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ON LARGE MATCHINGS AND CYCLES IN SPARSE RANDOM GRAPHS

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Let k be a fixed positive integer. A graph H has property M_k if it contains $\lfloor \frac{1}{2}k \rfloor$ edge disjoint hamilton cycles plus a further edge disjoint matching which leaves at most one vertex isolated, if k is odd. Let $p = c/n$, where c is a large enough constant. We show that $G_{n,p}$ a.s. contains a vertex induced subgraph H_k with property M_k and such that $|V(H_k)| = (1 - (1 + \varepsilon(c))c^{k-1}e^{-c}/(k-1)!)n$, where $\varepsilon(c) \to 0$ as $c \to \infty$. In particular this shows that for large c, $G_{n,p}$ a.s. contains a matching of size $\frac{1}{2}(1 - (1 + \varepsilon(c))e^{-c})n$ $(k = 1)$ and a cycle of size $(1 - (1 + \varepsilon(c))ce^{-c})n$ $(k = 2)$.

1. Introduction

In this paper we study the size of the largest matching and cycle in random graphs with edge probability c/n, where c is a large constant. We continue the analysis of Bollobás [2], Bollobás, Fenner and Frieze [3] and confirm the conjecture in the final paragraph of the latter paper.

We shall let $G_{n,p}$ denote a random graph with vertex set $V_n = \{1, 2, \ldots, n\}$ in which edges are chosen independently with probability p. We say that $G_{n,p}$ has a property Q **almost surely** (a.s.) if $\lim_{n\to\infty} \Pr(G_{n,p} \in Q) = 1$.

For $c > 0$ define $\alpha(c)$, $\beta(c)$ by

$$\alpha(c) = \sup(\alpha \geq 0)\text{: } G_{n,c/n} \text{ a.s. contains a matching of size at least } \tfrac{1}{2}\alpha n) \tag{1.1}$$

and

$$\beta(c) = \sup(\beta \geq 0)\text{: } G_{n,c/n} \text{ a.s. contains a cycle of size at least } \beta n). \tag{1.2}$$

Our main result is an improved estimate of $\beta(c)$.

In what follows $p = c/n$ and $\varepsilon_1(c)$, $\varepsilon_2(c)$ are unspecified functions satisfying $\lim_{c\to\infty} \varepsilon_i(c) = 0$, $i = 1, 2$.

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To prove (2.3) we observe that

$$\mathrm{Exp}(|\{v \in V_n : d_G(v) > 4\log n\}|) = n \sum_{k>4\log n} \binom{n-1}{k} p^k (1-p)^{n-k-1}$$
$$\leq n \sum_{k>4\log n} \left(\frac{ce}{k}\right)^k = \mathrm{o}(1)$$

as $ce \leq 3 \log n$.

Since the expectation of the number of cycles of length 3 or 4 is $\mathrm{o}(c^4)$ their contribution is easily absorbed into what follows.

Next let $P_k = \{$paths of length k in G with small endpoints$\}$. Now clearly

$$|W_k| \leq 2|P_k| \quad \text{for } k = 1, 2, 3, 4. \tag{2.7}$$

Furthermore

$$\mathrm{Exp}(|P_k|) = \binom{n}{2} p\lambda^2, \tag{2.8}$$

Where $\lambda = BS(c/10 - 1, n-2) \leq e^{-0.669c}$. Now

$$\mathrm{Exp}(|P_1|^2) = \mathrm{Exp}(|P_1|) + \binom{n}{2}\binom{n-2}{2} p^2 \lambda_1 + 2(n-2)\binom{n}{2} p^2 \lambda_2,$$

where

$$\lambda_1 = \Pr(\text{SMALL} \supseteq \{1,2,3,4\} \setminus E(G) \supseteq \{\{1,2\},\{3,4\}\})$$
$$\leq \Pr(|N_G(1) \cap \{5, 6, \ldots, n\}| \leq c/10 - 1)^4$$
$$\leq (\lambda(1-p)^{-2})^4$$

and

$$\lambda_2 = \Pr(\text{SMALL} \supseteq \{1,2,3\} \setminus E(G) \supseteq \{\{1,2\},\{2,3\}\})$$
$$\leq (\lambda(1-p)^{-1})^3.$$

This gives

$$\mathrm{Var}(|P_1|) \leq ce^{-4c/3} n \quad \text{for } n \text{ large}. \tag{2.9}$$

Similar calculations give

$$|P_k| = \tfrac{1}{2}(1 + \mathrm{o}(1)) n^{k+1} p^k \lambda^2 \quad \text{for } k = 2,3,4. \tag{2.10}$$

(2.4) now follows from (2.7), (2.8), (2.9) and (2.10).

To prove (2.5) we take $c \geq 20(l+1)\log(l+1)$ and first consider S for which $1 \leq s = |S| \leq n/(200e^3(l+1)^3)$. Let $T = S \cup N_G(S)$ and $t = |T|$. If (2.5) does not hold for S then $|T| \leq m_1 = \lceil n/(200e^3(l+1)^2 \rceil$ and T contains at least $m_2 =$

$\lceil ct/20(l+1))\rceil$ edges of G. The probability that such a T exists is no more than

$$\sum_{t=1}^{m_1}\binom{n}{t}\binom{\binom{t}{2}}{m_2}p^{m_2}\leq\sum_{t=1}^{m}\left(\frac{ne}{t}\right)^t\left(\frac{t^2ep}{2m_2}\right)^{m_2}$$

$$\leq\sum_{t=1}^{m_1}\left(\frac{ne}{t}\right)^t\left(\frac{10e(l+1)t}{n}\right)^{2t}$$

$$\leq\sum_{t=1}^{m_1}\left(\frac{100e^3(l+1)^2t}{n}\right)^t=o(1)$$

For $|S|\geq m_3=\lceil n/(300e^3(l+1)^3\rceil$ we can ignore the fact that the vertices of S are large. Let $m_4=\lceil n/2l\rceil$. The probability that such an S exists violating (2.5) is no more than

$$\sum_{s=m_3}^{m_4}\binom{n}{s}\binom{n}{ls}(1-p)^{s(n-ls)}\leq\sum_{s=m_3}^{m_4}\left(\frac{ne}{s}\right)^s\left(\frac{ne}{ls}\right)^{ls}e^{-3cs/7}$$

$$\leq\sum_{s=m_3}^{m_4}(300e^4(l+1)^3e^{-8(l+1)\log(l+1)})^{(l+1)s}=o(1)$$

which proves (2.5).

the probability that (2.6) does not hold is not more than

$$\sum_{s=m_4}^{\lfloor\frac{1}{2}n\rfloor}\binom{n}{s}BS(cs/3l,\,s(n-s))\leq 2\sum_{s=m_4}^{\lfloor\frac{1}{2}n\rfloor}\left(\frac{ne}{s}\right)^s\left(\frac{3ls(n-s)e}{cs}\right)^{cs/3l}\left(\frac{c}{n}\right)^{cs/3l}e^{-cs/3}$$

$(c,\ n$ large)

$$\leq 2\sum_{s=m_4}^{\lfloor\frac{1}{2}n\rfloor}(2le(3le)^{c/3l}e^{-c/3})^s=o(1).\quad\square$$

The proofs of our theorems rely on the removal of a certain set of vertices. We must show that this set is not too large. The following lemma deals with part of this set.

Lemma 2.2. *Let $X_0=$ SMALL and let the sequence of sets $X_1, X_2, \ldots, X_s$ be defined by*

$$X_i=\left\{v\in V_n:\left|N_G(v)\cap\bigcup_{t=0}^{i-1}X_t\right|\geq 2\right\}$$

and let s be the smallest $i\geq 1$ such that $X_{i+1}=X_i$. Let $X=\bigcup_{i=1}^{s}X_i$, then

$$|X|\leq 2e^4c^4e^{-4c/3}n\text{ a.s.}\tag{2.11}$$

Proof. For $x\in X\cup X_0$ let $i(x)=\min\{i: x\in X_i\}$ and let $D(x)=(V(x),A(x))$ denote a digraph inductively constructed as follows: for $x\in X_0$, $D(x)=(\{x\},\emptyset)$ and for $x\in X_0$ let y_1, y_2 be 2 distinct neighbours of x satisfying $i(x)>i(y_1)$, $i(y_2)$.

Then

$$D(x)=(V(y_1)\cup V(y_2)\cup\{x\}, A(y_1)\cup A(y_2)\cup\{(x,y_1),(x,y_2)\})$$

Each $D(x)$ is acyclic, (weakly) connected and satisfies

$$\text{each } v\in V(x) \text{ has outdegree 0 or 2 and } x \text{ is the unique vertex of indegree 0.} \tag{2.12}$$

Let

$k=$ the number of vertices of outdegree $2=|K(x)|$, where $K(x)=S(x)-X_0$,

and let

$l=$ the number of vertices of outdegree $0=|L(x)|$, where $l(x)=S(x)\cap X_0$.

It follows then that

$$|A(x)|=2k \tag{2.13a}$$

and we will show

$$l\leqslant k+1 \text{ and if } l=k+1, \text{ then } D(x) \text{ is a binary tree rooted at } x. \tag{2.13b}$$

This is most easily proved by induction on k. A digraph satisfying (2.12) has at least one vertex y whose outneighbours z_1, z_2 both have outdegree zero. Removing arcs (y,z_1) and (y,z_2) and any vertex which becomes isolated we obtain a smaller digraph satisfying (2.12).

We obtain from the above that we can associate with each $x\in X$, a set $V(x)$ of vertices and a partition of $V(x)$ into $K(x)$, $L(x)$ satisfying

$$x\neq x' \text{ implies } V(x)\neq V(x'); \tag{2.14a}$$

$$\text{if } k=|K(x)|,\ l=|L(x)|, \text{ then } 2\leqslant l\leqslant k+1; \tag{2.14b}$$

$$L(x)\subseteq \text{SMALL}; \tag{2.14c}$$

$$G(x)=G[V(x)] \text{ is connected and has at least } 2k \text{ edges;} \tag{2.14d}$$

$$\text{if } l=k+1 \text{ and } G(x) \text{ has } 2k \text{ edges, then } G(x) \text{ is a tree with leaves } L(x). \tag{2.14e}$$

We estimate $|X_s-X_0|$ by counting sets of vertices satisfying (2.14). For a given k, l, m let $\lambda_{k,l,m}$ be the expected number of sets K, L with $|K|=k$, $|L|=l$ satisfying (2.14) above, where $G[K\cup L]$ has m edges. Then

$$\begin{aligned}\lambda_{k,l,m}&\leqslant\binom{n}{k}\binom{n}{l}\binom{\binom{k+l}{2}}{m}p^m BS(c/10,n-k-l)^l\\&\leqslant\left(\frac{ne}{k}\right)^k\left(\frac{ne}{l}\right)^l\left(\frac{(k+l)^2e}{2m}\right)^m\left(\frac{c}{n}\right)^m e^{-2cl/3}\left(1-\frac{c}{n}\right)^{-l(k+l)}\\&=\mu_{k,l,m}.\end{aligned}$$

Now if $c \leq 2\log n$, k, $l \leq n^{1/3}$, then $\mu_{k,l,m+1}/\mu_{k,l,m} \leq n^{-1/4}$ for n large. Thus

$$\sum_{m=2k}^{\binom{k+l}{2}} \lambda_{k,l,m} \leq (1+o(1))\mu_{k,l,2k}. \tag{2.15}$$

With the same bounds on c, k, l and with n large and $l \leq k+1$ we have

$$\mu_{k,l,2k} \leq 21 n^{l-k}(e^4c^2k)^k l^{-l} e^{-2cl/3} \tag{2.16}$$

which implies

$$\sum_{l=2}^{k+1} \mu_{k,l,2k} \leq 21(e^4c^2k/n)^k \sum_{l=2}^{k+1} (n/le^{2c/3})^l$$
$$\leq n(e^4c^2)^k e^{-2ck/3}$$
$$\leq ne^{-ck/2} \quad \text{as } c \geq 300.$$

It follows that $s \leq \log n$ a.s., and we can assume $k \leq \log n$. Now, using (2.16),

$$\sum_{k=2}^{\log n}\sum_{l=2}^{k} \mu_{k,l,2k} \leq 21 \sum_{k=2}^{\log n} (e^4c^2)^k e^{-2ck/3}$$
$$\leq 22(e^4c^2)^4 e^{-4c/3}$$

and so

the number of sets, K, L with $2 \leq l \leq k$ is a.s. less than $n^{1/2}e^{-4c/3}$. (2.17)

We only need to consider the case $l = k+1$ from now on. But as $\mu_{k,k+1,m+1}/\mu_{k,k+1,m} \leq 3ck/n$ we have

$$\sum_{m \geq 2k} \mu_{k,k+1,m} \leq (1+o(1))\mu_{k,k+1,2k}. \tag{2.18}$$

So we are finally reduced to estimating

τ_k = the number of *vertex induced* binary trees with k leaves (*k-b-trees*) in which each leaf is small.

Let θ_k be the number of (vertex labelled) k-b-trees contained in a complete graph with $2k-1$ vertices. (Clearly $\theta_k \leq (2k-1)^{2k-3}$). Then

$$\operatorname{Exp}(\tau_k) = \binom{n}{2k-1}\theta_k p^{2k-2}(1-p)^{\binom{2k-1}{2}-2k+2} BS(c/10-1, n-2k+1)^k$$
$$\leq n(e^2c^2e^{-2c/3})^k \quad \text{for } n \text{ large}. \tag{2.19}$$

To estimate $\operatorname{Var}(\tau_k)$, let $\{T_1, T_2, \ldots, T_B\}$, $B = \binom{n}{2k-1}\theta_k$, be the set of $k-b$-trees contained in a complete graph with n vertices. Let A_i be the event that T_i is a vertex induced subgraph of $G_{n,p}$ in which all leaves are small.

Next let $Y_p = \{(i,j): |V(T_i) \cup V(T_j)| = p\}$ for $p = 2k-1, \ldots, 4k-2$ and let $Z_{p,q} = \{(i,j) \in Y_p: |E(T_i) \cup E(T_j)| = q\}$. Then

$$\operatorname{Exp}(\tau_k^2) = \operatorname{Exp}(\tau_k) + \Delta_1 + \Delta_2, \tag{2.20}$$

where

$$\Delta_1 = \sum_{(i,j)\in Y_{4k-2}} \Pr(A_i \cap A_j)$$

and

$$\Delta_2 = \sum_{p=2k-1}^{4k-3} \sum_{(i,j)\in Y_p} \Pr(A_i \cap A_j).$$

Now

$$\Delta_1 \leqslant \binom{n}{2k-1}^2 \left(\theta_k p^{2k-2}(1-p)^{\binom{2k-1}{2}-2k+2}\right)^2 \sigma,$$

where

$$\sigma = BS(c/10-1,\, n-2k+1)^k BS(c/10-1,\, n-4k+2)^k$$

is an estimate of the probability that all leaves of 2 particular disjoint trees are small. It follows that

$$\Delta_1 \leqslant \operatorname{Exp}(\tau_k)^2 (1-p)^{-2k^2}. \tag{2.21}$$

Now for $p \leqslant 4k-3$ we have

$$\begin{aligned} \sum_{(i,j)\in Y_p} \Pr(A_i \cap A_j) &= \sum_{q=p-1}^{4k-4} \sum_{(i,j)\in Z_{p,q}} \Pr(A_i \cap A_j) \\ &\leqslant \sum_{q=p-1}^{4k-4} \binom{n}{p} \binom{\binom{p}{2}}{q} \binom{q}{2k-1}^2 \left(\frac{c}{n}\right)^q e^{-2ck/3}(1-p)^{-8k^2} \\ &\leqslant n e^{-ck/2} \quad \text{for } n \text{ large.} \end{aligned} \tag{2.22}$$

(2.19), (2.20), (2.21), (2.22) plus the Chebycheff inequality implies that τ_k is a.s. within a factor $(1+o(1))$ of the right-hand side of (2.19). This together with (2.17) and (2.18) proves the result. □

For a positive integer k, the *k-core* $V_k(G)$ is defined to be the largest set $S \subseteq V_n$ such that $\delta(G[S]) \geqslant k$. This is well defined, for if $\delta(G[S_i]) \geqslant k$ for $i = 1, 2$, then $\delta(G[S_1 \cup S_2]) \geqslant k$. We let G_k denote the subgraph of G induced by $V_k(G)$.

The k-core can be constructed using the following algorithm.

begin
  $H := G$;
  while $\delta(H) < k$ **do**
  begin
    $Y := \{v \in V(H) : d_H(v) < k\}$;
    $H := H[V(H) - Y]$
  end
end

On termination $H = G_k$. This is because one can easily show inductively that each iteration removes vertices that are not in $V_k(G)$ and as $\delta(H) \geq k$ we have $V(H) \subseteq V_k(G)$.

Clearly any matching of G is contained in G_1 (= G minus isolated vertices) and any cycle of G is contained in G_2.

Now for $k \geq 1$ let $A_k = A_k(G_{n,p}) = V_k(G_{n,p}) - (W \cup X \cup Y_k)$, where W, X are as defined in Lemmas 2.1, 2.2 respectively and

$$Y_k = \{y \in V_n \colon d_{G_{n,p}}(y) = k \quad \text{and} \quad N_{G_{n,p}}(y) \cap X \neq \emptyset\}.$$

Let $H_k = H_k(G_{n,p}) = G_{n,p}[A_k]$, then we have

Lemma 2.3. *For $k \geq 1$ let M be any matching of $G_{n,p}[A_k]$ which is not incident with any small vertex. Let $\hat{H}_k = H_k - M$, then for large c*

$$\emptyset \neq S \subseteq A_k,\ |S| \leq n/(2k+8) \quad \textit{implies} \quad |N_{\hat{H}_k}(S)| \geq k\,|S| \text{ a.s.} \tag{2.23}$$

Proof. Let $G = G_{n,p}$, $H = \hat{H}_k$ and for a given S let $S_1 = S \cap \text{SMALL}$ and $S_2 = S - S_1$. Now

$$|N_H(S)| \geq |N_H(S_1)| - |S_2| + |N_H(S_2)| - \min(|S_1|, |S_2|). \tag{2.24}$$

This follows from $S \cap (W \cup X) = \emptyset$.

Also, we claim

$$|N_H(S_1)| \geq k\,|S_1|. \tag{2.25}$$

Note first that $v \in S_1$ implies $d_G(v) \geq k$ and no pair of vertices of S_1 are adjacent, since $S_1 \cap W_1 = \emptyset$. Note that no pair of vertices of S_1 have a common neighbour as $S_1 \cap W_2 = \emptyset$. Also $N_G(S_1) \cap (W \cup Y_k) = \emptyset$ as $S_1 \cap W_1 = \emptyset$. Furthermore $v \in S_1$ implies $|N_G(v) \cap X| \leq 1$ as $S_1 \cap X = \emptyset$. Thus to prove (2.25) we need only show that if $v \in S_1$ and $d_G(v) = k$, then $N_G(v) \cap X = \emptyset$. But this follows from $S_1 \cap Y_k = \emptyset$.

We claim next that if (2.5) holds with $l = k + 4$, then

$$|N_H(S_2)| \geq (k+2)\,|S_2|. \tag{2.26}$$

For then $|N_G(S_2)| \geq (k+4)\,|S_2|$ and for each $v \in S_2$, $|(N_G(v)| \leq |N_H(v)| + 2$. This is because v is incident with at most one edge of M and is adjacent to at most one vertex of $W \cup X \cup Y_k$. It is a simple matter to verify (2.23) from (2.24), (2.25) and (2.26). □

Lemma 2.4.

$$|A_k| \geq n\left(1 - (1 + \varepsilon(c))\frac{e^{k-1}}{(k-1)!}e^{-c}\right) \text{ a.s.}, \tag{2.27}$$

where $\varepsilon(c) \to 0$ as $c \to \infty$.

Proof.

$$|A_k| \geq |V_k(G)| - |W| - |X| - |Y_k - W \cup X|.$$

We show first that

$$|Y_k - W \cup X| \leq |X|. \tag{2.28}$$

For $y \in Y_k - X$ there is, by definition, a unique $x(y) \in X$ such that y is adjacent to $x(y)$ in G. Now for distinct $y_1, y_2 \in Y_1 - W$ we have $x(y_1) \neq x(y_2)$ else $y_1 \in W_2$ and (2.28) follows.

Now let Z_0 be the set of vertices of degree $\leq k-1$ in G and let $Z_1, Z_2, \ldots$ be the sequence of sets removed in each iteration of the k-core finding algorithm. Now, it is well known that

$$|Z_0| = (1 - o(1))n\left(1 - \sum_{i=0}^{k-1} \frac{c^i e^{-c}}{i!}\right) \text{ a.s.}$$

We show that

$$Z_i \subseteq X \cup W_1 \cup Y_k \quad (i = 1, 2, \ldots)$$

Thus assume inductively that $Z_1, Z_2, \ldots, Z_{i-1} \subseteq X \cup W_1 \cup Y_k$ for some $i \geq 1$ (true vacuously for $i = 1$) and let $T = \bigcup_{t=0}^{i-1} Z_t$. Then $y \in Z_i$ implies $d_G(y) \geq k$ but $|N_G(y) - T| \leq k - 1$.

Case 1. $|N_G(y) \cap T| \geq 2$
By assumption $T \subseteq X \cup$ SMALL and so $y \in X$.

Case 2. $|N_G(y) \cap T| = 1$
Then $d_G(y) = k$ implies $y \in X \cup W_1 \cup Y_k$.
Hence $|V_k(G)| \geq |Z_0| - |X \cup W_1 \cup Y_k|$ and the lemma follows. □

Lemma 2.5. *Let c be large and G satisfy the conditions in Lemmas* 2.1, 2.2 *and* 2.3. *Let X be a t-factor of H_k where, $t < k$. Then $H = (A_k, E(A_k) - X)$ is connected.*

Proof. If H is not connected, then there exists a nonempty $S \subseteq A_k$ such that $N_H(S) = \emptyset$. We show that this is not possible for c large enough. (2.23) implies that $|S| \geq n/(2k+8)$. (2.27) implies that, for c large, fewer than $2c^{k-1}e^{-c}n$ vertices are deleted from G in producing H. Then (2.2) implies that at most $8c^k e^{-c} n$ edges are lost in the construction. But then (2.6) with $l = k + 4$ implies that not all edges with one vertex in S have been deleted. □

Suppose a graph G contains h edge-disjoint hamilton cycles. Let the graph obtained from G by deleting the edges in these cycles be referred to as an h-subgraph of G.

Define $\phi(G) = (h, p)$ by

$$h = \text{maximum number of disjoint hamilton cycles in } G;$$

$$p = \begin{cases} 0 & \text{if } k \leqslant 2h \\ \text{maximum cardinality of a matching in any } h\text{-subgraph of } G & \text{if } k = 2h+1 \\ \text{maximum length of a path in any } h\text{-subgraph of } G & \text{if } k \geqslant 2h+2 \end{cases}$$

If $\phi(G) = (h, p)$ we define a ϕ-subgraph H of G to be any h-subgraph of G containing either a matching of size p or a path of length p as the case may be. Let the edges in $E(G) - E(H)$ be referred to as a ϕ-set.

Lemma 2.6. *Let H be a graph which cannot be disconnected by the removal of a t-factor, $t < k$. Suppose that H does not have property M_k. Then there exists $U = \{u_1, u_2, \ldots, u_t\} \subseteq V(H)$ and for each $u_i \in U$, a set $U_i \subseteq V(H)$ such that*

(i) *$u_i \in U$, $w \in U_i$ implies $(u_i, w) \notin E(H)$ and $\phi(\hat{H}) > \gamma(H)$ (in the lexicographic ordering), where $\hat{H}$ is obtained from H by adding the edge (u_i, w).*

(ii) $|N_H(U_i)| < k\,|U_i|$, $i = 1, 2, \ldots, t$.

Proof. Let $(h, p) = \phi(H)$ and H' be a ϕ-subgraph of H. We deduce that H' is connected.

Case 1. $h < \lfloor \frac{1}{2}k \rfloor$

Let $U = \{u_1, u_2, \ldots, u_t\}$ be the set of vertices which are endpoints of longest paths of H'. Posa [12] has shown that for each $u_i \in U$ there exists a set $U_i \subseteq U$ such that

(a) for each $w \in U_i$ there is a longest path in H' with endpoints u_i, w;

(b) $|N_{H'}(U_i)| < 2\,|U_i|$.

Since H' is connected and non-hamiltonian no edge joins the endpoints of any longest path. Adding such an edge must increase ϕ (in the lexicographic sense).

Case 2. $h = \lfloor \frac{1}{2}k \rfloor$, k odd

Let $\mathcal{M}$ be the set of maximum cardinality matchings of H. Let $U = \{u_1, u_2, \ldots, u_t\}$ be the set of vertices left isolated by some $M \in \mathcal{M}$.

Let $u_i \in U$ and let some $M_i \in \mathcal{M}$ leave u_i isolated. Let $S_i \neq \emptyset$ be the set of vertices, different from u_i, left isolated by M_i. Let U_i' be the set of vertices reachable from S_i by an even length alternating path w.r.t. M_i. Let $U_i = S_i \cup U_i' \subseteq U$. It is clear that (1) holds.

If $u \in N_H(U_i)$, then $u \notin S_i$ and so there exists y_1 such that $\{u, y_1\} \in M_i$. We show that $y_1 \in U_i$ which will prove that $|N_{H'}(U_i)| < |U_i|$ and the lemma. Now there exists $y_2 \in U_i$ such that $\{u, y_2\} \in E(H)$. Let P be an even length alternating path from some $s \in S_i$ terminating at y_2. If P contains $\{u, y_1\}$ we can truncate it to terminate with $\{u, y_1\}$, otherwise we can extend it using edges $\{y_2, x\}$ and $\{x, y_1\}$.

We are now ready for the

Proof of Theorem 1.3. We use a coloring argument that was introduced in Fenner and Frieze [6]. Suppose that after generating $G = G_{n,p}$ all its edges are colored blue, and then each edge of G is re-colored green with probability $p' = (\log n)/cn$ and left blue with probability $1 - p'$. These recolourings are done independently of each other.

Let E^b, E^g denote the blue and green edges respectively and let $G^b = (V_n, E^b)$, $H_k = H_k(G)$ and $H_k^b = H_k(G^b)$.

Remark 2.7. It is important to note that for a fixed value of E^b, E^g is a random subset of $\bar{E}^b$, where each $e \in \bar{E}^b$ is independently included in E^g with probability $p_1 = pp'/(1 - p(1 - p'))$ and excluded with probability $1 - p_1$.

Consider next the following 2 events:

$\mathscr{G} \equiv G = G_{n,p}$ satisfies the conditions of Lemmas 2.1, 2.2, 2.3 and

$$\phi(H_k) < (\lfloor \tfrac{1}{2}k \rfloor, (\tfrac{1}{2}a)(k - 2\lfloor \tfrac{1}{2}k \rfloor)), \quad \text{where } a = |A_k(G)|.$$

$\mathscr{E} \equiv$ (a) $\emptyset \neq S \subseteq A_k(G^b)$, $|S| \leqslant n/(2k+8)$ implies $|N_{H_k^b}(S)| \geqslant k\,|S|$;

(b) there does not exist $e = \{v, w\} \in E^g$, $e \subseteq A_k(G^b)$ such that $\phi(H_k^b + e) > \phi(H_k^b)$.

In consequence of what has already been proved, we need only prove

$$\lim_{n \to \infty} \Pr(\mathscr{G}) = 0. \tag{2.29}$$

To prove (2.29) we shall prove that for c large

$$\Pr(\mathscr{E} \mid \mathscr{G}) \geqslant (1 - o(1))(1 - p')^{kn}, \tag{2.30a}$$

$$\Pr(\mathscr{E}) \leqslant (1 - p_1)^{n^2/(2(2k+8)^2)}, \tag{2.30b}$$

which together imply (2.29).

Proof of (2.30a). Let $G_0 \in \mathscr{G}$ be fixed and let F_0 be any fixed ϕ-set of H_k. We prove

$$\Pr(\mathscr{E} \mid G_{n,p} = G_0) \geqslant (1 - p')^{kn} - 16(\log n)^4/c^2 n. \tag{2.31}$$

We can readily verify this once we have shown that

$$\mathscr{E} \cap \mathscr{G} \supseteq \mathscr{E}_1 \cap \mathscr{E}_2 \cap \mathscr{E}_3 \cap \mathscr{G},$$

where

$\mathscr{E}_1 \equiv E^g$ is a matching of G_0;

$\mathscr{E}_2 =$ no green edge meets any vertex of degree less than $c/10 + 2$ in G_0 or any vertex in $W \cup X \cup Y_k$;

$\mathscr{E}_3 = F_0 \cap E^g = \emptyset$.

For $\mathscr{E}_1 \cap \mathscr{E}_2$ implies

$$A_k(G_0^b) = A_k(G_0)$$

and then $\mathscr{E}_1$ implies (see Lemma 2.3) that (2.23) holds, which verifies $\mathscr{E}(a)$. $\mathscr{E}_3$ implies $\mathscr{E}(b)$.

Now it follows from (2.3) that

$$\Pr(\bar{\mathscr{E}}_1) \leq 16(\log n)^4/c^2n.$$

From Lemmas 2.1, 2.2 and (2.27) we find that the total number of edges of G_0 that are excluded by the conditions in $\mathscr{E}_2$, $\mathscr{E}_3$ is no more than

$$n((c/10+1)e^{-2c/3} + 4c^k e^{-c}) + \tfrac{1}{2}kn \leq kn$$

Thus

$$\Pr(\bar{\mathscr{E}}_1 \cup \bar{\mathscr{E}}_2 \cup \bar{\mathscr{E}}_3) \leq 1 - (1-p')^{kn} + 16(\log n)^4/c^2n,$$

which proves (2.31). □

Proof of (2.30b). Now

$$\Pr(\mathscr{E}) = \sum_{\Gamma} \Pr(\mathscr{E} \mid G^b = \Gamma)\Pr(G^b = \Gamma),$$

where Γ is an arbitrary graph with vertices V_n.

Now if $H_k(\Gamma)$ fails to satisfy $\mathscr{E}(a)$, then $\Pr(\mathscr{E} \mid G^b = \Gamma) = 0$. So let us assume that $\mathscr{E}(a)$ holds.

Now if $U, U_1, \ldots, U_t$ are as defined in Lemma 2.6 with $H = H_k$, then each set is of size at least $n/(2k+8)$ and for $\mathscr{E}(b)$ to hold no green edge can join $u_i \in U$ to $w \in U_i$. But then in view of Remark 2.7 and $\mathscr{E}(a)$ we have

$$\Pr(\mathscr{E}(b) \mid G^b = \Gamma) \leq (1-p_1)^{n^2/(2(2k+8)^2)},$$

which implies (2.30b). □

Finally, let us consider what happens when $c \to \infty$. The above proof shows that H_k a.s. has property M_k. For $k=1$ and $c = \log n + x$, x constant, one can easily show that A_1 a.s. comprises all non-isolated vertices of G. Thus we obtain Erdös and Renyi's result [5] as a corollary. Similarly, when $k=2$ and $c = \log n + \log\log n + x$, A_2 a.s. comprises all vertices of degree at least 2 and so we obtain Komlós and Szemerédi's result [9] as well. (Tomasz Luczak pointed out an error in an earlier statement of these last two results). □

Corollary 1.2 follows directly from Theorem 1.2 and the Percolation Theorem of McDiarmid [11].

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