Karp’s patching algorithm on random perturbations of dense digraphs

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Abstract

We consider the following question. We are given a dense digraph $D_0$ with minimum in- and out-degree at least $\alpha n$, where $\alpha > 0$ is a constant. We then add random edges $R$ to $D_0$ to create a digraph $D$. Here an edge $e$ is placed independently into $R$ with probability $n^{-\epsilon}$ where $\epsilon > 0$ is a small positive constant. The edges of $D$ are given edge costs $C(e), e \in E(D)$, where $C(e)$ is an independent copy of the exponential mean one random variable $\text{EXP}(1)$ i.e. $\mathbb{P}(\text{EXP}(1) \geq x) = e^{-x}$. Let $C(i,j), i,j \in [n]$ be the associated $n \times n$ cost matrix where $C(i,j) = \infty$ if $(i,j) \notin E(D)$. We show that w.h.p. the patching algorithm of Karp finds a tour for the asymmetric traveling salesperson problem that is asymptotically equal to that of the associated assignment problem. Karp’s algorithm runs in polynomial time.

1 Introduction

Let $\mathcal{D}(\alpha)$ be the set of digraphs with vertex set $[n]$ and with minimum in- and out-degree at least $\alpha n$. We are given a digraph $D_0 \in \mathcal{D}(\alpha)$ and then we add random edges $R$ to $D_0$ to create a digraph $D$. Here an edge $e$ is placed independently into $R$ with probability $n^{-\epsilon}$ where $\epsilon > 0$ is a small positive constant. The edges of $D$ are given independent edge costs $C(e), e \in E(D)$, where $C(e)$ is a copy of the exponential mean one random variable $\text{EXP}(1)$ i.e. $\mathbb{P}(\text{EXP}(1) \geq x) = e^{-x}$. Let $C(i,j), i,j \in [n]$ be the associated $n \times n$ cost matrix where $C(i,j) = \infty$ if $(i,j) \notin E(D)$. One is interested in using the relationship between the Assignment Problem (AP) and the Asymmetric Traveling Salesperson Problem (ATSP) associated with a cost matrix $A(i,j), i,j \in [n]$ to asymptotically solve the latter.

The problem AP is that of computing the minimum cost perfect matching in the complete bipartite graph $K_{n,n}$ when edge $(i,j)$ is given a cost $C(i,j)$. Equivalently, when translated to the complete digraph $\vec{K}_n$ it becomes the problem of finding the minimum cost collection of vertex disjoint directed cycles that cover all vertices. The problem ATSP is that of finding a single cycle of minimum cost that covers all vertices. As such it is always the case that $v(\text{ATSP}) \geq v(\text{AP})$ where $v(\bullet)$ denotes the optimal cost. Karp \cite{Karp} considered

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the case where \(D = \tilde{K}_n\). He showed that if the cost matrix is comprised of independent copies of the uniform \([0, 1]\) random variable \(U(1)\) then w.h.p. \(v(\text{ATSP}) = (1 + o(1))v(\text{AP})\). He proves this by the analysis of a patching algorithm. Karp’s result has been refined in [21], [28] and [34].

**Karp’s Patching Algorithm:** First solve AP to obtain a minimum cost perfect matching \(M\) and let \(A_M = \{C_1, C_2, \ldots, C_\ell\}\) be the associated collection of vertex disjoint cycles covering \([n]\). Then patch two of the cycles together, as explained in the next paragraph. Repeat until there is one cycle. A pair \(e = (x, y), f = (u, v)\) of edges in different cycles \(C_1, C_2\) are said to be a patching pair if the edges \(e' = (u, y), f' = (x, v)\) both exist. In which case we can replace \(C_1, C_2\) by a single cycle \((C_1 \cup C_2 \cup \{e', f'\}) \setminus \{e, f\}\). The edges \(e, f\) are chosen to minimise the increase in cost of the set of cycles.

**Theorem 1.** Suppose that \(D_0 \in \mathcal{D}(\alpha)\), \(\alpha > 0\) where \(\alpha\) is constant. Suppose that \(D\) is created by adding random edges \(R\) to \(D_0\) and that each edge of \(D\) is given an independent \(\text{EXP}(1)\) cost. Here an edge \(e\) is placed independently into \(R\) with probability \(n^{-\varepsilon}\) where \(\varepsilon > 0\) is a small positive constant. Then w.h.p. \(v(\text{ATSP}) = (1 + o(1))v(\text{AP})\) and Karp’s patching algorithm finds a tour of the claimed cost in polynomial time.

The use of \(\text{EXP}(1)\) as opposed to \(U(1)\) is an artifact of our proof, but see Section 5.

This model for instances of the ATSP arises in the following context: It is often the case that adding some randomness to a combinatorial structure can lead to significant positive change. Perhaps the most important example of this and the inspiration for a lot of what has followed, is the seminal result of Spielman and Teng [45] on the performance of the simplex algorithm, see also Vershynin [47] and Dadush and Huiberts [15].

The paper [45] inspired the following model of Bohman, Frieze and Martin [10]. They consider adding random edges to an arbitrary member \(G\) of \(\mathcal{G}(\alpha)\). Here \(\alpha\) is a positive constant and \(\mathcal{G}(\alpha)\) is the set of graphs with vertex set \([n]\) and minimum degree at least \(\alpha n\). They show that adding \(O(n)\) random edges to \(G\) is enough to create a Hamilton cycle w.h.p. This is in contrast to the approximately \(\frac{1}{2}n \log n\) edges needed if we rely only on the random edges. Research on this model and its variations has been quite substantial, see for example [6], [8], [9], [11], [12], [16], [20], [29], [36], [37], [38], [42], [43], [44], [46].

Anastos and Frieze [5] introduced a variation on this theme by adding color to the edges. They consider rainbow Hamiltonicity and rainbow connection in the context of a randomly colored dense graph with the addition of randomly colored edges. Aigner-Horev and Hefetz [2] strengthened the Hamiltonicity result of [5] and Balogh, Finlay and Palmer strengthened the rainbow connectivity result of [5]. The model in this paper was introduced by Frieze [24].

**Notation** Let \(G\) denote the bipartite graph with vertex partition \(A = \{a_1, a_2, \ldots, a_n\}, B = \{b_1, b_2, \ldots, b_n\}\) and an edge \(\{a_i, b_j\}\) for every directed edge \((i, j) \in D\). A matching \(M\) of \(G\) induces a collection \(\mathcal{A}_M\) of vertex disjoint paths and cycles in \(D\) and vice-versa. If the matching is perfect, then there are only cycles.

The proof requires a number of definitions of graphs, digraphs and trees. It might be helpful to the reader if we list them now along with their definitions. This will only be useful for referring back to and so some of the
items here are not yet defined.

$G_r$ : This is the bipartite subgraph of $G$ induced by $A_r, B_r$.

$G_r(u,v)$ : This is the subgraph of $G_r$ induced by the edges $(a_i,b_j)$ for which $u_i + v_j > 0$.

$\Gamma_r$ : This is the graph obtained from $G_r(u,v)$ by contracting $M_r$.

$\Gamma_r^+$ : This is the subgraph of $\Gamma_r$ induced by the edges $(a_i, a_j)$ for which $u_i + v_{\phi(a_j)} > 0$.

$\bar{G}_r$ : This is digraph obtained by orienting the edges of $G_{r+1}$ from $A_{r+1}$ to $B_{r+1}$, except for the edges of $M_r$, which are oriented from $B_r$ to $A_r$.

$\bar{\Gamma}_r$ : This is digraph obtained from $\bar{G}_r$ by contracting the matching edges.

$T_r$ : This is the spanning tree of $G_r$ corresponding to an optimal basis.

$\hat{T}_r$ : This is the tree obtained from $T_r$ by contracting $M_r$.

$\tilde{T}_r$ : This is a tree obtained from $\hat{T}_r$ by orienting edges.

$DT_k$ : This is the tree comprising the first $k$ vertices selected by Dijkstra’s algorithm below.

2 Proof of Theorem 1

We begin by solving AP. We prove the following:

**Lemma 2.** W.h.p., the solution to AP contains only edges of cost $C(i,j) \leq \gamma_n = n^{-(1-2\varepsilon)}$.

**Lemma 3.** W.h.p., after solving AP, the number $\nu_C$ of cycles is at most $r_0 \log n$ where $r_0 = n^{1-3\varepsilon}$.

Bounding the number of cycles has been the most difficult task. Karp proves $O(\log n)$ w.h.p. for the complete digraph $\tilde{K}_n$. Karp’s proof is very clean but rather fragile. It relies on the key insight that if $D = \tilde{K}_n$ then the optimal assignment comes from a uniform random permutation. This seems unlikely to be true in general and this requires building a proof from scratch.

Given Lemmas 2, 3, the proof is straightforward. We can begin by temporarily replacing costs $C(e) > \gamma_n$ by infinite costs in order to solve AP. Lemma 2 implies that w.h.p. we get the same optimal assignment as we would without the cost changes. Having solved AP, the memoryless property of the exponential distribution, implies that the unused edges in $E(D)$ of cost greater than $\gamma_n$ have a cost which is distributed as $\gamma_n + EXP(1)$.

Let $C = C_1, C_2, \ldots, C_\ell$ be a cycle cover and let $k_i = |C_i|$ where $k_1 \leq k_2 \leq \cdots \leq k_\ell$, $2 \leq \ell \leq r_0 \log n$. (There is nothing more to do if $\ell = 1$.) Different edges in $C_i$ give rise to disjoint patching pairs. We ignore the saving associated with deleting $e, f$ and only look at the extra cost $C(e') + C(f')$ incurred. We will also only consider the random edges $R$ when looking for a patch. The number of possible patching pairs $\nu_C$ satisfies

$$\nu_C \geq \sum_{i<j} k_ik_j = \frac{1}{2} \left( n^2 - \sum_{i=1}^{\ell} k_i^2 \right) \geq \frac{1}{2} \left( n^2 - ((n-\ell+1)^2 + \ell - 1) \right) \geq \frac{\ell n}{2}.$$  

Each of these $\nu_C$ pairs uses a disjoint set of edges. We define the sets

$$R_\ell = \left\{ e \in R : C(e) \leq \gamma_n + \frac{1}{(\ell n^{1-5\varepsilon/2})^{1/2}} \right\}, 1 \leq \ell \leq r_0.$$
Each edge of $E(\bar{K}_n) \setminus E(D_0)$ appears in $R_\ell$ with probability at least $p_\ell = n^{-\varepsilon} (\frac{1}{\ell n^{1-5\varepsilon/2}})^{1/2}$, independent of other edges. Let $E_{\ell}$ be the event that $|C| = \ell$ and that there is no patch using only edges in $R_\ell$. If $E_{\ell}$ does not occur then we reduce the number of cycles by at least one. We have

$$\mathbb{P}(E_{\ell}) \leq (1 - p_\ell^2)^{\ell n/2} \leq \exp \left\{ -\frac{\ell n}{2} \cdot n^{-2\varepsilon} \cdot \frac{1}{\ell n^{1-5\varepsilon/2}} \right\} = e^{-n^{\varepsilon/2}/2}.$$  

It follows that

$$\mathbb{P}(\exists 2 \leq \ell \leq r_0 : E_{\ell}) \leq \sum_{\ell = 2}^{r_0} \mathbb{P}(E_{\ell} \mid \neg E_{\ell+1} \wedge \cdots \wedge \neg E_{r_0}) \leq \sum_{\ell = 2}^{r_0} \frac{\mathbb{P}(E_{\ell})}{1 - \sum_{k = \ell+1}^{r_0} \mathbb{P}(E_k)} \leq \sum_{\ell = 2}^{r_0} \frac{e^{-n^{\varepsilon/2}/2}}{1 - r_0 e^{-n^{\varepsilon/2}/2}} = o(1).$$

And w.h.p. the patches involved in these cases add at most the following to the cost of the assignment:

$$\sum_{\ell = 1}^{r_0 \log n} \left( \gamma + \frac{1}{(\ell n^{1-5\varepsilon/2})^{1/2}} \right) \leq r_0 \gamma + \left( \frac{2r_0}{n^{1-5\varepsilon/2}} \right)^{1/2} = o(1). \quad (1)$$

Given the last equality and the fact that w.h.p. $v(AP) > (1 - o(1))\zeta(2) > 1$ we see that Karp’s patching heuristic is asymptotically optimal. The lower bound of $(1 - o(1))\zeta(2)$ on $v(A)$ comes from [4].

**Remark 1.** When $\alpha > 1/2$ we believe that can modify the proofs of Lemmas 2 and 3 and avoid the need for the set R. At the present moment there are some difficulties in modifying this section.

### 3 Proof of Lemma 2

We show that w.h.p. for any pair of vertices $a \in A, b \in B$ and any perfect matching between $A$ and $B$ that there is an $M$-alternating path from $a$ to $b$ that only uses non-$M$ edges of cost at most $\gamma_n$. And for which the difference in cost between added and deleted edges is also at most $\gamma_n$. We need to prove a slightly more general version where $r \geq r_0$ replaces $n$, see Lemma 6.

The idea of the proof is based on the fact that w.h.p. the sub-digraph induced by edges of low cost is a good expander. There is therefore a low cost path between every pair of vertices. Such a path could be used to replace and expensive edge.

Assume now that $a_1, a_2, \ldots, a_n$ is a random permutation of $A$ and similarly for $B$. For $r \geq r_0$ we let $A_r = \{a_1, a_2, \ldots, a_r\}$ and $B_r = \{b_1, b_2, \ldots, b_r\}$. We let $G_r$ denote the subgraph of $G$ induced by $A_r \cup B_r$.

**Lemma 4.** If $r \geq r_0$ then $G_r$ is (i) connected and (ii) contains a perfect matching with probability $1 - o(n^{-1})$.

**Proof.** If $p = m/2n^2 = n^{-\varepsilon}/2$ then adding edges to $D_0$ with probability $p$ will add fewer than $m$ random edges w.h.p. On the other hand the probability that $K_{r,r,p}$ has a perfect matching is $1 - o(n^{-1})$ if $r \geq r_0$. This is because, as shown by Erdős and Rényi [22], the probability there is no perfect matching is dominated by the probability that there is an isolated vertex. And this is at most $2r(1-p)^r \leq 2ne^{-rn^{-\varepsilon}} = o(n^{-1})$. A similar argument deals with connectivity. \hfill \square
**Proof.** For a fixed $T$ suppose that the solution $\phi^{-1}(T)$ contains only edges of cost $C(i, j) \leq \beta_r$. The existence of such a path with $k \leq 5\varepsilon^{-1}$ contradicts the optimality of $M_r$. Therefore, we have that $C(i, j) > \beta_r$ for all $i, j \in V$.

Now consider the sequence of sets $S_0 = \{a_1\}, S_1, S_2, \ldots \subseteq A, T_1, T_2, \ldots \subseteq B$ defined as follows:

$T_i = N_0\left(\bigcup_{j<i} S_j\right)$ and $S_i = \phi^{-1}(T_i)$, where $M_r = \{(a_i, \phi(a_i)) : i = 1, 2, \ldots, r\}$. It follows from (2) that w.h.p.

$$|S_i| = |T_i| \leq r^{1-\varepsilon} \text{ implies that } |S_i| \geq \left(\frac{\varepsilon r^\varepsilon}{40}\right)^i.$$

So define $i_0$ to be the smallest integer $i$ such that $\left(\frac{\varepsilon r^\varepsilon}{40}\right)^i \geq r^{1-\varepsilon}$. Note that $i_0 \geq 2/\varepsilon$. Thus w.h.p. $|S_{i_0}| \geq r^{1-\varepsilon}$. Replace $S_{i_0}$ by a subset of $S_{i_0}$ of size $r^{1-\varepsilon}$ and then after this, we have that w.h.p. $|T_{i_0+1}| \geq \frac{\varepsilon r^\varepsilon}{40}$.

For a set $T \subseteq B$, we let

$$\tilde{N}_0(T) = \{a_i \in A_r : \exists b_j \in T \text{ such that } (a_i, b_j) \in E(D) \text{ and } C(i, j) \leq \beta_r\}.$$
We then define $\hat{T}_0 = \{b_1\}, \hat{T}_1, \hat{T}_2, \ldots, \hat{T}_{i_0+1} \subseteq B, \hat{S}_1, \hat{S}_2, \ldots, \hat{S}_{i_0+1} \subseteq A_r$ by $\hat{S}_i = \tilde{N}_0 \left( \bigcup_{j<i} \hat{T}_j \right)$ and $\hat{T}_i = \phi(\hat{S}_i)$ and argue that $|\hat{T}_{i_0+1}| \geq \frac{e^r}{40}$ with probability $1 - e^{-\Omega(\varepsilon r^4)}$.

For $S \subseteq A_r, T \subseteq B_r$ let $E_R(S, T) = \{a_i \in S, b_j \in T : (i, j) \in R, C(i, j) \leq \beta_r \}$.

Then,

$$
\mathbb{P} \left( \exists S \subseteq A_r, T \subseteq B_r : |S|, |T| \geq \frac{\varepsilon r}{40}, E_R(S, T) = \emptyset \right) \leq 2^{2^r} \exp \left\{ -\frac{\varepsilon^2 r^2}{16000r^{1-2\varepsilon}} \right\} = e^{-\Omega(r^{1+2\varepsilon})}.
$$

It follows that there will be an edge in $E_R(S_{i_0+1}, \hat{T}_{i_0+1})$ and we have found an alternating path of length at most $2i_0 + 3$ using edges of cost at most $\beta_r$ and this completes the proof of Lemma 6 and hence Lemma 2.

### 4 Proof of Lemma 3

The proof is quite complicated and so we outline our strategy.

1. We analyse the sequential shortest path algorithm for solving the assignment problem. We mean by this that given $M_r$, we obtain $M_{r+1}$ by solving a shortest path problem.

2. In this shortest path problem there will edges of negative arc length corresponding to the edges of $M_r$. We modify costs in a standard way to make them non-negative. For this we need to use the optimal variables $u_i, v_j, i, j = 1, 2, \ldots, r$ to the linear program $D_r$ dual to the linear programming formulation $LP_r$ of the assignment problem for computing $M_r$.

3. Once we have made the costs non-negative, we can use Dijkstra’s algorithm. We then estimate the number of short cycles created during the execution of Dijkstra’s algorithm. We bound the number of short cycles only, as there cannot be many long cycles.

4. To analyse Dijkstra’s algorithm, we need to understand the distribution of amended costs. This involves understanding the structure of basic solutions to $LP_r$. Basic and non-basic variables have very different amended costs $\hat{C}(i, j) = C(i, j) - u_i - v_j$. The former have $\hat{C}(i, j) = 0$ and the latter are independent (shifted) exponentials.

5. Basic variables are problematic in that they will appear in the Dijkstra tree, if they are oriented in the right way. We need to know that they are unlikely to create a short cycle. Basic solutions to $LP_r$ correspond to random spanning trees of the graph induced by edges such that $u_i + v_j \geq 0$. We have to show that this graph has high minimum degree. Then we can obtain the necessary information about random spanning trees by utilising the random walk algorithm of Aldous [3] and Broder [13].

As you can see there several ingredients to our proof and we now proceed to construct them.

We consider the linear program $LP_r$ that underlies the assignment problem and its dual $D_r$. We show that we can choose optimal dual variables of absolute value at most $2\gamma_r = 2r^{-1-2\varepsilon}$. We obtain $M_{r+1}$ from $M_r$ via an augmenting path $P_r$, and we examine the expected number of short cycles created by this path. A simple accounting then proves Lemma 3.
We consider the linear program $\mathcal{LP}_r$ for finding $M_r$. To be precise we let $\mathcal{LP}_r$ be the linear program
\[
\text{Minimise } \sum_{i,j \in [r]} C(i,j)x_{i,j} \text{ subject to } \sum_{j \in [r]} x_{i,j} = 1, i \in [r], \sum_{i \in [r]} x_{i,j} = 1, j \in [r], x_{i,j} \geq 0.
\]
The linear program $\mathcal{D}_r$ dual to $\mathcal{LP}_r$ is given by:
\[
\text{Maximise } \sum_{i=1}^r u_i + \sum_{j=1}^r v_j \text{ subject to } u_i + v_j \leq C(i,j), i,j \in [r].
\]

4.1 Trees and bases

An optimal basis of $\mathcal{LP}_r$ can be represented by a spanning tree $T_r$ of $G_r$ that contains the perfect matching $M_r$, see for example Ahuja, Magnanti and Orlin [1], Chapter 11. We have that for $T = T_r$,
\[
C(i,j) = u_i + v_j \text{ for } (a_i, b_j) \in E(T)
\]
and
\[
C(i,j) \geq u_i + v_j \text{ for } (a_i, b_j) \notin E(T).
\]
Note that if $\lambda$ is arbitrary then replacing $u_i$ by $\hat{u}_i = u_i - \lambda, i = 1, 2, \ldots, r$ and $v_i$ by $\hat{v}_i = v_i + \lambda, i = 1, 2, \ldots, r$ has no affect on these constraints. We say that $\mathbf{u}, \mathbf{v}$ and $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ are equivalent.

For a fixed tree $T$ and $\mathbf{u}, \mathbf{v}$ let $\mathbf{C}(T, \mathbf{u}, \mathbf{v})$ denote the set of cost matrices $\mathbf{C}$ such that the edges of $T$ satisfy (3). The following lemma implies that the space of cost matrices (essentially) partitions into sets defined by $T, \mathbf{u}, \mathbf{v}$. As such, we can prove Lemma 3 by showing that there are few cycles for almost all $\mathbf{u}, \mathbf{v}$ and spanning trees satisfying (3), (4).

**Lemma 7.** (a) Fix $\mathbf{u}, \mathbf{v}$. If $T_1, T_2$ are distinct spanning trees of $G_r$ then $\mathbf{C}(T_1, \mathbf{u}, \mathbf{v}) \cap \mathbf{C}(T_2, \mathbf{u}, \mathbf{v})$ has conditional measure zero.
(b) If $u_1 = u'_1 = 0$ and $(\mathbf{u}, \mathbf{v}) \neq (\mathbf{u}', \mathbf{v}')$ then for any spanning tree $T$ of $G_r$, we have that $\mathbf{C}(T, \mathbf{u}, \mathbf{v}) \cap \mathbf{C}(T, \mathbf{u}', \mathbf{v}') = \emptyset$.

**Proof.** (a) Suppose that $C \in \mathbf{C}(T_1, \mathbf{u}, \mathbf{v}) \cap \mathbf{C}(T_2, \mathbf{u}, \mathbf{v})$. We root $T_1, T_2$ at $a_1$ and let $u_1 = 0$ (This is allowed as we can always replace $u_i$ by $u_i + \lambda$ and $v_j$ by $v_j - \lambda$ for $i,j = 1,2,\ldots,r$ where $\lambda$ is arbitrary. After this transformation the values $u_i + v_j$ are unchanged.) and then the equations (3) imply that for $i \in [r]$, $u_i$ is the alternating sum and difference of costs on the path $P_{i,k}$ from $a_1$ to $a_i$ in $T_k$. So, unless $P_{i,1} = P_{i,2}$ for all $i$, there will be an additional non-trivial linear combination of the $C(i,j)$ that equals zero.
(b) The costs of the tree edges are determined by $\mathbf{u}, \mathbf{v}$.

Let $\mathcal{T}_r(\mathbf{u}, \mathbf{v})$ denote the set of spanning trees of $G_r(\mathbf{u}, \mathbf{v})$, where $G_r(\mathbf{u}, \mathbf{v})$ is the subgraph of $G_r$ induced by the edges $(a_i, b_j)$ for which $u_i + v_j \geq 0$.

**Lemma 8.** If $T \in \mathcal{T}_r(\mathbf{u}, \mathbf{v})$ then
\[
\mathbb{P}(T_r = T \mid \mathbf{u}, \mathbf{v}) = \prod_{(a_i, b_j) \in G_r(\mathbf{u}, \mathbf{v})} e^{-(u_i + v_j)},
\]
which is independent of $T$. 


Proof. Fixing \( u, v \) and \( T_r \) fixes the lengths of the edges in \( T_r \). If \((a_i, b_j) \notin E(T_r)\) then \( P(C(i, j) \geq u_i + v_j) = e^{-(u_i+v_j)^+} \) where \( x^+ = \max\{x, 0\} \). Thus,

\[
P(T_r = T \mid u, v) = \prod_{(a_i, b_j) \notin E(T)} e^{-(u_i+v_j)^+} \prod_{(a_i, b_j) \in E(T)} e^{-(u_i+v_j)} = \prod_{(a_i, b_j) \in E_r(u, v)} e^{-(u_i+v_j)}. \tag{6}
\]

We need to know that w.h.p. each vertex \( a_i \) is connected to \( b_j \) for which \( u_i + v_j \geq 0 \). We fix a tree \( T \) and condition on \( T_r = T \). For \( i = 1, 2, \ldots, r \) let \( L_{i, +} = \{ j : (i, j) \in E(G) \} \) and let \( L_{j, -} = \{ i : (i, j) \in E(G) \} \). Then for \( i = 1, 2, \ldots, r \) let \( A_{i, +} \) be the event that \( |\{ j \in L_{i, +} : u_i + v_j \geq 0 \}| \geq \eta r \) and let \( A_{j, -} \) be the event that \( |\{ i \in L_{j, -} : u_i + v_j \geq 0 \}| \geq \eta r \) where \( \eta \) will be some small positive constant.

Lemma 9. Fix a spanning tree \( T \) of \( G_r \).

\[
P(\neg A_{i, +} \lor \neg A_{j, -} \mid T_r = T) = O(n^{-10}) \quad \text{for } i, j = 1, 2, \ldots, r.
\]

Proof. In the following analysis \( T \) is fixed and our conditioning is that (3), (4) hold and so we do not need to mention \( T_r = T \) explicitly in probabilistic statements. Let \( B \) be the event that \( C(i, j) \geq u_i + v_j \) for \((a_i, b_j) \notin E(T)\). Throughout the proof we assume that the costs \( C(i, j) \) for \((a_i, b_j) \in T \) are distributed as independent \( EXP(1) \), conditional on \( C(i, j) \leq \gamma_r \). Lemma 6 is the justification for this in that we can solve the assignment problem, only using edges of cost at most \( \gamma_r \).

For each \( i \in [r] \) there is some \( j \in [r] \) such that \( u_i + v_j = C(i, j) \). This is because of the fact that \( a_i \) meets at least one edge of \( T \) and we assume that (3) holds. We also know from \( B \) that \( u_i + v_j \leq C(i', j) \) for all \( i' \neq i \). It follows that \( u_i - u_i' \geq C(i, j) - C(i', j) \geq -\gamma_r \) for all \( i' \neq i \). Since \( i \) is arbitrary, we deduce that \( |u_i - u_i'| \leq \gamma_r \) for all \( i, i' \in [r] \). We deduce by a similar argument that \( |v_j - v_{j'}| \leq \gamma_r \) for all \( j, j' \in [r] \).

Fix \( s \) and put \( u_s = 0 \). The remaining values \( u_i, i \neq s \), \( v_j \) are then determined by the costs of the edges of the tree \( T \). Let \( E \) be the event that \( |u_i|, |v_j| \leq 2\gamma_r \) for all \( i, j \). It follows from the argument in the previous paragraph that \( B \subseteq E \). (After putting \( u_s = 0 \) we immediately see that \( |u_i| \leq \gamma_r \). Then for each \( j \) there is an \( i \) such that \( v_j = C(i, j) - u_i \).)

Let \( Y = \{ C(i, j) : (a_i, b_j) \in E(T) \} \) and let \( \delta_1(Y) \) be the indicator for \( \neg A_{i, +} \land E \). We write

\[
P(\neg A_{i, +} \mid B) = P(\neg A_{i, +} \land E \mid B) = \frac{\sum_y \delta_1(Y)P(Y)P(B \mid Y)}{\sum_y P(Y)P(B \mid Y)} \tag{7}
\]

Then we note that

\[
P(B \mid Y) = \prod_{(a_i, b_j) \notin E(T)} \exp\{-u_i(Y) + v_j(Y)^+\} = \exp\left\{- \sum_{(a_i, b_j) \notin E(T)} (u_i(Y) + v_j(Y))^+\right\}. \tag{8}
\]

Now consider the random variable \( W = W(Y) = \sum_{(a_i, b_j) \notin E(T)} (u_i(Y) + v_j(Y))^+ \). It is a function of \( 2r - 1 \) independent random variables \( Y_i \) and changing the value of one \( Y_i \) can change the value \( W \) by at most \( 2r \gamma_r \). This is because we are changing the cost of one edge and the \( u_i, v_j \)'s are the costs of the paths. Changing the cost of one edge by \( x \) can change the cost of a path by at most \( x \). So applying McDiarmid’s inequality \( \text{[41]} \) we get

\[
P(|W - E(W)| \geq \xi) \leq \exp\left\{-\frac{2\xi^2}{4(2r-1)r^2\gamma_r^2}\right\}. \tag{9}
\]
Putting $\xi = r^{1-3\epsilon/2}$ we see that
\[
\mathbb{P}(|W - \mathbb{E}(W)| \geq r^{1-3\epsilon/2}) \leq e^{-r^{1-\epsilon}/4}. \tag{10}
\]
Next let $\Omega_1 = \{Y : W \geq \mathbb{E}(W) - r^{1-3\epsilon/2}\}$. Then we have
\[
\sum_{Y \in \Omega_1} \delta_Y(\mathcal{Y}) \mathbb{P}(\mathcal{Y} | Y) \leq e^{-\mathbb{E}(W) + r^{1-3\epsilon/2}} \mathbb{P}(\neg \mathcal{A}_{1,+} \land \mathcal{E}). \tag{11}
\]
\[
\sum_{Y \notin \Omega_1} \delta_Y(\mathcal{Y}) \mathbb{P}(\mathcal{Y} | Y) \leq e^{-\mathbb{E}(W)} \mathbb{E}(e^{-(W-\mathbb{E}(W))} 1_{Y \notin \Omega_1})
\leq e^{-\mathbb{E}(W)} \int_{\xi=r^{1-3\epsilon/2}}^{\infty} \exp \left\{ \xi - \frac{2\xi^2}{4(2r-1)r^2\gamma^2} \right\} d\xi
\leq 2e^{-\mathbb{E}(W)-r^{\epsilon}/2}.
\tag{12}
\]
\[
\sum_{Y} \mathbb{P}(\mathcal{Y}) \mathbb{P}(\mathcal{B} | Y) = \mathbb{E}(e^{-W}) \geq e^{-\mathbb{E}(W)}. \tag{13}
\]
It then follows from (7), (11), (12), (13) that
\[
\mathbb{P}(\neg \mathcal{A}_{1,+} | \mathcal{B}) \leq e^{r^{1-3\epsilon/2}} \mathbb{P}(\neg \mathcal{A}_{1,+} | \mathcal{E}) + 2e^{-r^{\epsilon}/2}. \tag{14}
\]
We argue next that w.h.p. the maximum degree in $T$ is at most $2r^{2\epsilon}$. This is because the number of edges incident to a vertex of cost at most $\gamma_r$ is dominated by $\text{Bin}(r, \gamma_r)$ and the claim follows from the Chernoff bounds.

Let $b_j$ be a neighbor of $a_s$ and let $P_j = (i_1 = s, j_1, i_2, j_2, \ldots, i_k, j_k = j)$ define the path from $a_s$ to $b_j$ in $T$. Then it follows from (3) that $v_{j_1} = v_{j_1-1} - C(i_{t-1}, j_{t-1}) + C(i_t, j_t)$. Thus $v_j$ is the final value $S_k$ of a random walk $S_t = X_0 + X_1 + \cdots + X_t, t = 0, 1, \ldots, k$, where $X_0 \geq 0$ and each $X_t, t \geq 1$ is the difference between two independent copies of $\text{EXP}(1)$ that are bounded above by $\gamma_r$. Given $\mathcal{E}$ we can assume that the partial sums $S_i$ satisfy $|S_i| \leq 2\gamma_r$ for $i = 1, 2, \ldots, k-1$. Assume for the moment that $k \geq 4$ and let $x = u_{ik-4} \in [-2\gamma_r, 2\gamma_r]$. Given $x$ we see that there is some positive probability $p_0 = p_0(x)$ that $S_k > 0$. Indeed,
\[
p_0 = \mathbb{P}(S_k > 0) = \mathbb{P}(x + Z_1 - Z_2 > 0), \tag{15}
\]
where $Z_i = Z_{i,1} + \cdots, Z_{i,4}, i = 1, 2$ are the sums of $4$ independent $\text{EXP}(1)$ random variables, each conditioned on being bounded by $\gamma_r$ and such that $\abs{x + \sum_{t=1}^{4} (Z_{1,t} - Z_{2,t})} \leq 2\gamma_r$ for $t = 1, 2, 3, 4$. We claim that the absolute constant $\eta_0 = p_0(-2\gamma_r) > 0$ is such that $\min \{ x \geq -2\gamma_r : p_0(x) \geq \eta_0 \}. \]

At first sight we may be guilty of conditioning on the future by assuming that $|u_i|, |v_j| \leq 2\gamma_r$ for all $i, j$. On the other hand we need to only condition up to $k - 4$ steps and then the history reduces to the value $x$.

We now partition the neighbors of $a_i$ into $N_1, N_2, N_3$ where $N_t = \{b_j : k \geq 4, k \mod 4 = t\}$, $k$ being the number of edges in the path $P_j$ from $a_s$ to $b_j$. Now because $T$ has maximum degree $2r^{\epsilon}$, we know that there exists $t$ such that $|N_t| \geq (\alpha_0 r - (2r^2\delta)^3)/4 \geq \alpha r/5$. It then follows from (15) that $\bar{L}_{j,+}$ dominates $\text{Bin}(\alpha r/5, \eta_0)$ and the claim for $\bar{L}_{1,+}$ with $\eta = \alpha \eta_0/6$ follows from the Chernoff bounds. Similarly for $\bar{L}_{1,-}$. Applying the union bound over $r$ choices for $s$ and applying (14) gives the lemma. \hfill \Box

We let $\bar{G}_r$ be the orientation of $G_{r+1}$ with edges oriented from $A_{r+1}$ to $B_{r+1}$ except for the edges of $M_r$ which are oriented from $B_r$ to $A_r$. In $\bar{G}_r$, the cost of edge $(a_i, b_j) \notin M_r$ is $C(i, j)$ and the cost of edge $(b_j, a_i) \in M_r$ is $-C(i, j)$. We obtain $M_{r+1}$ from $M_r$ by finding a minimum cost (augmenting) path $P_r = (x_1 = a_{r+1}, y_1, x_2, \ldots, x_{\sigma}, y_{\sigma} = b_{r+1})$ from $a_{r+1}$ to $b_{r+1}$ in $\bar{G}_r$. In which case
\[
M_{r+1} = (M_r \cup \{(x_j, y_j) : i = 1, 2, \ldots, \sigma\}) \setminus \{(x_{i+1}, y_i) : i = 1, 2, \ldots, \sigma\}.
\]
To find $P_r$, we let
\[
u_{r+1} = \min \{ C(r + 1, j) - v_j(T_r) : j \in [r] \} \quad \text{and} \quad
u_{r+1} = \min \{ C(r + 1, r + 1) - u_{r+1}, \min \{ C(i, r + 1) - u_i(T_r) : i \in [r] \} \}.
\]

We then use costs $\tilde{C}(i, j) = C(i, j) - u_i - v_j$ in our search for a shortest augmenting path. Our choice of $u_{r+1}, v_{r+1}$ and \([3], [4]\) implies that $\tilde{C}(i, j) \geq 0$ and that matching edges have cost zero. This idea for making edge costs non-negative is well known, see for example Kleinberg and Tardos \([35]\). The $\tilde{C}$ cost of a path $P$ from $a_{r+1}$ to $b_{r+1} \in B$ differs from its $C$ cost by $-(u_{r+1} + v_{r+1})$, independent of $P$.

4.2 Analysis of the augmenting path

We now introduce some more conditioning, $C$. We fix $M_r = \{(a_i, b_{\phi(i)}), i = 1, 2, \ldots, r \}$ and $u, v \notin U$. We then have the constraints that
\[
C(i, \phi(i)) = u_i + v_{\phi(i)} \quad \text{for} \quad i = 1, 2, \ldots, r
\]
\[
C(i, j) \geq u_i + v_j, \quad \text{otherwise}.
\]

Note that with this conditioning, the tree $T_r$ of basic variables is not completely determined. We define $u_{r+1}, v_{r+1}$ as in \([16]\).

We use Dijkstra’s algorithm to find the shortest augmenting path from $a_{r+1}$ to $b_{r+1}$ in the digraph $\tilde{G}_r$ obtained from $G_r$ by contracting the matching edges. This digraph has vertex set $A_{r+1} \cup \{b_{r+1}\}$. Edge $(a_i, a_j)$ is oriented from $i$ to $j$ assuming it comes from the path $(a_i, \phi(a_j), a_j)$. (We sometimes have both $(a_i, a_j)$ and $(a_j, a_i)$ as edges.) Edges incident with $b_{r+1}$ will be directed into $b_{r+1}$. We use costs $D(i, j) = \tilde{C}(i, \phi(i_2)) = \tilde{C}(i, \phi(i_2)) + C(\phi(i_2), i_2)$ in our search for a shortest augmenting path. Our choice of $u_{r+1}, v_{r+1}$ and \([17]\) implies that $\tilde{C}(i, j) \geq 0$ and that $\tilde{C}(\phi(i_2), i_2) = 0$.

The tree $T_r$ will not be exposed all at once, but we will expose it as necessary and denote the tree obtained from $T_r$ by contracting $M_r$ by $\tilde{T}_r$. At the end of the algorithm, $T_r$ will only be partially exposed, but the choices for the unexposed parts will not affect the number of small cycles produced.

The Dijkstra algorithm produces a sequence of values $0 = d_1 \leq d_2 \leq \cdots \leq d_{r+1}$. The $d_i$ are the costs of shortest paths. Suppose that after $k$ rounds we have found a shortest path of length $d_i$ to $a_i \in S_k$ and that $d_i$ for $a_i \notin S_k$ is our current estimate for the cost of a shortest path from $a_{r+1}$ to $a_i$. The algorithm chooses $a_l \notin S_k$ to add to $S_k$ to create $S_{k+1}$. Here $l^*$ minimises $d_{l^*} + \tilde{C}(i, l)$ over $a_i \in S_k, a_l \notin S_k$. It then updates the $d_i$, $a_l \notin S_{k+1} = S_{k+1}$ appropriately.

Let $\theta_{i, \ell} = d_k - d_i + u_i - u_{\ell} + C(\ell, \phi(\ell))$. Note that $d_i + \tilde{C}(i, \ell) - d_k = C(i, \ell) - \theta_{i, \ell}$. The memoryless property of the exponential distribution implies that the non-basic/non-matching values $C(i, \ell)$ are independently distributed as follows:

If $\theta_{i, \ell} \geq 0$ then $d_i + \tilde{C}(i, \ell) - d_k$ is distributed as $\text{EXP}(1)$.

Otherwise, $d_i + \tilde{C}(i, \ell) - d_k$ is distributed as $-\theta_{i, \ell} + \text{EXP}(1) \leq u_{\ell} - u_i + \text{EXP}(1)$.

If we increase any subset of non-basic $C(i, \ell)$ by arbitrary amounts then $T_r$ will continue to define an optimal basis. This justifies our claim of independence.

The augmenting path $P_r = (x_1 = a_{r+1}, y_1, x_2, \ldots, x_\sigma, y_\sigma = b_{r+1})$. Suppose that $1 \leq \tau < \sigma$ and that $M_{r, \tau}$ is the matching obtained from $M_r$ by adding the edges $\{(x_i, y_i) : i = 1, 2, \ldots, \tau \}$ and deleting the edges
\{x_i, y_i\}, i = 1, 2, \ldots, \tau$. Suppose now that \(x_\tau = a_i\) and \(y_\tau = b_j\). Observe that in the digraph \(D\), vertex \(i\) is the head of a path, \(Q\) say, in the set of paths and cycles \(A_{\hat{M}_{r,\sigma}}\). (\(Q\) is directed towards \(i\).) We say that vertex \(x_\tau\) creates a short cycle if \(j\) lies on \(Q\) and the sub-path of \(Q\) from \(j\) to \(i\) has length at most \(\ell_1 = n^{4\epsilon}\). Extending the notation, we say that \(x_\sigma\) creates a short cycle if \(r+1\) \((y_\sigma = b_{r+1})\) is the tail of \(Q\) and the length of \(Q\) is at most \(\ell_1\). For \(r \geq r_0\) we only count the creation of a small cycle by an edge \((x, y)\) if this is the first such edge involving \(x\). Call this a virgin short cycle. Let \(\chi_r\) denote the number of virgin short cycles created in iteration \(r\). We then have that

\[
\mathbb{E}(\nu_C) \leq \frac{r_0}{2} + \frac{n}{\ell_1} + \sum_{r=r_0}^{n} \mathbb{E}(\chi_r). \tag{19}
\]

Here \(n/\ell_1\) bounds the number of large cycles induced by \(M_n\) and the sum bounds the expected number of small cycles ever created. The \(r_0/2\) term bounds the contributions from the matching \(M_{r_0}\).

We claim that

\[
\Sigma_C = \sum_{r=r_0}^{n} \mathbb{E}(\chi_r) \leq 10\ell_1 n^{1-8\epsilon}. \tag{20}
\]

Assume (20) for the moment. Then we have,

\[
\mathbb{E}(\nu_C) \leq \frac{r_0}{2} + \frac{n}{\ell_1} + 10\ell_1 n^{1-8\epsilon} \leq r_0. \tag{21}
\]

Lemma 3 now follows from the Markov inequality. It only remains to prove (20).

4.2.1 Proof of (20)

The costs of edges incident with \(a_{r+1}, b_{r+1}\) are unconditioned at the start of the search for \(P_r\). As such the probability they create short cycles is \(O(\ell_1/r)\) and can be ignored in the analysis below.

A path \((a, \phi(a'), a')\) in the tree \(T_r\) gives rise to a basic edge \((a, a')\) of \(\hat{\Gamma}_r\). We treat the basic and non-basic non-\(M_r\) edges separately.

Basic Edges Basic edges have \(\hat{C}\) value zero and so if there are basic edges oriented from \(DT_k\) to \(A_{r+1} \setminus DT_k\) then one of them will be added to the shortest path tree and we will have \(d_{k+1} = d_k\). We need to argue that they are unlikely to create short cycles. At this point we will only have exposed basic edges that are part of \(DT_k\).

Fix \(a_i \in DT_k\). We want to show that given the history of the algorithm, the probability of creating a short cycle via an edge incident with \(a_i\) is sufficiently small. At the time \(a_i\) is added to \(DT_r\) there will be a set \(L_1\) of size at most \(\ell_1\) for which \((a_i, a), a \in L_1\) creates a short cycle. This set is not increased by the future execution of the algorithm. At this point we only know edges of \(\hat{\Gamma}_r\) pointing into \(a_i\). Let \(e = (a_i, x), x \in A_r\) and let \(\Gamma_r\) be obtained from \(G_r(u, v)\) by contracting \(M_r\). Let \(\hat{T_r}\) be obtained from \(T_r\) by contracting \(M_r\). Now Lemma 5 implies that \(\hat{T_r}\) is a uniform random spanning tree of \(\Gamma_r^+\), the subgraph of \(\Gamma_r\) induced by the edges \((a_i, a_j)\) for which \(u_i + v_{\phi(a_i)} > 0\). We claim that

\[
\mathbb{P}((a_i, x) \in \hat{T_r}) = O\left(\frac{1}{r}\right) \tag{22}
\]

from which we can deduce that

\[
\mathbb{P}(\text{an added basic edge is bad}) = O\left(\frac{\ell_1}{r}\right). \tag{23}
\]
To prove (22) we use two well known facts: (i) if e = \{a, b\} is an edge of a connected graph G and T denotes a uniform random spanning tree then \( P(e \in T) = R_{\text{eff}}(a, b) \) where \( R_{\text{eff}} \) denotes effective resistance, see for example [40]; (ii) \( R_{\text{eff}}(a, b) = \frac{\tau(a, b) + \tau(b, a)}{2E(G)} \) where \( \tau(x, y) \) is the expected time for a random walk starting at \( x \) to reach \( y \), see for example [19]. We note that in the context of (22), we may have exposed some edges of \( T_r \). Fortunately, edge inclusion in a random spanning tree is negatively correlated i.e. \( P(e \in T_r \mid f_1, \ldots, f_s \in T_r) \leq P(e \in T_r) \), see for example [40].

Given (i) and (ii) and Lemma 8 it only remains to show that with \( G = \Gamma^+ \) that \( \tau(a, x) = O(r) \). For this we only have to show that the mixing time for a random walk on \( \Gamma^+ \) is sufficiently small. After this we can use the fact that the expected time to visit a vertex \( a \) from stationarity is \( 1/\pi_a \leq \eta/\alpha \) where \( \eta \) is from Lemma 9 and where \( \pi \) denotes the stationary distribution, see for example [39]. We estimate the mixing time of a walk by its conductance.

Let \( \deg_v \geq \eta r \) denote degree in \( \Gamma^+ \). For \( S \subseteq A_r \), let \( \Phi_S = e(S, S)/\deg(S) \) where \( e(S, S) \) is the number of edges of \( \Gamma^+ \) with one end in \( S \) and \( \deg(S) = \sum_{v \in S} \deg_v \). Let \( \Phi = \min \{ \Phi_S : \deg(S) \leq \deg(A_r)/2 \} \). Note that if \( \deg(S) \leq \deg(A_r)/2 \) then \( \deg(S) \geq \deg(A_r)/2 \geq \eta r^2/2 \) which implies that \( |S| \geq \eta/2 \) and so \( |S| \leq (1 - \eta/2)r \).

Asume first that \( |S| \leq \eta r/2 \). Then

\[
\Phi_S \geq \frac{\sum_{v \in S} (\deg_v - |S|)}{\deg(S)} \geq \frac{(\eta r/2)|S|}{r|S|} = \frac{\eta}{2}.
\]

If \( \eta r/2 \leq |S| \leq (1 - \eta/2)r \) then we use the random edges \( R \). Lemma 9 (and its proof, particularly (15)) and the Chernoff bounds imply that w.h.p. \( e(S, S) \geq \eta(1 - \eta/2)r^{2-\epsilon}/3 \). So,

\[
\Phi_S \geq \frac{\eta(1 - \eta/2)r^{2-\epsilon}/3}{r^2/2} = \frac{2\eta(1 - \eta/2)r^{2-\epsilon}}{3}.
\]

It then follows that after \( r \) steps of the random walk the variation distance between the walk and the steady state is \( e^{-\Omega(r^{1-2\epsilon})} \), see for example [39]. This completes our verification of (23).

We will also need a bound on the number of basic edges in any path in the tree \( DT_r \) constructed by Dijkstra’s algorithm. Starting with \( T_r \), we orient its edges from \( A_r \) to \( B_r \) and then contract \( M_r \) to obtain an oriented tree \( \tilde{T}_r \), where the path \( a_i, b_{\phi(i)}, a_j \) gives rise to a directed edge \( (a_i, a_j) \) in \( \tilde{T}_r \). Suppose that at some stage we see \( a_i \in DT_k \) and \( a_j \notin DT_{k} \) and either (i) \( (a_i, a_j) \in \tilde{T}_r \) or (ii) \( (a_j, a_i) \in \tilde{T}_r \) or (iii) both (i) and (ii). In case (i) we will add \( (a_i, a_j) \) to the shortest path tree, but in case (ii) we will not.

Our next task is to show that it is very unlikely for \( \tilde{T}_r \) to contain a long path consisting of edges of type (i).

**Lemma 10.** Let \( H \) be a graph with \( r \) vertices and minimum degree at least \( dr \). Suppose also that each edge \( e = \{v, w\} \) has a label \( \lambda(e) \in \{v, w\} \). Suppose that each vertex \( v \) is incident with at least \( d\zeta r \) edges with label \( v \) where \( 0 < \zeta < 1 \) is a constant independent of \( r \). Let \( T \) be a random spanning tree of \( H \). Then there exists a constant \( C > 0 \) such with probability \( 1 - O(r^{-2}) \), \( T \) contains no path of length at least \( C\log r \) where successive edge labels are distinct.

**Proof.** We consider a simple random walk \( W \) on \( H \) started in the steady state. Aldous [3] and Broder [13] show that \( T \) can be computed as the tree consisting of the set of first edges incident with each vertex. We run the chain for \( r^4 \) steps. The expected time for a random walk to visit all vertices of an \( r \)-vertex graph is \( O(r^3) \), see for example Feige [23]. It follows that \( H \) will be covered in time \( r^4 \) with probability \( 1 - e^{-\Omega(r)} \).
Fix some vertex $x$ as the start of the walk and consider those times when a new vertex $z$ is added to the tree via an edge $\{y, z\}$. If $y$ already has degree at least $\log r$ in the tree then we declare failure. Otherwise, the probability that $z$ has the same label as all the vertices in the path from $x$ to $y$ is at most $\frac{\min(\zeta, 1-\zeta)}{dr - \log r} \leq 1 - \zeta' / 2$ where $\zeta' = \min \{\zeta, 1 - \zeta\}$. So the probability that there is a path from $x$ in $T$ of length $C \log r$ is at most $r(1 - \zeta'/2)^{C \log r}$.

We observe that if $\Delta$ is the maximum degree in $T$ then $\Delta \leq \log r$ w.h.p. It follows from (22) and negative correlation that for some $c > 0$,

$$\mathbb{P}(\Delta \geq \log r) \leq r \left( \frac{r}{\log r} \right)^{\log r} \leq r \left( \frac{ce}{\log r} \right)^{\log r} = r^{-\Omega(\log \log r)}.$$ 

It follows that the probability that $T$ contains a path is at most $r^5(1 - \zeta'/2)^{C \log r} + r^{-\Omega(\log \log r)}$ and the lemma follows by choosing $C = 20/\zeta'$.

We apply Lemma 10 to the graph $\Gamma_i^+$ obtained from $\Gamma_+^+$ by contracting $M_r$. We label an edge $e = \{a_i, a_j\}$ with $a_i$ if $e$ arises from the path $(a_i, \phi(a_j), a_j)$. Lemma 9 implies that $\zeta > 0$ here. In a path where all edges are of type (i), the successive edge labels will be distinct. We can therefore assert that if $r \geq r_0$ then

$$\mathbb{P}(T_r \text{ contains a path of length } C \log r \text{ for which all edges are of type (i)}) = o(n^{-2}).$$

(24)

Non-Basic Edges Each $a_i \in DT_k$ corresponds to an alternating path $P_i$. As such there are at most $\ell_1$ choices of $\ell$ such that $(i, \ell)$ would create a bad edge. This is true throughout an execution of the Dijkstra algorithm. Also, while we initially only know that the $C(i, \ell, \ell \neq \phi(i)$ are $EXP(1)$ subject to (17), as Dijkstra’s algorithm progresses, we learn lower bounds on $C(i, \ell)$ through (18). The costs $C(i, \ell)$ will still be independent, seeing as the constraints added are all of the form $C(i, \ell) \geq \theta_{i, \ell}$.

Suppose that vertices are added to $DT_r$ in the sequence $i_1, i_2, \ldots, i_r$. For $r_0 \leq k < r - r_0$ let

$$F(i, k) = \{|t < k : u_{i_t} \leq u_{i_k} + \gamma_r \varepsilon_r\} \text{ where } \varepsilon_r = r^{-11\varepsilon}.$$ 

We bound $G_r(i) = \{|k \leq r : F(i, k) \leq r \varepsilon_r^2\}$ from above.

Claim 1. $G_r(i) \leq 4r \varepsilon_r$.

Proof. Assume without loss that $i_j = j$. We show that we can assume that $u_1 \leq u_2 \leq \ldots \leq u_r$. Assume that there exist $k$ such that $u_k > u_{k+1}$ and let $\tilde{i}$ be obtained from $i$ by interchanging $k$ and $k+1$. Let $\tilde{u}$ be obtained from $u$ by interchanging the values $u_k, u_{k+1}$. We claim that $F(i, j) \supseteq F(\tilde{i}, j)$ for all $j$. Indeed, suppose that $i \in F(\tilde{i}, j) \setminus F(i, j)$. Then we have $i > j$ and $\tilde{u}_i \leq \tilde{u}_j + \gamma_r \varepsilon_r$ and $u_i > u_j + \gamma_r \varepsilon_r$. So $i = k + 1$ or $j = k$ or both. In either case we have a contradiction through $\tilde{u}_k < u_k$ and $\tilde{u}_{k+1} > u_{k+1}$.


Next let $k_1$ be the smallest index $k$ in $G_r(i)$ and let $J_1 = [u_{k_1}, u_{k_1} + \gamma_r \varepsilon_r]$. The interval $J_1$ contains at most $r \varepsilon_r^2$ of the values $u_i$. Then let $k_2$ be the smallest index $k$ in $G_r(i)$ with $k > u_{k_1} + \gamma_r \varepsilon_r$ and let $J_2 = [u_{k_2}, u_{k_2} + \gamma_r \varepsilon_r]$ and so on. Using the fact that $\mathbf{u} \notin \mathbf{U}$ we see that in this way we cover $G_r(i)$ with at most $4 \varepsilon_r^{-1}$ intervals each containing at most $r \varepsilon_r^2$ of the values $u_i$.

Let

$$K_r = \{k : |G_r(i) \cap [k - r \varepsilon_r^2/2, k]| \geq r \varepsilon_r^2/4\}.$$
We argue that
\[ |K_r| \leq 2|G_r|(i)| \leq 8r\varepsilon_r. \] (25)
Indeed, let \( z_{j,k} \) be the indicator for \((j, k)\) satisfying \( k - r\varepsilon_r^2/2 \leq j \leq k \) and \( j \in G_r(i) \). Then if \( z = \sum_{j,k} z_{j,k} \) we have
\[
\begin{align*}
  z &\geq \sum_{k \in K_r} r\varepsilon_r^2/4 = |K_r|r\varepsilon_r^2/4. \\
  z &\leq \sum_{j \in G_r(i)} r\varepsilon_r^2/2 \leq r\varepsilon_r^2|G_r(i)|/2.
\end{align*}
\]
and (25) follows.

It follows from the definition of \( K_r \) that if \( k \notin K_r \) then there are at least \( r\varepsilon_r^2/4 \times r\varepsilon_r^2 \) pairs \((i, \ell)\) such that \( i \leq k < \ell \) and \( u_{\ell} \leq u_i + \gamma_r\varepsilon_r \). Note that \( \theta_{i,\ell} \geq -\varepsilon_r\gamma_r \) for each such pair. We next estimate for \( k \notin K_r \) and \( r_0 \leq k \leq r \) and \( j \leq k < m \leq r \) the probability that \((j, m)\) minimises \( d_i + \widehat{C}(i, \ell) \). The Chernoff bounds imply that w.h.p. \( \varepsilon_r^2 n^{-\varepsilon}/5 \) of these pairs appear as edges in the random edge set \( R \). Given this, it follows from the final inequality in (18) that
\[
\mathbb{P}(\text{an added non-basic edge is bad}) \leq \ell_1 \left( \varepsilon_r\gamma_r + \frac{5n^\varepsilon}{r^2\varepsilon_r^4} \right) \tag{26}
\]

Explanation: There are at most \( \ell_1 \) possibilities for a bad edge \( e = (a_j, a_m) \) being added. The term \( \varepsilon_r\gamma_r \) bounds the probability that the cost of edge \( e \) is less than \( \varepsilon_r\gamma_r \). Failing this, \( e \) will have to compete with at least \( r^2\varepsilon_r^4 n^{-\varepsilon}/5 \) other pairs for the minimum.

We will now put a bound on the length \( L \) of a sequence \((t_k, x_k), k = 1, 2, \ldots, L \) where \( t_k, k \notin K_r \) is an iteration index where a non-basic edge \((y_k, x_k)\) is added to \( DT_r \). The expected number of such sequences can be bounded by
\[
\sum_{t_1 < t_2 < \cdots < t_L} \left( \varepsilon_r\gamma_r + \frac{5n^\varepsilon}{r^2\varepsilon_r^4} \right)^L \leq \left( \frac{r}{L} \right)^{(2\varepsilon_r\gamma_r)r^2} L \leq \left( \frac{2c\varepsilon_r\gamma_r r^2}{L} \right) = o(n^{-2}), \tag{27}
\]
if \( L \geq 3c\varepsilon_r\gamma_r r^2 \).

Explanation: We condition on the tails \( y_k \) of the edges added at the given times. Then there are at most \( r \) possibilities for the head \( x_k \) and then \( \varepsilon_r\gamma_r + \frac{5n^\varepsilon}{r^2\varepsilon_r^4} \) bounds the probability that \((y_k, x_k)\) is added.

Combining (24), (25), (27) and Lemma 10 we obtain a bound of \( r^{1-12\varepsilon} \) on the diameter of \( DT_r \).

Let \( \zeta_{r,k} \) be the 0,1 indicator for \( e_k \) being a virgin bad edge i.e. one that creates a virgin short cycle. Note that \( \sum_{r=r_0}^r \sum_{k=1}^r \zeta_{r,k} \leq n \).

We have that with \( C \) equal to the hidden constant in (23),
\[
\sum_{r=r_0}^r \sum_{k=1}^r \mathbb{P}(e_k \text{ is bad} \mid C) \zeta_{r,k} \leq C\ell_1 \sum_{r=r_0}^n \frac{4r^{1-12\varepsilon}}{r} + \sum_{r=r_0}^n \ell_1 \sum_{k=k_0}^r \left( \gamma_r\varepsilon_r + \frac{5n^\varepsilon}{r^2\varepsilon_r^4} \right) \zeta_{r,k}.
\]

Explanation: For each \( a_i \in DT_k \), the set of possible bad edges does not increase for each \( k' > k \). This is because each \( a_i \in DT_k \) is associated with an alternating path that does not change with \( k' \). The first term
bounds the expected number of bad basic edges, using (23) and our bound on the diameter of $DT_r$. The second sum deals with non-basic edges and uses (26).

Now
\[
\ell_1 \sum_{r=r_0}^{n} \frac{r^{1-12\epsilon}}{r} \leq \ell_1 r^{1-11\epsilon}
\]
and
\[
\sum_{r=r_0}^{n} \sum_{k=1}^{r} \zeta_{r,k} \ell_1 \gamma_r \varepsilon_r \leq \ell_1 \gamma_{r_0} \varepsilon_{r_0} \sum_{r=r_0}^{n} \sum_{k=1}^{r} \zeta_{r,k} \leq \ell_1 \gamma_{r_0} \varepsilon_{r_0} n = o(1).
\]

and
\[
\sum_{r=r_0}^{n} \frac{5\ell_1 n^\epsilon}{r^2 \varepsilon_r^4} \sum_{k=1}^{r} \zeta_{r,k} \leq \frac{5\ell_1 n^1+\epsilon}{r^2 \varepsilon_r^4} \sum_{r=r_0}^{n} \sum_{k=1}^{r} \zeta_{r,k} \leq \frac{5\ell_1 n^{1+\epsilon}}{r^2 \varepsilon_r^4 n} = 5n^{-1+55\epsilon}.
\]

Finally, it follows from (25) and the fact that only edges of cost at most $\gamma_r$ are added that
\[
\sum_{r=r_0}^{n} \sum_{k=1}^{r} \Pr(e_k \text{ is bad } | C) \zeta_{r,k} \leq 8\ell_1 \sum_{r=r_0}^{n} r \gamma_r \varepsilon_r \leq 8\ell_1 n^{1-8\epsilon}.
\]

This completes the justification for (20) and the proof of Lemma 3.

5 Final Remarks

We have extended the proof of the validity of Karp’s patching algorithm to random perturbations of dense graphs with minimum in- and out-degree at least $\alpha n$ and independent $\text{EXP}(1)$ edge weights. It is a routine exercise to extend the analysis to costs with a distribution function $F(x)$ that satisfies $F(x)/x \downarrow 1$ as $x \to 0$. Janson [31] describes a nice simple coupling in the case of shortest paths.

References


