Rainbow Hamilton Cycles in Random Geometric Graphs

Alan Frieze* and Xavier Pérez-Giménez†

March 5, 2020

Abstract

Let $X_1, X_2, \ldots, X_n$ be chosen independently and uniformly at random from the unit square $[0,1]^d$. Let $r$ be given and let $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$. The random geometric graph $G = G_{\mathcal{X},r}$ has vertex set $\mathcal{X}$ and an edge $X_iX_j$ whenever $|X_i - X_j| \leq r$. We show that if each edge of $G$ is colored independently from one of $n + o(n)$ colors and $r$ has the smallest value such that $G$ has minimum degree at least two, then $G$ contains a rainbow Hamilton cycle a.a.s.

1 Introduction

Given a graph $G = (V,E)$ plus an edge coloring $c : E \to [q]$, we say that $S \subseteq E$ is rainbow colored if no two edges of $S$ have the same color. There has been a substantial amount of research on the question as to when does an edge colored graph contain a Hamilton cycle. The early research was done in the context of the complete graph $K_n$ when restrictions were placed on the colorings. In this paper we deal with the case where we have a random geometric graph and the edges are colored randomly.

In the case of the Erdős-Rényi random graph $G_{n,m}$, Cooper and Frieze [3] proved that if $m \geq 21n \log n$ and each edge of $G_{n,m}$ is randomly given one of at least $q \geq 21n$ random colors then w.h.p. there is a rainbow Hamilton cycle. Frieze and Loh [6] improved this result to show that if $m \geq \frac{1}{2}(n + o(n)) \log n$ and $q \geq (1 + o(1))n$ then w.h.p. there is a rainbow Hamilton cycle. This was further improved by Ferber and Krivelevich [4] to $m = n(\log n + \log \log n + \omega)/2$ and $q \geq (1 + o(1))n$, where $\omega \to \infty$ with $n$. This is best possible in terms of the number of edges. The case $q = n$ was considered by Bal and Frieze [2]. They showed that $O(n \log n)$ random edges suffice.

Let $X_1, X_2, \ldots, X_n$ be chosen independently and uniformly at random from the unit square $[0,1]^d$ where $d \geq 2$ is constant. Let $r$ be given and let $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$. The random geometric graph $G_{\mathcal{X},r}$ has vertex set $\mathcal{X}$ and an edge $X_iX_j$ whenever $|X_i - X_j| \leq r$. Here $|\cdot|$ refers to an arbitrary $\ell_p$-norm, where $1 < p \leq \infty$. Suppose now that each edge of $G_{\mathcal{X},r}$ is given a random color from $[q]$. Call the resulting edge colored graph $G_{\mathcal{X},r,q}$. Bal, Bennett, Pérez-Giménez and Pralat [1] considered the problem of the existence of a rainbow

*Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA, USA, 15213. Research supported in part by NSF grant DMS1661063.
†Department of Mathematics, University of Nebraska-Lincoln, Lincoln NE, USA, 68588. Research supported in part by Simons Foundation Grant #587019.
Hamilton cycle $G_{X,r,q}$. They show that for $r$ at the threshold for Hamiltonicity, $q = O(n)$ random colors are sufficient to have a rainbow Hamilton cycle w.h.p. The aim of this paper is to show that $q = n + o(n)$ colors suffice in this context.

Let $\theta = \theta(d,p)$ denote the volume of the unit $\ell_p$-ball in $d$ dimensions and let $r$ satisfy

$$r^d = \frac{(2/d) \log n + (4 - d - 2/d) \log \log n + \omega}{22^d},$$

where $\omega = o(\log \log n) \to \infty$.

**Theorem 1.** Let $r$ be as in (1), let $\eta > 0$ be arbitrarily small and $q = (1 + \eta)n$. Then w.h.p. $G_{X,r,q}$ contains a rainbow Hamilton cycle.

We actually have the stronger hitting time result. Let

$$\hat{r} = \inf \{ r \geq 0 : G_{X,r} \text{ has minimum degree at least 2} \}.$$

**Theorem 2.** $G_{X,\hat{r},q}$ has a rainbow Hamilton cycle a.a.s.

## 2 Notation and Structure

Let $\varepsilon \ll \eta$ be a constant. We will convention let the set of colors $Q = [m]$ where $m = (1 + 2\eta)n$ and that the colors of the edges of the random geometric graph, $G_{X,r,q}$ are randomly chosen from $Q$. Let $Q_0 = [1, (1 + \eta)n]$ and $Q_1 = [n + \eta n + 1, n + 2\eta n]$.

We divide $[0,1]^d$ into a set $C$ of $N = \frac{1}{(\varepsilon r)^d}$ cells of side $s = \varepsilon r$ where $\varepsilon \ll \eta$. We remark that

$$N \approx \frac{\theta n}{2^{d-1} \varepsilon^d \log n}.$$

The graph of cells $G_C$ is a graph with vertex set $C$ where two cells are adjacent in $G_C$ if their centres are at $\ell_p$-distance at most $r - 2ds$. A cell $C$ is dense if $|C \cap \mathcal{X}| \geq \varepsilon^3 \log n$. Otherwise it is sparse. The set of dense cells is denoted by $\mathcal{D}$ and $G_D$ is the subgraph of $G$ induced by the good cells. The paper [1] shows that a.a.s.

$$G_D \text{ contains a unique giant component } K_g \text{ containing } N - o(N) \text{ cells.}$$

The cells in $K_g$ are good. A cell that is not good, but is adjacent to a cell in $K_g$ is called bad. The remaining cells are called ugly. The following two lemmas describe properties that occur a.a.s and their proofs are in [1]:

**Lemma 3.**

P1 $|C \cap \mathcal{X}| \leq \log n$ for all $C \in C$.

P2 There are at most $n^{1-\varepsilon/2}$ bad cells.

P3 There are at most $n^{O(\varepsilon^{1/d})}$ ugly cells.

P4 The maximum degree in $G_{X,r,q}$ is at most $2d \log n$.

**Lemma 4.** Let $\mathcal{X}_U$ denote the set of points in ugly cells. Then there is a collection of paths $\mathcal{P}$ such that
Q1 $\mathcal{P}$ covers $\mathcal{X}_U$.

Q2 $\mathcal{P}$ covers at most two vertices inside any non-ugly cell.

Q3 Every vertex in $\mathcal{X}$ that is covered by $\mathcal{P}$ is at graph-distance at most $2(20d)^d$ from some vertex in $\mathcal{X}_U$ with respect to the graph $G_C$.

Q4 For each path $P \in \mathcal{P}$, there is a good cell $C_P$ such that the two endvertices of $P$ lie in cells that are adjacent in $G_C$ to $C_P$.

Q5 Every two different paths in $\mathcal{P}$ are at $\ell_p$-distance at least $Ar$ from each other, $A$ arbitrary.

Q6 The total number of edges on the paths in $\mathcal{P}$ is $n^{O(\varepsilon^{1/d})}$.

For a cell $C$ we let $V(C) = C \cap \mathcal{X}$ and $E(C) = (C \cap \mathcal{X})^2 \subseteq E(G)$. A cell is rainbow if $E(C)$ is rainbow.

We now prove some lemmas related to the colorings of cells.

**Lemma 5.** A.a.s. (a) there are at most $\log^4 n$ non-rainbow cells, (b) no cell contains 3 repetitions of a color, (c) all non-rainbow cells are good, (d) there are no non-rainbow cells within distance $10r$ of the boundary of $[0,1]^d$ and (e) there are no bad cells adjacent to a non-rainbow cell.

**Proof.** Now for a fixed cell $C$,

$$\mathbb{P}(C \text{ is not rainbow colored} \mid P1) \leq (1 + 2\eta)n\mathbb{P}\left(\text{Bin}\left(\log^2 n, \frac{1}{(1 + 2\eta)n}\right) \geq 2\right) = O\left(\frac{\log^4 n}{n}\right). \quad (3)$$

(a) We then have, by the Markov inequality that

$$\mathbb{P}(\neg (a) \mid P1) \leq O\left(\frac{N \log^4 n}{n \log^4 n}\right) = O\left(\frac{1}{\log n}\right).$$

**Explanation:** we choose a cell $C$ and a color $c$. Then the number of edges of color $c$ in cell $C$ is dominated by the stated binomial.

(b) We have that

$$\mathbb{P}(\neg (b) \mid P1) \leq (N + 2\eta)n\mathbb{P}\left(\text{Bin}\left(\log^2 n, \frac{1}{(1 + 2\eta)n}\right) \geq 3\right) + N(1 + 2\eta)^2 n^2\mathbb{P}\left(\text{Bin}\left(\log^2 n, \frac{1}{(1 + 2\eta)n}\right) \geq 2\right)^2 = O\left(\frac{\log^7 n}{n}\right).$$

**Explanation:** The first term in the upper bound for 3 edges of the same color and the second term accounts for two pairs of edges with the same color.

(c) We have that

$$\mathbb{P}(\neg (c) \mid P1, P2, P3) \leq 2n^{1 - \varepsilon/2}(1 + 2\eta)n \times O\left(\frac{\log^8 n}{n^2}\right) = O(n^{-\varepsilon/3}).$$

(d) There are $O(1/r)$ cells within $10r$ of the boundary and so

$$\mathbb{P}(\neg (d)) = O\left(\frac{\log^4 n}{rn}\right) = o(1).$$

3
(e) The graph $G_C$ has maximum degree $O(1)$ and so
\[
\mathbb{P}(\neg (e) \mid P2) \leq O\left(\frac{n^{1-\varepsilon/2} \log^4 n}{n}\right) = o(1).
\]
\[\square\]

We add the non-rainbow cells of Lemma 5 to the set of bad cells. This will remove some cells from $K_g$. We argue next that $K_g$ remains connected.

**Lemma 6.** $K_g$ remains connected a.a.s. after the deletion of the non-rainbow cells.

**Proof.** We now refer to the cells in $C$ as $C(i) : i \in [M]^d$ where $M = N^{1/d}$. We first claim the following.

If $C(i), C(i')$ are not rainbow, then $||i - i'||_1 > 2$. \hspace{1cm} (4)

For a fixed $i$, observe that there at most $d^2$ cells $i'$ such that $||i' - i||_1 \leq 2$. It follows from (3) that
\[
\mathbb{P}(\neg (4)) \leq O\left(\frac{d^2 \log^4 n}{n}\right) = o(1).
\]

We also have
\[
\mathbb{P}(\exists C, C' \in C : C \text{ is good but non-rainbow}, C' \text{ is not good}, \{C, C'\} \in E(G_C) \mid P2, P3) = O\left(\frac{n^{1-\varepsilon/2} 5^d \log^4 n}{n}\right) = o(1). \hspace{1cm} (5)
\]

So, suppose now that $C$ is a non-rainbow good cell. Referring to $||i' - i||_1$ as the $i$-distance between $C(i), C(i')$, we let $N_2(C)$ denote the cells with $i$-distance 2 of $C$ in $G_C$, excluding $C$ itself. Note that the cells in $N_2(C)$ are neighbors of $C$ in $G_C$. Let $K_g^b$ denote $K_g$ before we remove the non-rainbow cells and let $K_g^a$ denote $K_g^b$, less the cells that have been removed. We can assume by [5] that $N_2(C) \subseteq K_g^a$. Suppose there is a path $P$ in $K_g^b$ that contains $C$ as an interior vertex. Let $C_1 = C(i_1), C_2 = C(i_2)$ be the neighbors of $C = C(i)$ on $P$ where $i_k = i + \theta_k, k = 1, 2$. The lemma now follows from the fact that there is a path $Q$ from $C_1$ to $C_2$ that only uses cells at $i$-distance at most one from $C$. If $\theta_1 \neq -\theta_2$ then we consider the sequence $(i_1, i_1 + \theta_2, i_2 + \theta_1, i_2)$ which defines a path of length 3 in $K_g^b$ from $C_1$ to $C_2$. If $\theta_1 = -\theta_2$ then we use the sequence $i_1, i_1 + \theta_3, i + \theta_3, i + \theta_3 - \theta_1, i_2$ where w.l.o.g. $\theta_1 = (1, 0, \ldots, 0)$ and $\theta_3 = (0, 1, 0, \ldots, 0)$. \[\square\]

**Lemma 7.** The edges on $P$ are rainbow colored a.a.s.

**Proof.** The probability the statement of the lemma fails can be bounded by $\left(\frac{n^{O(1/d)}}{2}\right) \times n^{-1} = o(1)$. \[\square\]

[XP: moved here: ] Let $B_1$ denote the colors on edges that are incident with vertices in bad or ugly cells or lie on a path in $P$.

**Lemma 8.** No cell $C$ is incident with $k_0 = 20/\varepsilon$ edges with colors from $B_1$, excluding colors from $C$ if it is bad or ugly.

**Proof.** Then
\[
\mathbb{P}(\exists C \mid P1, P2, P3, P4) \leq N\left(\frac{2^d \log^2 n}{k_0}\right) \left(\frac{2^d n^{1-\varepsilon/3} \varepsilon^3 \log^2 n}{(1 + \eta)n}\right)^{k_0} \leq n^{1+o(1)-k_0 \varepsilon/10} = o(1).
\]

**Explanation:** P1 and P4 imply that no cell is incident to more than $2^d \log^2 n$ edges. P2, P3 and P4 imply that $|B_1| \leq 2^d n^{1-\varepsilon/3} \varepsilon^3 \log n$. \[\square\]
We now prove some lemmas concerning the existence of paths in $G_{m,p}$.

**Lemma 9.** Suppose that $0 < p < 1$ is constant. Then,

(a) $\mathbb{P}(G_{m,p} \text{ is not Hamiltonian}) \leq e^{-mp/4}$ for $m$ sufficiently large.

(b) Let $\psi(G)$ denote the minimum number of vertex disjoint paths that cover the vertices of $G$. Then if $k = O(1)$ then

$$\mathbb{P}(\psi(G_{m,p}) \geq k) \leq e^{-kmp/4}.$$  

Proof.

(a) We first observe that if $N_G(S)$ is the disjoint neighborhood of $S$ in a graph $G$ then with $G = G_{m,p}$,

$$\mathbb{P}(\exists S, 1 \leq |S| \leq m/6 : |N_G(S)| \leq 2|S|) \leq \sum_{s=1}^{m/6} \binom{m}{s} \left( \frac{m}{2s}(1-p)^{s(m-3s)} \leq \sum_{s=1}^{m/6} \frac{me}{s} \cdot \frac{m^2e^2}{4s^2} \cdot e^{-mp/2} \right)^s \leq e^{-mp/3}.$$  

So, writing $G_{m,p} \supseteq G_{m,p/2} \cup G_{m,p/2}$ and applying Pósa’s argument, we see that, if $q = p/2$ then

$$\mathbb{P}(G_{m,p} \text{ is not Hamiltonian}) \leq e^{-mq/3} + m \exp \left\{ - \left( \frac{m/6}{2} \right)^q \right\} \leq e^{-mp/4}.$$  

Outline explanation: Given that $|N_G(S)| > 2|S|$ for $G = G_{m,q}, |S| \leq m/6$, Pósa’s lemma implies that there are at least $\binom{m}{2}^6$ non-edges, whose addition to a connected non-Hamiltonian graph will increase the length of the longest path length by one. (If the graph has a Hamilton path, then there are this number of edges that create a Hamilton cycle.) See [5], Chapter 6 for more details.

(b) Let $V_\ell$ be the set of vertices of degree at most $\ell = O(1)$ in $G_{m,p}$. Then for $r = O(1),$

$$\mathbb{P}(|V_\ell| \geq r) \leq \binom{m}{r} \mathbb{P}(\text{Bin}(m - r, p) \leq \ell)^r \leq m^r \left( \sum_{i=0}^{\ell} \binom{m-r}{i} p^i (1-p)^{m-r-i} \right)^r \leq m^{r+\ell r} e^{r(r+\ell-m)p} \leq e^{-(m-o(m))pr}.$$  

Suppose now that we arbitrarily add edges to vertices of degree at most $3k$ in $G_{m,p}$ so that the new graph $H$ has minimum degree $\ell = 3k$. Arguing as in (a) we see that

$$\mathbb{P}(\exists S, 1 \leq |S| \leq m/6 : |N_H(S)| \leq 2|S|) \leq \sum_{s=k}^{m/6} \binom{m}{s} \left( \frac{m}{2s}(1-p)^{s(m-3s)} \leq \sum_{s=k}^{m/6} \frac{me}{s} \cdot \frac{m^2e^2}{4s^2} \cdot e^{-mp/2} \right)^s \leq e^{-kmp/3}.$$  

It then follows that

$$\mathbb{P}(H \text{ is not Hamiltonian}) \leq e^{-kmp/3} + m \exp \left\{ - \left( \frac{m/6}{2} \right)^q \right\} \leq e^{-kmp/4}.$$  

Now if $|V_{3k}| \leq k$ and $H$ is Hamiltonian, we have $\psi(G_{m,p}) \leq k$.  

\[\square\]
3 Coloring procedure

We now describe how we select our rainbow Hamilton cycle.

3.1 Good cells

Let $T_C$ be a spanning tree of the giant $K_p$. Suppose that $C_1, C_2, \ldots, C_M$ is an enumeration of the good cells that follows from a depth first search of $T_C$. We examine them in this order and when we reach $C_i$ we will have constructed a cycle $H_{i-1}$ through $V(C_1) \cup \cdots \cup V(C_{i-1})$. Let $C_\pi(i)$ denote the parent of $C_i$ in this search. Let $A$ denote the allowable colors at this point, i.e. those colors not used in $H_{i-1}$ or in $B_1$. Let $G_i$ denote the edge-colored subgraph of $G$ induced by $V(C_i)$ and the edges with colors in $A$. We note that $G_i$ contains a copy of $G_{m,p}$ where $m \geq \varepsilon^3 \log n$ and $p = \eta/2$. We first try to construct a cycle $D_i$ using edges of color in $A \cap Q_0$. Note that $D_i$ is necessarily a rainbow cycle. It follows from Lemma 9(a) that we succeed with probability at least $1 - n^{-\varepsilon^3/4}$. So, a.a.s. we fail to construct $D_i$ at most $o(n)$ times overall.

For $j < i$, let $E_{i,j}$ denote the edges of $H_{i-1}$ that are incident with exactly one point in $C_j$. We justify the following claim later.

Claim A: A.a.s., at all times in the coloring process, if $j < i$, then $|E_{i,j}| = o(\log n)$.

(a) If $C_i$ contains a spanning cycle $D_i$ then we try to patch it into $H_{i-1}$ as follows: we search for an edge $e_1 = \{a, b\}$ of $D_i$ and an edge $e_2 = \{c, d\}$ of $H_{i-1} \cap E(C_\pi(i-1))$ such that (i) $f_1 = \{a, c\}, f_2 = \{b, d\}$ are both edges of $G$ and (ii) $f_1, f_2$ use colors from $Q_1 \setminus B_1$ that have not been used in $H_{i-1}, D_i$. We then delete $e_1, e_2$ and replace them with $f_1, f_2$ creating $H_i$.

We note that to this point we have used $o(n)$ patching colors from $Q_1$ and that the probability of failing to patch $D_i$ is at most $(1 - \eta^2/4)^{\Omega(\log^2 n)} = n^{-\omega(1)}$. To obtain this bound, we see that there are $\Omega(\log n)$ choices for each of $e_1$ and for each such choice there are $\Omega(\log n)$ choices for $e_2$. We can make these choices so that an edge can occur at most once as $f_1, f_2$ and so all these possibilities are independent. Finally, the probability that both of $f_1, f_2$ are both acceptable is at least $(\eta/2)^2$. We need Claim A to justify the number of choices for $e_2$. So, in this case we can assume that $D_i$ is patched into $H_{i-1}$.

We can select the cycle $D_i$ by exposing the edges that use available colors, but without exposing the actual color. Of course, having selected $D_i$ we expose the colors of its edges. We call this deferred coloring.

(b) There is no $D_i$. Lemma 9(b) implies that if $\psi_0 = \frac{5}{\varepsilon^3 \eta}$ then

\[ \mathbb{P}(\exists \text{ good cell } C : \psi(G_i) \geq \psi_0) \leq n e^{-\psi_0 \varepsilon^3 \eta \log n / 40} = o(1). \] (6)

We expose the set of paths using deferred coloring. Let $P$ be one of the at most $\psi_0$ paths and let $c, d$ be the endpoints of $P$. We now consider all edges $f_i = \{c_i, d_i\}, i = 1, 2, \ldots, k$ of $H_{i-1}$ that are within distance $r$ of $c, d$. Here $k \geq \alpha \log n$, $\alpha \geq \varepsilon^3 / 2 - o(1)$. We consider adding $P$ to $H_{i-1}$ by deleting $f_i$ and adding edges $\{c_i, c\}, \{d_i, d\}$. We say that such a swap is a failure, if the new cycle is not rainbow. The probability that all swaps lead to a failure is at most $(1 - \eta^2/4)^{\alpha \log n / 2} \leq n^{-\xi_0}, \xi_0 = \eta^2 \varepsilon / 8$. (We estimate $\alpha \log n / 2$ by taking every other pair and the $o(1)$ accounts for the edges of $H_{i-1}$ alluded to in Claim A.

If all swaps lead to non-rainbow cycles, then we consider making the swap involving $f = f_1$ and then deleting the unique edges $g_l = \{x_l, y_l\}, l = 1, 2$ of the same color as $\{c, c\}, \{d, d\}$ in $H_{i-1}$. First consider the
path \( \hat{P} \) that covers all the vertices of \( H_{i-1} \) and \( P \) that is obtained by adding both edges \( \{c, c_1\}, \{d, d_1\} \) and then deleting \( g_1 \). We will subsequently delete \( g_2 \) using the method proposed next. Suppose that \( x = x_1 \) lies in cell \( C_x \) and \( y = y_1 \) lies in cell \( C_y \). What we do now is to search for an edge \( \{u, v\} \) of \( E(\hat{P}) \) with \( u \in E(C_x), v \in E(C_y) \) such that the cycle \( \hat{P} + \{u, x\} + \{v, y\} - \{u, v\} \) is rainbow. In this attempt, we can restrict ourselves to edges \( \{u, x\}, \{v, y\} \) with colors in \( Q_1 \). Because (i) we will only search for \( o(n) \) edges of color in \( Q_1 \), and (ii) because

**Claim B:** a.a.s., no cell occurs as \( C_x \) or \( C_y \) more than \( o(\log n) \) times,

and (iii) because every point in \( C_x \) is within \( \ell_p \) distance \( r \) of every point in \( C_y \), we see that we succeed with probability at least \( 1 - n^{-\xi_0} \). Having dealt with \( g_1 \) we follow the same procedure with \( g_2 \). So, with probability at least \( 1 - 2n^{-\xi_0} \) we succeed in adding \( P \) to \( H_{i-1} \) to create a rainbow cycle. If we fail, then we are left with a cycle containing one or two repeated colors. We think of this as the first stage in a branching process that creates repeated colors.

We repeat the same process to try to remove any repeated colors. The expected number of repeated colors after \( l \) stages is \((4n^{-\xi_0})^l\). So, with probability at least \( 1 - n^{-2+o(1)} \) we can say that after at most \( 2/\xi_0 \) stages, we have created a rainbow cycle through the vertices of \( P \) and \( H_{i-1} \). Furthermore, we will have used at most \( 2^{1/\xi_0} \) colors from \( Q_1 \). We repeat the same process to absorb all the paths of \( C_i \) to create \( H_i \).

We must now deal with the claims A and B. We note that Claim B implies Claim A. The only way that \( E_{i,j} \) can get too large is through a cell being chosen too many times as \( C_x \). It cannot get too large because it occurs many times as \( C_{\pi(i)} \). This follows from the fact that \( G_c \) has maximum degree \( \varepsilon^{-3} \). At any stage, for a cell \( C \), the probability it is chosen in the addition of \( C_i \) is equal to

\[
P((b) \text{ occurs and edge } f_1 \text{ uses a repeated color}) \times P(\text{color appears in } C) = O \left( \frac{i \log n}{n^{1+\eta^{-3}/4}} \cdot \frac{1}{i} \right) = O \left( \frac{1}{N n^{\eta^{-3}/4}} \right). \tag{7}
\]

So, using the Markov inequality, a.a.s. the number of times that \( C \) occurs as \( C_x \) is dominated by the maximum size of a box when \( O(N n^{-\eta^{-3}/8}) \) balls are thrown randomly into \( N \) boxes. And we note that a.a.s. this maximum is \( O(1) \).

Let now \( H \) denote the rainbow cycle that uses all the points in good cells.

### 3.2 Bad cells

We note that \( H \) does not use any color incident with a point in non-good cells. We add the points in bad cells, cell by cell, growing \( H \) as we go. We avoid using the colors in the paths \( \mathcal{P} \) promised in Lemma 8. If the cell \( C \) contains more than \( 2k_0 + 1 \) points then because it forms a clique in \( G \), it contains \( k_0 \) edge disjoint Hamilton cycles (w.r.t. \( X \cap C \)) and Lemma 8 implies that at least one of these \( H_C \) is free from colors repeated in other non-good cells. We can then patch \( H_C \) into \( H \) avoiding edges from \( B_1 \). This follows from the fact that bad cells have at least one good neighbor. If \( C \) contains at most \( 2k_0 \) vertices then we insert these vertices one by one. We can do this because for a good cell \( C \) there are at most \( \pi/\varepsilon^2 \) bad cells contained in the disk with the same center as \( C \) and radius \( r \).
3.3 Ugly cells

We patch the points in the ugly cells into $H$ by using the paths $\mathcal{P}$ of Lemma 4. By avoiding colors repeated in other non-good cells, we have avoided using the colors of $\mathcal{P}$. We also require the claim in Lemma 7.

References


