], Frieze and Loh [5], and Ferber and Krivelevich [3] considered the existence of rainbow Hamilton cycles. Here the results are not so precise. Basically we know that if there are at least $n + o(n)$ colors available, and the graph has minimum degree at least two, then there is a rainbow Hamilton cycle.

Ferber, Nenadov and Peter [4] consider the threshold for finding rainbow copies of an arbitrary spanning subgraph H. They prove that if $d \leq \Delta(H) = O(1)$, and $p \geq n^{-1/d} \log^{5/d} n$ then w.h.p. $\mathbb{G}_{n,p}^{[\kappa]}$ contains a rainbow colored copy of H, provided $\kappa \geq (1+\delta)E(H)$ where δ is an arbitrary constant.

In this paper we modify the argument in [2] and prove the following two theorems:

Theorem 1. *Let H be a fixed sequence of graphs with $n = |V(H)| \to \infty$ and maximum degree Δ where $(\Delta^2+1)^2 < n$. Suppose that $\kappa \geq (1+\varepsilon)\Delta n/2$, and that*

$$p > \max\left\{ \left(\frac{10 \log \lfloor n/\Delta^2 + 1) \rfloor}{\lfloor n/\Delta^2 + 1) \rfloor} \right)^{1/\Delta}, \frac{10\varepsilon^{-2}\Delta \log n}{n} \right\}. \tag{1}$$

Then w.h.p. $\mathbb{G}_{n,p}^{[\kappa]}$ contains a rainbow colored copy of H.

The original (uncoloured) result of Alon and Furédi [1] differs from Theorem 1 in that the condition $p > 10\varepsilon^{-2}\Delta \log n/n$ is not required, and can be stated as

$$\text{There is an (uncolored) copy of } H \text{ w.h.p. when } p > \left(\frac{10 \log \lfloor n/\Delta^2 + 1) \rfloor}{\lfloor n/\Delta^2 + 1) \rfloor} \right)^{1/\Delta}. \tag{2}$$

Let H be as in Theorem 1. Let

$$e_H(x) = \max\left\{e(F) : F \subset H, v(F) = x\right\}.$$
$$\gamma = \max_{3 \leq x \leq n} \left\{ \frac{e_H(x)}{x-2} \right\}.$$

Theorem 2. *Suppose that $e(H) \geq n$ and $\min\{e(H)p, (1-p)n^{1/2}\} \to \infty$. Suppose that $\kappa \geq (1+\varepsilon)\Delta n/2$, and that*

$$p > \frac{10\varepsilon^{-2}\Delta \log n}{n} \quad \textit{and} \quad np^{\gamma}/\Delta^4 \to \infty.$$

Then w.h.p. $\mathbb{G}_{n,p}^{[\kappa]}$ contains a rainbow colored copy of H.

The corresponding result of Riordan [7] is that with the same definitions as above and in Theorem 2,

$$\text{There is an (uncolored) copy of } H \text{ w.h.p. when } np^{\gamma}/\Delta^4 \to \infty. \tag{3}$$

We consider two examples. In both cases Theorem 2 gives a better result than Theorem 1. First suppose that $n = m^2$ and that H is the $m \times m$ square grid so that $\Delta = 4$. Let $\kappa = 2(1+\varepsilon)n$. Theorem 2 implies that w.h.p. if $\kappa \geq (1+\varepsilon)|E(H)|$ and $np^2 \to \infty$ then G_n, p contains a rainbow copy of H w.h.p. Suppose next that $n = 2^d$ and that H is the d-dimensional hypercube. Theorem 2 implies that w.h.p. if $\kappa \geq (1+\varepsilon)|E(H)|$ and $p \geq \frac{1}{4} + \frac{5 \log d}{d}$ then G_n, p contains a rainbow copy of H w.h.p.

2 Proof of Theorems 1 and 2

The proof idea is quite simple. Let $\mathbb{D}_{d-out}$ be the random digraph with vertex set $[n]$ where each vertex v independently chooses d random neighbors from $[n] \setminus \{v\}$. Let $\mathbb{G}_{d-out}$ be obtained by ignoring orientation. We let $\mathbb{D}_{d-out}^{[\kappa]}$denote a randomly edge colored version of $\mathbb{D}_{d-out}$.

We start with $\mathbb{G}_{n,p}^{[\kappa]}$, we randomly orient the edges to obtain $\mathbb{D}_{n,p/2}^{[\kappa]}$. We show that w.h.p. $\mathbb{D}_{n,p/2}^{[\kappa]}$ contains a rainbow copy of $\mathbb{D}_{d-out}^{[\kappa]}$. We then argue that w.h.p. $\mathbb{D}_{d-out}$ contains a copy of the digraph $\mathbb{D}_{n,p/2}$. So, w.h.p. $\mathbb{D}_{d-out}^{[\kappa]}$ contains a rainbow colored copy of $\mathbb{D}_{n,p/2}^{[\kappa]}$. We ignore orientation of edges to obtain a rainbow colored copy of $\mathbb{G}_{n,p}$, for a relevant value of p. We then apply the relevant theorems of Alon-Furèdi and of Riordan to show the existence of (a copy of) the required subgraph H. This copy must be rainbow coloured because $\mathbb{G}_{n,p}$ is rainbow colored.

We begin with the following lemma, which tightens a lemma from Cooper and Frieze, [2].

Lemma 3. *Suppose that $d \geq 1$, $np \geq 10d\varepsilon^{-2}\log n$ and $\kappa = (1+\varepsilon)dn$. Then w.h.p. $\mathbb{G}_{n,p}^{[\kappa]}$ contains a rainbow colored copy of $\mathbb{G}_{d-out}$.*

Proof. Let p_1 satisfy $1-p = (1-p_1)^2$, so that $p_1 \sim p/2$. Let $\mathbb{D}_{n,p_1}$ be the random digraph where each edge occurs independently with probability p_1. Suppose now that we randomly colour the edges of $\mathbb{D}_{n,p_1}$ with κ colours to obtain the random coloured digraph $\mathbb{D}_{n,p_1}^{[\kappa]}$. Ignoring orientation gives us the random graph $\mathbb{G}_{n,p}^{[\kappa]}$, provided we make a random choice from the two possible colours when coalescing the edges of any directed 2-cycles.

We define a flow network $\mathcal{N}$ as follows. $\mathcal{N}$ has source σ and sink τ. The vertex set W consists of σ, τ, the set of colours $C = [\kappa]$ and the set $V = [n]$ of vertices of the $\mathbb{D}_{n,p_1}^{[\kappa]}$ under consideration.

For each colour $x \in C$ there is an edge (σ, x) in $\mathcal{N}$ of capacity 1. There is an edge (x, v) in $\mathcal{N}$ of infinite capacity for every $v \in V$ for which there is an edge (v, w) in $\mathbb{D}_{n,p_1}^{[\kappa]}$ with colour x. Finally, for each vertex $v \in V$ there is an edge (v, τ) of capacity d.

For $S \subseteq C$, let $N(S) = \{v : \ x \in S,\ v \in V,\ (x, v) \in \mathcal{N}\}$ be the out-neighbour set of S in $\mathcal{N}$. A cut of finite capacity can be obtained from a set $S \subseteq C$ and $N(S) \subseteq V$ as follows. Let $T = N(S)$, $W = \{\sigma\} \cup S \cup T$, and let $\overline{W} = (C \setminus S) \cup (V \setminus T) \cup \{\tau\}$. The capacity of the cut $(W : \overline{W})$ is $\kappa - |S| + d|T|$. Applying the max-flow min-cut theorem we see that $\mathcal{N}$ admits a flow of value dn if and only if, for all $S \subseteq C$,

$$\kappa - |S| + d|N(S)| \geq dn. \tag{4}$$

We estimate the probability that (4) is not true because, for some set S, $|N(S)| < n - (\kappa - |S|)/d$, i.e. there exists a set of colors S of size s and a set of vertices $\overline{T} = V \setminus N(S)$ of size $|\overline{T}| > (\kappa - s)/d$ such that every edge of $D = \mathbb{D}_{n,p_1}^{[\kappa]}$ whose tail is in $\overline{T}$ has a colour in $C \setminus S$.

Since p_1 satisfies $1-p=(1-p_1)^2$ we have $p_1 \geq p/2$, for $p \geq 0$. We see from (1) that $np_1 \geq 5d\varepsilon^{-2}\log n$.

Let $\mathcal{E}$ denote the event that $\delta^+(\mathbb{D}_{n,p_1}) > (1-\varepsilon)np_1$. The Chernoff bounds imply that

$$\mathbb{P}(\overline{\mathcal{E}}) \leq n\mathbb{P}(Bin(n,p_1) \leq (1-\varepsilon)np_1) \leq ne^{-\varepsilon^2 np_1/2} = o(1). \tag{5}$$

Let

$$L(s) = 2\binom{\kappa}{s}\binom{n}{\lceil(\kappa-s)/d\rceil}\left(\frac{\kappa-s}{\kappa}\right)^{(\kappa-s)(1-\varepsilon)np_1/d}$$

be an upper bound on the probability that some set of size s does not satisfy (4) conditional on $\mathcal{E}$. The range of s we need to consider is between $\kappa-dn+1$ and $\kappa-1$. For, if $|S| < \kappa-dn$ then (4) is true with $N(S)=\emptyset$, and if $s=\kappa$ then as $\delta^+(D) \geq (1-\varepsilon)np_1$, $\overline{T}=\emptyset$.

The probability that (4) is not satisfied is therefore bounded by Θ where

$$\Theta = \mathbb{P}(\overline{\mathcal{E}}) + \sum_{s=\kappa-dn+1}^{\kappa-1} L(s). \tag{6}$$

As $\mathbb{P}(\overline{\mathcal{E}}) = o(1)$, we can concentrate on the summation term in (6).

Now, choosing $\kappa \geq (1+\varepsilon)dn$, and putting $\lceil(\kappa-s)/d\rceil = (\kappa-s)/d + f_s$, $0 \leq f_s < 1$,

$$\begin{aligned}
\sum_{s=\kappa-dn+1}^{\kappa-1} L(s) &\leq 2ned \sum_{s=\kappa-dn+1}^{\kappa-1}\left(\frac{\kappa e}{\kappa-s}\left(\frac{ned}{\kappa-s}\right)^{1/d}\left(\frac{\kappa-s}{\kappa}\right)^{n(1-\varepsilon)p_1/d}\right)^{\kappa-s} \\
&\leq 2ned \sum_{s=\kappa-dn+1}^{\kappa-1}\left(e^2\left(\frac{\kappa-s}{\kappa}\right)^{n(1-\varepsilon)p_1/d-2}\right)^{\kappa-s} \\
&\leq 2ned \sum_{s=\kappa-dn+1}^{\kappa-1}\left(\frac{e^2}{(1+\varepsilon)^{n(1-\varepsilon)p_1/d-2}}\right)^{\kappa-s} \\
&= o(1).
\end{aligned} \tag{7}$$

In the second line we used $np_1 \geq 5d\log n/\varepsilon^2$ to imply that $n(1-\varepsilon)p_1/d \gg 1+1/d$, and as $\kappa > n$ to replace $(n/\kappa)^{1/d} < 1$. In the third line we substituted $\kappa-s \leq dn-1$ into $(\kappa-s)/\kappa$. In the last line we used $\kappa-s \geq 1$ in the exponent and $d=O(n/\log n)$ from the bound on np_1 to obtain the final $o(1)$.

Thus w.h.p. $\mathcal{N}$ contains a flow of value of nd. The capacities of $\mathcal{N}$ are integers and so we can assume this flow is integer valued. It decomposes into nd paths from σ to τ each of which assigns a colour x to a vertex v. By construction a colour can be assigned at most once to an edge and each vertex is assigned d colours. This defines a rainbow colored digraph D in which each vertex has out-degree d. It is easy to argue that D is distributed as $\mathbb{D}_{d-out}^{[\kappa]}$. Indeed we could start with $\mathbb{D}_{n,p_1}^{[\kappa]}$ and then replace each edge (v,w) by $(v,\pi_v(w))$ where the π_v, $v \in V$ are independent permutations of $V \setminus \{v\}$. After this transformation the digraph is still distributed as $\mathbb{D}_{n,p_1}^{[\kappa]}$. We run the network flow algorithm and w.h.p. we obtain a rainbow colored digraph D in which each vertex has out-degree d. By replacing each edge (v,w) by $(v,\pi_v^{-1}(w))$ we obtain a rainbow subgraph of the original $\mathbb{D}_{n,p_1}^{[\kappa]}$ which is distributed as $\mathbb{D}_{d-out}^{[\kappa]}$. Ignoring orientation, we have a rainbow colored copy of $\mathbb{G}_{d-out}$. $\square$

Lemma 4. *For every $\varepsilon>0$ there exists $C_\varepsilon>0$ such that if $d\geq C_\varepsilon \log n$ then w.h.p. $\mathbb{G}_{d-out}$ contains a copy of $\mathbb{G}_{n,(2-\varepsilon)d/n}$.*

Proof. Let $p=(2-\varepsilon)d$ and let $1-p=(1-p_1)^2$ and let $D_1=\mathbb{D}_{n,p_1}$ be as in Lemma 3. In the digraph D_1, each $v\in[n]$ independently chooses $Bin(n,p_1)$ random out-neighbors and for large enough C_ε the Chernoff bounds imply that w.h.p. $Bin(n,p_1)\leq(1+\varepsilon/2)np_1\leq d$ for all $v\in[n]$. So, w.h.p. $\mathbb{D}_{d-out}$ contains a copy of $\mathbb{D}_{n,(1-\varepsilon/2)d/n}$ and so after ignoring orientation, $\mathbb{G}_{d-out}$ contains a copy of $\mathbb{G}_{n,(2-\varepsilon)d/n}$. $\square$

We can now prove Theorems 1 and 2. It follows from Lemmas 3 and 4 with $d=\Delta/2$ that w.h.p. $\mathbb{G}_{n,p}^{[\kappa]}$ contains a rainbow copy Γ of $\mathbb{G}_{n,p_1}$ where $p_1=(1-\varepsilon/2)p$. For Theorem 1 we apply the result (2) of Alon and Furédi [1] and for Theorem 2 we apply the result (3) of Riordan [7]. (There is enough slack in the statements of Theorems 1 and 2 to absorb a factor $1+\varepsilon$ in the lower bounds on p.)

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