Randomly coloring simple hypergraphs

Alan Frieze*   Páll Melsted
Department of Mathematical Sciences,
Carnegie Mellon University,
Pittsburgh PA15213.
Email: alan@random.math.cmu.edu; pmelsted@gmail.com.

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Abstract

We study the problem of constructing a (near) uniform random proper q-coloring of a simple k-uniform hypergraph with n vertices and maximum degree Δ. (Proper in that no edge is mono-colored and simple in that two edges have maximum intersection of size one). We show that if for some $\alpha < 1$ we have $\Delta \geq n^\alpha$ and $q \geq \Delta^{(1+\alpha)/k\alpha}$ then Glauber dynamics will become close to uniform in $O(n \log n)$ time from a random (improper) start. Note that for $k > 1 + \alpha^{-1}$ we can take $q = o(\Delta)$.

1   Introduction

Markov Chain Monte Carlo (MCMC) is an important tool in sampling from complex distributions. It has been successfully applied in several areas of Computer Science, most notably for estimating the volume of a convex body [3], [8], [9] and estimating the permanent of a non-negative matrix [7].

Generating a (nearly) random q-coloring of a n-vertex graph $G = (V, E)$ with maximum degree Δ is a well-studied problem in Combinatorics [1] and Statistical Physics [10]. Jerrum [6] proved that a simple, popular Markov chain, known as the *Glauber dynamics*, converges to a random q-coloring after $O(n \log n)$ steps, provided $q/\Delta > 2$. This led to the challenging problem of determining the smallest value of q/Δ for which a random q-coloring can be generated in time polynomial in n. Vigoda [11] gave the first significant improvement over Jerrum's result, reducing the lower bound on q/Δ to 11/6 by analyzing a different Markov chain. There has been no success in extending Vigoda's approach to smaller values of q/Δ, and it remains the best bound for general graphs. There are by now several papers giving improvements on [11], but in special cases. See Frieze and Vigoda [5] for a recent survey.

In this paper we consider the related problem of finding a random coloring of a simple k-uniform hypergraph. A k-uniform hypergraph $H = (V, E)$ has vertex set V and $E = \{e_1, e_2, \ldots, e_m\}$ are the edges. Each edge is a k-subset of V. Hypergraph H is simple if $|e_i \cap e_j| \leq 1$ for $i \neq j$. A coloring of H is proper if every edge contains two vertices of a different color. The chromatic number $\chi(H)$ is the smallest number of colors in a proper coloring of H. In the case of graphs $k = 2$ we have $\chi(G) \leq \Delta + 1$ but for hypergraphs $(k \geq 3)$ we have much smaller bounds. For example a simple application of the local lemma implies that $\chi(H) = O(\Delta^{1/(k-1)})$. In fact a recent result of Frieze and Mubayi [4], is that for simple hypergraphs $\chi(H) = O((\Delta/\log \Delta)^{1/(k-1)})$. The proof of [4] is somewhat more involved. It relies on a proof technique called the "nibble". The aim of this short paper is to study randomly coloring simple hypergraphs when there are fewer than Δ colors available. The goal is to describe an MCMC algorithm for (near) uniform sampling.

Before formally stating our theorem we will define the Glauber dynamics. All of the afore-mentioned results on coloring graphs (except Vigoda [11]) analyze the Glauber dynamics, which is a simple and popular Markov chain for generating a random q-coloring.

Let $\mathcal{Q}$ denote the set of proper q-colorings of H. For a coloring $X \in \mathcal{Q}$ we define

$$B(v, X) = \{c \in [q] : \exists e \ni v \text{ such that } X(x) = c \text{ for all } x \in e \setminus \{v\}\}$$

be the set of colors unavailable to v.

Then let $Q = \{1, 2, \ldots, q\}$ and

$$A(v, X) = Q \setminus B(v, X).$$

For technical purposes, the state space of the Glauber dynamics is $\Omega = Q^V \supseteq \mathcal{Q}$. From a coloring $X_t \in \Omega$, the evolution $X_t \to X_{t+1}$ is defined as follows:

Glauber Dynamics
(a) Choose $v = v(t)$ uniformly at random from V.

(b) Choose color $c = c(t)$ uniformly at random from $A(v, X_t)$. If $A(v, X_t)$ is empty we let $X_{t+1} = X_t$.

(c) Define X_{t+1} by

$$X_{t+1}(u) = \begin{cases} X_t(u) & u \neq v \\ c & u = v \end{cases}$$

We will assume from now on that

$$q \leq 2\Delta \tag{1}$$

If $q > 2\Delta$ then we defer to Jerrum's result [6].

Let Y denote a coloring chosen uniformly at random from $\mathcal{Q}$. We will prove the following:

Theorem 1 *Let H be a k-uniform simple hypergraph with maximum degree Δ where $k \geq 3$. Suppose that* (1) *holds and that*

$$\Delta \geq n^\alpha \tag{2}$$

2

for some α satisfying

$$\frac{2}{k-2} < \alpha < 1 \qquad\qquad\qquad k \geq 5$$

$$\frac{1}{k-1} < \alpha < 1 \qquad\qquad\qquad k = 3, 4$$

Suppose that for a sufficiently large constant K,

$$q \geq K\Delta^{(1+\alpha)/\alpha k}. \tag{3}$$

Suppose that the initial coloring X_0 is chosen randomly from Ω. Then for an arbitrary constant $\delta > 0$ we have

$$d_{TV}(X_t, Y) \leq \delta \tag{4}$$

for $t \geq t_\delta$, where $t_\delta = 2n \log(2n/\delta)$.

Here d_{TV} denotes variational distance i.e. $\max_{S \subseteq \mathcal{Q}} |\mathbf{Pr}(X_t \in S) - \mathbf{Pr}(Y \in S)|$.

Our assumptions on α, Δ imply some inequalities that we put here for future reference:

$$q^k \gg n\Delta. \tag{5}$$
$$q^{k-2} \gg \Delta \qquad for\ k \geq 5. \tag{6}$$
$$q \gg \Delta^{2/3} \qquad for\ k = 3 \tag{7}$$
$$q \gg \Delta^{1/2} \qquad for\ k = 4 \tag{8}$$

Here we use $A_n \gg B_n$ to mean that $A_n/B_n \to \infty$ as $n \to \infty$.

Note that H simple implies that

$$\Delta \leq \frac{n-1}{k-1} < \frac{n}{k}.$$

Note that we do not claim rapid mixing from an arbitrary start. Indeed, since we are using relatively few colors, it is possible to choose an initial coloring from which there is no Glauber move i.e. we do not claim that the chain is ergodic, see Section 5 for examples of blocked colorings.

The algorithm can be used in a standard way, [6], to compute an approximation to the number of proper colorings of H.

We can also prove the following. We can consider Glauber Dynamics as inducing a graph $\Gamma_{\mathcal{Q}}$ on $\mathcal{Q}$ where two colorings are connected by an edge if there is a move taking one to the other. Note that if Glauber can take X to Y in one step, then it can take Y to X in one step.

Corollary 1 *The graph $\Gamma_{\mathcal{Q}}$ contains a giant component $\mathcal{Q}_0$ of size $(1 - o(1))|\mathcal{Q}|$.*

2 Good and bad colorings

Let $X \in \Omega$ be a coloring of V. For a vertex v and $1 \leq i \leq k - 1$ let

$$E_{v,i,X} = \{e : \ v \in e \ and \ | \{X(w) : w \in e \setminus \{v\}\} | = i\}$$

be the set of edges e containing v in which $e \setminus \{v\}$ uses exactly i distinct colors under X. Let $y_{v,i,X} = |E_{v,i,X}|$ so that $|B(v, X)| \leq y_{v,1,X}$ for all v, X. Let

$$\epsilon = \frac{1}{100(2k)^k}.$$

We divide our analysis into three cases: $k \geq 5, k = 3$ and $k = 4$.

2.1 $\ k \geq 5$

We define the sequence $\boldsymbol{\epsilon} = \epsilon, \epsilon^2, \ldots, \epsilon^{k-2}$.

Definition 1 *We say that X is $\theta\epsilon$-bad if $\exists v \in V, 1 \leq i \leq k - 2$ such that*

$$y_{v,i,X} \geq \theta\mu_i \ where \ \mu_i = \min\left\{(\epsilon q)^i, \frac{\Delta}{(10k)^{k-2-i}\epsilon q}\right\}. \tag{9}$$

Otherwise we say that X is $\theta\boldsymbol{\epsilon}$-good. (If $\theta = 1$ then we drop it).

Given this definition, we have

$$10k\mu_i \leq \mu_{i+1} \leq \epsilon q\mu_i \qquad for \ 1 \leq i \leq k - 3. \tag{10}$$

It is convenient to define

$$\mu_{k-1} = \Delta. \tag{11}$$

Note that if X is $\boldsymbol{\epsilon}$-good then $|A(v, X)| \geq (1 - \epsilon)q$ for all $v \in V$.

In this section we will show that almost all colorings of Ω are $\boldsymbol{\epsilon}$-good and almost all colorings in $\mathcal{Q}$ are $\boldsymbol{\epsilon}$-good. Consider a random coloring $X \in \Omega$. We first estimate the probability that it is properly colored. We use the local lemma.

Fix an edge $e \in H$. Then using $\mathbf{Pr}_\Omega$ to indicate that the random choice is from Ω,

$$p = \mathbf{Pr}_\Omega(e \text{ is not properly colored by } X) = \frac{1}{q^{k-1}}.$$

Now consider a dependency graph, in the context of the local lemma. The events are $\mathcal{B}_e = \{e$ is not properly colored$)\}$. $\mathcal{B}_e$ and $\mathcal{B}_f$ are indpendent if $e \cap f = \emptyset$. Thus the maximum degree Δ_1 in the dependency graph is bounded by $k\Delta$. Then

$$4\Delta_1 p \leq 4k\Delta p \leq \frac{4k\Delta}{K^{k-1}\Delta^{(k-1)(1+\alpha)/(\alpha k)}} = \frac{4k}{K^{k-1}\Delta^{(k-1-\alpha)/(\alpha k)}} < 1. \tag{12}$$

4

So, by the local lemma, if $m \leq \Delta n/k$ is the number of edges in H,

$$\mathbf{Pr}_\Omega(X \text{ is proper}) \geq (1 - 2p)^m \geq (1 - 2p)^{\Delta n/k} \geq e^{-2p\Delta n/k(1-2p)} \geq e^{-3\Delta n/(kq^{k-1})}. \quad (13)$$

For the third inequality we use $(1 - x) \geq e^{-x/(1-x)}$ whenever $0 < x < 1$.

Given this, we consider the probability that there is a bad vertex. For a fixed vertex v and an edge e containing v the probability that $e \in E_{v,i,X}$ is at most $\binom{k-1}{i}\left(\frac{i}{q}\right)^{k-1-i}$. Indeed, there will be a set of i vertices S, for which there are $\binom{k-1}{i}$ possibilities such that the remaining $k - 1 - i$ vertices have their colors all used by S. The probability of this being at most $\left(\frac{i}{q}\right)^{k-1-i}$. Thus the value $y_{v,i,X}$ has distribution dominated by the binomial $Bin\left(\Delta, \binom{k-1}{i}\left(\frac{i}{q}\right)^{k-1-i}\right)$.

So, from Chernoff bound:

$$Pr(B(n,p) \geq \lambda np) \leq \left(\frac{e}{\lambda}\right)^{\lambda np} \quad (14)$$

we see that

$$\mathbf{Pr}(y_{v,i,X} \geq \mu_i) \leq \left(\frac{e\binom{k-1}{i}(i/q)^{k-1-i}\Delta}{\mu_i}\right)^{\mu_i}$$

$$\leq \left(\frac{e(2k)^k\Delta}{\min\left\{\epsilon^i q^{k-1}, \epsilon^{-1}(q/(10k))^{k-2-i}\Delta\right\}}\right)^{\mu_i}$$

$$\leq e^{-\mu_i}$$

$$\leq e^{-\epsilon \min\{q,\Delta/q\}}$$

for $i = 1, 2, \ldots, k - 2$, since $\mu_i \geq \mu_1 \geq \epsilon \min\{q, \Delta/q\}$. (We use (6) and our definition of ϵ to go from the second to the third line).

It follows that if X is chosen randomly from Ω then

$$\mathbf{Pr}_\Omega(X \text{ is } \epsilon - bad) \leq kne^{-\epsilon \min\{q,\Delta/q\}}. \quad (15)$$

So, using $\mathbf{Pr}_\mathcal{Q}$ to indicate the random choice is from $\mathcal{Q}$, and using (13),

$$\mathbf{Pr}_\mathcal{Q}(X \text{ is } \epsilon - bad) = \mathbf{Pr}_\Omega(X \text{ is } \epsilon - bad \mid X \text{ is a proper coloring})$$

$$\leq kn \exp\left\{-\epsilon \min\{q, \Delta/q\} + \frac{3n\Delta}{kq^{k-1}}\right\}$$

$$\leq e^{-\epsilon \min\{q,\Delta/q\}/2} = o(1),$$

after using (5) and (6).

Thus **whp**, a random proper or improper q-coloring of H is **ϵ**-good.

Remark 1 *We observe that if $q^{k-1}/n\Delta \to \infty$ then a random coloring (from Ω) is proper* **whp***, in which case we won't need to use the local lemma.*

5

Remark 2 *It is reasonable to conjecture that* $\mathbf{Pr}_{\mathcal{Q}}(X$ *is* $\epsilon - bad)$ *is not much larger than* $\mathbf{Pr}_{\Omega}(X$ *is* $\epsilon - bad)$. *Using the Local Lemma inflates the first proability by too much, seemingly. A better estimate of the first probability would allow us to consider smaller* q.

2.2 $k = 3$

In this case we let $\mu_1 = \epsilon q$. $y_{v,1,X}$ has distribution dominated by $Bin(\Delta, 1/q)$ and so

$$\mathbf{Pr}(y_{v,1,X} \geq \mu_1) \leq \left(\frac{e\Delta}{q\mu_1} \right)^{\mu_1} \leq e^{-q}$$

since $q \gg \Delta^{2/3}$ here, see (7). So

$$\mathbf{Pr}_{\mathcal{Q}}(X \text{ is } \epsilon - bad) \leq 3n \exp\left\{ -q + \frac{n\Delta}{q^2} \right\} = o(1),$$

using (5).

2.3 $k = 4$

In this case we let $\mu_1 = \epsilon q$ and $\mu_2 = 10k\mu_1$. $y_{v,1,X}$ has distribution dominated by $Bin(\Delta, 3/q^2)$ and $y_{v,2,X}$ has distribution dominated by $Bin(\Delta, 3/q)$ so

$$\mathbf{Pr}(y_{v,1,X} \geq \mu_1) \leq \left(\frac{3e\Delta}{q^2\mu_1} \right)^{\mu_1} \leq e^{-q}$$

$$\mathbf{Pr}(y_{v,2,X} \geq \mu_2) \leq \left(\frac{3e\Delta}{q\mu_2} \right)^{\mu_2} \leq e^{-q}$$

since $q \gg \Delta^{1/2}$ here, see (8). So

$$\mathbf{Pr}_{\mathcal{Q}}(X \text{ is } \epsilon - bad) \leq 4n \exp\left\{ -q + \frac{3n\Delta}{4q^3} \right\} = o(1),$$

using (5).

3   Persistence of goodness

We once again divide our analysis into three cases: $k \geq 5, k = 3$ and $k = 4$.

3.1 $k \geq 5$

We show first that

Lemma 1

$$\mathbf{Pr}(X_t \ is \ 2\epsilon - good \ for \ t \leq t_0 \mid X_0 \ is \ \epsilon - good) \geq 1 - 2^{-\mu_1/2}. \tag{16}$$

where

$$t_0 = \frac{n}{4k^2 e}. \tag{17}$$

Proof   For vertices $x, y \in V$ that share an edge, let $e(x, y)$ be that edge.

Observe that if X_t is 2ϵ-good and $x \in V$ then

$$\mathbf{Pr}(y_{x,i,t+1} - y_{x,i,t} = 1 \mid X_t) \leq \frac{(k-2)^2 y_{x,i+1,t}}{(1-2\epsilon)qn} + \frac{ky_{x,i-1,t}}{n} \tag{18}$$

for $1 \leq i \leq k - 2$. (Here $y_{x,0,t} = 0$).

Indeed, we have $y_{x,i,t+1} - y_{x,i,t} = 1$ **only if** with $v = v(t)$ we have (i) $e(v, x) \in E_{x,i+1,X_t}$ and (ii) $X_t(v)$ is a color used once on $e(v, x) \setminus \{x\}$ and (iii) $c(t)$ is used on $e(v, x) \setminus \{x, v\}$ or we have (ia) $e(v, x) \in E_{x,i-1,X_t}$.

Now

$$\mathbf{Pr}((i), (ii), (iii)) = \mathbf{Pr}((i)) \cdot \mathbf{Pr}((ii) \mid (i)) \cdot \mathbf{Pr}((iii) \mid (i), (ii)) \leq$$
$$\frac{(k-1)y_{x,i+1,t}}{n} \cdot \frac{i}{k-1} \cdot \frac{i}{(1-2\epsilon)q} \leq \frac{(k-2)^2 y_{x,i+1,t}}{(1-2\epsilon)qn}$$

and

$$\mathbf{Pr}((ia)) \leq \frac{(k-1)y_{x,i-1,t}}{n}.$$

We consider the following sequence of events for $1 \leq i \leq k - 2$:

$$\mathcal{B}_i(t) = \{\exists s \leq t, v \in V : X_\tau \ is \ 2\epsilon - good \ for \ \tau < s \ and \ y_{v,i,s} \geq y_{v,i,0} + \mu_i\}$$

Let $\mathcal{B}(t) = \bigcup_{i=1}^{k-2} \mathcal{B}_i(t)$ and note that if $\neg\mathcal{B}(t)$ then X_t is 2ϵ-good.

Now X_0 is ϵ-good and so (18) implies that so long as X_τ is 2ϵ-good for $\tau < s \leq t$, we have

$$y_{v,k-2,s} \leq y_{v,k-2,0} + Bin\left(s, \frac{(k-2)^2 y_{v,k-1,t}}{(1-2\epsilon)qn} + \frac{ky_{v,k-3,t}}{n}\right)$$
$$\leq y_{v,k-2,0} + Bin\left(s, \frac{k^2}{n}\left(\frac{\Delta}{q} + \frac{2\mu_{k-3}}{k}\right)\right). \tag{19}$$

The probability in the first binomial above is the RHS of (18) and in the second we bound $y_{v,k-1,t}$ by Δ and $y_{v,k-3,t}$ by $2\mu_{k-3}$. Then note that $y_{v,k-2,i+1}$ is $y_{v,k-2,i} + 1$ with at most this probability. So, on using (14),

$$\mathbf{Pr}(\mathcal{B}_{k-2}(t)) \leq tn\left(\frac{2etk^2}{n}\left(\frac{\Delta}{\mu_{k-2}q} + \frac{2\mu_{k-3}}{k\mu_{k-2}}\right)\right)^{\mu_{k-2}}$$
$$\leq tn\left(\frac{2etk^2}{n}\left(\frac{\Delta}{\epsilon^{k-2}q^{k-1}} + \epsilon + \frac{1}{5k^2}\right)\right)^{\mu_{k-2}} \leq tn\left(\frac{2etk^2}{n}\right)^{\mu_{k-2}} \leq \frac{t_0 n}{2^{\mu_{k-2}}}. \tag{20}$$

We have dealt with the minimum in the definition of μ_{k-2} by using the inequality $\frac{A}{\min X,Y} \leq \frac{A}{X} + \frac{A}{Y}$. The reader will observe that we have not shown that $t_0 \geq \mu_{k-2}$. We do not claim this and when $t_0 < \mu_{k-2}$ we can replace the RHS of (20) by zero.

For $i < k - 2$, (18) implies that so long as X_τ is 2ϵ-good for $\tau < s \leq t$ and $\bigcup_{j=i+1}^{k-2} \mathcal{B}_j(t)$ does not occur, we have $y_{v,i,s} \leq y_{v,i,0} + Bin(s, p)$ where

$$p = \frac{2(k-2)^2 \mu_{i+1}}{(1-2\epsilon)qn} + \frac{2k\mu_{i-1}}{n}.$$

So, on using (10),

$$\mathbf{Pr}\left(\mathcal{B}_i(t) \middle| \neg \bigcup_{j=i+1}^{k-2} \mathcal{B}_j(t)\right) \leq t_0 n \left(\frac{2e(k-2)^2 \mu_{i+1} t}{(1-2\epsilon)qn\mu_i} + \frac{2ke\mu_{i-1} t}{n\mu_i}\right)^{\mu_i} \leq t_0 n \left(\frac{2k^2 et}{n}\right)^{\mu_i} \leq \frac{t_0 n}{2^{\mu_i}}.$$

(21)

Equation (16) and the lemma follows from (20) and (21). $\square$

We show next that

Lemma 2

$$\mathbf{Pr}(X_{t_0} \text{ is } \epsilon - good \mid X_0 \text{ is } \epsilon - good) \geq 1 - e^{-c\mu_1} \qquad for \ some \ c > 0. \qquad (22)$$

Proof For this we use the fact that if X_t is 2ϵ-good and $x \in V$ then

$$\mathbf{Pr}(y_{x,i,t+1} - y_{x,i,t} = -1 \mid X_t) \geq \frac{y_{x,i,t}}{n} \qquad (23)$$

for $1 \leq i \leq k - 2$.

Indeed, we have $y_{x,i,t+1} - y_{x,i,t} = -1$ if (i) $e(v,x) \in E_{x,i,X_t}$, (ii) $X_t(v)$ is used more than once on $e(v,x) \setminus \{x\}$ and (iii) $c(t)$ is a color not used $e(v,x) \setminus \{x,v\}$. Now

$$\mathbf{Pr}((i),(ii),(iii)) = \mathbf{Pr}((i)) \cdot \mathbf{Pr}((ii) \mid (i)) \cdot \mathbf{Pr}((iii) \mid (i),(ii)) \geq$$
$$\frac{(k-1)y_{x,i,t}}{n} \cdot \frac{k-i}{k} \cdot \frac{q-i}{q} \geq \frac{2(k-1)(q-i)y_{x,i,t}}{kqn} \geq \frac{y_{x,i,t}}{n}.$$

We couple $y_{x,i,t}$ with a biased random walk $Y_t, t \geq 0$ on $\{0,1,2,\ldots,\}$. Here $Y_0 = \mu_i$ and

$$Y_{t+1} = \begin{cases} Y_t + 1 & Probability \ \frac{2(k-2)^2 \mu_{i+1}}{(1-2\epsilon)qn} + \frac{2k\mu_{i-1}}{n} \\ Y_t - 1 & Probability \ \frac{Y_t}{n} \\ Y_t & Probability \ 1 - \frac{2(k-2)^2 \mu_{i+1}}{(1-2\epsilon)qn} - \frac{2k\mu_{i-1}}{n} - \frac{Y_t}{n} \end{cases} \qquad (24)$$

If $\mathcal{B}(t)$ does not occur then Y_t has no lower a chance of increasing by one than $y_{x,i,t}$ and when $Y_t = y_{i,x,t}$ it has no greater a chance of decreasing by one. We can therefore, conditional on $\neg\mathcal{B}(t)$, couple $y_{x,i,t}, Y_t$ so that $y_{x,i,t} \leq Y_t$ always. We can therefore prove (22) by proving

$$\mathbf{Pr}(Y_{t_0} > \mu_i) \leq e^{-c\mu_i} \qquad (25)$$

8

Let $I = \{t \leq t_0 : Y_t \geq \mu_i/2\}$ and $J = \{t \leq t_0 : Y_{t+1} \neq Y_t\}$. If $I = \emptyset$ then $Y_{t_0} < \mu_1/2$. Furthermore, if $t \in I$ then (i) $\mathbf{Pr}(t \in J) \geq \mu_i/(2n)$ and (ii) $\mathbf{Pr}(Y_{t+1} = Y_t - 1 \mid Y_{t+1} \neq Y_t) \geq \frac{2}{3}$. To verify (ii) use (10) to show that

$$\frac{2n}{\mu_i} \cdot \left(\frac{2k^2 \mu_{i+1}}{(1-2\epsilon)qn} + \frac{2k\mu_{i-1}}{n} \right) \leq \frac{4\epsilon k^2}{1-2\epsilon} + \frac{4k}{10k} \leq \frac{1}{2}. \tag{26}$$

Now consider two cases:

Case 1: $I = [t_0]$: It follows from (i), (ii) above and the Chernoff bounds that

$$\mathbf{Pr}\left(|J| \leq \frac{\mu_i t_0}{4n} \right) \leq e^{-c_1 \mu_i} \qquad \textit{for some } c_1 > 0.$$

And

$$\mathbf{Pr}\left(\sum_{t \in J} (Y_{t+1} - Y_t) \geq -|J|/6 \,\Big|\, |J| \geq \frac{\mu_i t_0}{4n} \right) \leq e^{-c_2 \mu_i} \qquad \textit{for some } c_2 > 0.$$

But $Y_{t_0} \leq \mu_i + \sum_{t \in J} (Y_{t+1} - Y_t)$ and so (25) is verified for this case.

Case 2: $I \neq [t_0]$: Let $t_1 = \max\{t : t \notin I\}$ and let $J_1 = \{t \in J : t > t_1\}$. If $|J_1| < \mu_i/2$ then $Y_{t_0} \leq \frac{\mu_1}{2} + |J_1| < \mu_1$. So assume that $|J_1| \geq \mu_i/2$. But then with probability $1 - e^{-c_3 \mu_i}$ we find that $\sum_{t \in J_1} (Y_{t+1} - Y_t) < 0$ and (25) is verified for this case too.

This completes the proof of (25), and hence (22) and the lemma. $\qquad \square$.

3.2 $\quad k = 3$

Going back to (19) we see that

$$y_{v,1,s} \leq y_{v,1,0} + Bin\left(s, \frac{k^2 \Delta}{nq} \right).$$

And then (20) becomes

$$\mathbf{Pr}(\mathcal{B}_1(t)) \leq tn \left(\frac{18e\Delta t}{nq\mu_1} \right)^{\mu_1} \ll 2^{-\mu_1/2}$$

for $t \leq t_0$ of (17).

We then define Y_t as in (24) and (see (26)) compute $\frac{2n}{\mu_1} \cdot \frac{2\Delta}{(1-2\epsilon)qn} = \frac{4\Delta}{(1-2\epsilon)\epsilon q^2} \ll 1$. We then argue as in the case $k \geq 5$.

3.3 $\quad k = 4$

Going back to (20) we see that

$$\mathbf{Pr}(\mathcal{B}_2(t)) \leq tn \left(\frac{32et}{n} \left(\frac{\Delta}{\epsilon q^2} + \frac{2}{4} \right) \right)^{\mu_2} \ll 2^{-\mu_2/2}.$$

for $t \leq t_0$ of (17).

Going back to (21) we see that

$$\mathbf{Pr}\left(\mathcal{B}_1(t)\middle|\neg\mathcal{B}_2(t)\right) \leq t_0 n \left(\frac{8et}{(1-2\epsilon)qn}\right)^{\mu_1} \ll 2^{-\mu_1/2}$$

for $t \leq t_0$ of (17).

We then define Y_t as in (24) and (see (26)) compute $\frac{2n}{\mu_1} \cdot \frac{8\mu_2}{(1-2\epsilon)qn} = \frac{160k}{(1-2\epsilon)\epsilon q} \ll 1$ (remember that $\mu_2 = 10k\mu_1$ here) and $\frac{2n}{\mu_2}\left(\frac{8\Delta}{(1-2\epsilon)qn} + \frac{2k\mu_1}{n}\right) \leq \frac{1}{2}$. We then argue as in the case $k \geq 5$.

4   Coupling Argument

Now consider a pair X, Y of copies of our Glauber chain. Let

$$h(X_t, Y_t) = |\{v \in V : X_t(v) \neq Y_t(v)\}|$$

be the Hamming distance between X_t, Y_t. We use describe a simple coupling between the chains and show that

$$\mathbf{E}(h(X_{t+1}, Y_{t+1}) \mid X_t, Y_t) \leq \left(1 - \frac{1}{2n}\right) h(X_t, Y_t) \tag{27}$$

if X_t, Y_t are both 2ϵ-good.

Our coupling is the same as that used by Jerrum [6]. The choice of vertex $v(t)$ will be the same in both chains. We maximally couple the choice of color in each chain. Then, with $v = v(t)$,

$$\mathbf{Pr}(X_{t+1}(v) \neq Y_{t+1}(v) \mid X_t, Y_t) \leq (k-1)\frac{|E_{v,1,X_t}| + |E_{v,1,Y_t}|}{(1-2\epsilon)q}.$$

Hence, assuming that X_t, Y_t are both 2ϵ-good for $1 \leq t \leq t_0$ we see that

$$
\begin{aligned}
\mathbf{E}(h(X_{t+1}, Y_{t+1}) \mid X_t, Y_t) &= \sum_{v \in V} \mathbf{Pr}(X_{t+1}(v) \neq Y_{t+1}(v)) \\
&= \sum_{w \in V} \mathbf{Pr}(v(t) \neq w \text{ and } X_t(w) \neq Y_t(w)) \\
&\quad + \sum_{w \in V} \mathbf{Pr}(v(t) = w \text{ and } X_{t+1}(w) \neq Y_{t+1}(w)) \\
&= \frac{n-1}{n} h(X_t, Y_t) + \frac{1}{n} h(X_t, Y_t) \frac{4(k-1)\epsilon}{1-2\epsilon} \\
&\leq \left(1 - \frac{1}{2n}\right) h(X_t, Y_t).
\end{aligned}
$$

Summarising, we have shown that with probability at least $1 - e^{-c\mu_1}$ for some positive constant c, we have that both of X_0 and Y_0 are ϵ-good, both X and Y are 2ϵ-good for

10

t_0 steps and both of X_{t_0} and Y_{t_0} are ϵ-good. If we run the chain for $t_0 t^*$ steps, where $t^* = e^{c\mu_1/2}$ steps then the probability that either chain stops being 2ϵ-good is at most $t^* e^{-c\mu_1} \leq e^{-c\mu_1/2}$. Conditional on these events, $\mathbf{E}(h(X_{t_\delta}, Y_{t_\delta}) \leq \delta/2$ and together with the fact that the variation distance between X_t and Y_t is monotone non-increasing, this implies (4). (Note that δ includes the probability that either of X_0, Y_0 are not 2ϵ-good). This completes the proof of Theorem 1.

4.1   Proof of Corollary 1

The proof of Theorem 1 shows that if $X, Y \in \mathcal{Q}$ are both ϵ-good then there is a path from X to Y in $\mathcal{Q}$ of length $O(n \log n)$. Since almost all of $\mathcal{Q}$ is ϵ-good, we are done. $\square$

5   Blocked example

In this section we give an example of an edge colored hypergraph for which there are no Glauber moves.

Lemma 3 *Let $k \geq 3$ and let m, q be sufficiently large. Suppose that $\epsilon^k \leq \frac{1}{10k!}$ and $\rho = \frac{\epsilon^k}{(qm)^{k-2}}$. Then there exists a hypergraph H with qm vertices and maximum degree $\Delta \in [\frac{\epsilon^k qm}{2(k-1)!}, \frac{2\epsilon^k qm}{(k-1)!}]$ and a coloring with q colors so that there are no Glauber moves.*

Proof   We first choose a simple $(k-1)$-uniform hypergraph H_1 with m vertices and qm edges and maximum degree at most $2k^4 q$. The existence of such a hypergraph is easy to show via the probabilistic method. Fix some $0 < p < 1$ and choose each possible edge to include with probability p. Let Z_1 be the number of edges chosen and let Z_2 be the number of pairs of edges that share two or more vertices. We show is that there is a p such that $\mathbf{E}(Z_1 - Z_2 \mid D) \geq qm$ where D is the event that the maximum degree is at most $2k^4 q$. Now $\mathbf{E}(Z_1) = \binom{m}{k-1} p$ and $\mathbf{E}(Z_2) \leq \binom{m}{k-1}\binom{k-1}{2}\binom{m-2}{k-3} p^2$. Putting $p = \frac{k^4 q}{m\binom{k-1}{2}\binom{m-2}{k-3}}$ gives $\mathbf{E}(Z_1 - Z_2) \geq \frac{k^4 q(m-1)}{(k-1)^2(k-2)^2}$. Now the degree of a vertex is $Bin\left(\binom{m-1}{k-2}, p\right)$ which has mean $\frac{2k^4 q(m-1)}{m(k-1)(k-2)^2}$ and so the probability its degree is greater than $2k^4 q$ is exponentially small in q. Thus $\mathbf{E}(Z_1 - Z_2 \mid D) \geq qm$ and our hypergraph exists.

We build a vertex colored k-uniform simple hypergraph H for which the coloring is proper and for which there are no Glauber moves. We choose disjoint sets $V_1, V_2, \ldots, V_q$ of size m and let $V = V_1 \cup V_2 \cup \cdots V_q$. The vertices in V_i are given color i. We let $H_i = (V_i, E_i)$ be a copy of the hypergraph H_1. Then for each i we define an injective map f_i from $V \setminus V_i \to E_i$. This is possible as $|E_i| = qm \geq |V|$. Then for each $x \in V_j$ and $i \neq j$ we add the edge $F_{x,i} = \{x\} \cup f_i(x)$ to H. These edges block all Glauber moves. Furthermore, we have (i) $F_{x,i} \cap F_{x,i'} = \{x\}$ for $i \neq i'$ and (ii) $|F_{x,i} \cap F_{y,i}| \leq 1$ for $x \neq y$ and (iii) $F_{x,i} \cap F_{y,i} = \emptyset$ for $x \neq y, i \neq j$. Thus the hypergraph created is simple. Denote the set of edges added so far by F_1 and note that $|F_1| \leq q^2 m$.

At the moment the degree of a vertex lies in $[q-1, (k^4+1)q-1]$. We now add random edges F_2 so that we have more flexibility with the maximum degree. We only consider edges with at most one vertex in each V_i and we add these with probability ρ. Now let A_1 denote the number of pairs of edges in F_2 that share two or more vertices and let A_2 denote the number of pairs of edges, one from F_1 and one from F_2 that share two or more vertices.

Now $\mathbf{E}(|F_2|) = \binom{q}{k}m^k\rho$ and our choice of ρ is such that $\mathbf{E}(|F_2|) \geq 2\mathbf{E}(A_1 + A_2)$. Now

$$\mathbf{E}(A_1 + A_2) \leq$$
$$\binom{q}{k}m^k\binom{k}{2}\binom{q-2}{k-2}m^{k-2}\rho^2 + q^2m\binom{k}{2}\binom{q-2}{k-2}m^{k-2}\rho \leq (qm)^{2k-2}\rho^2 + k^2q^km^{k-1}\rho.$$

Concentration of measure now ensures that Δ is in the range required. $\square$

6 Open Questions

(a) Can we replace the lower bound in (2) by $\Delta \geq \Delta_0$?

(b) Can we replace (3) by $q^{k-1} \geq K\Delta$ for sufficiently large K?

(c) Can we extend the result to arbitrary k-uniform hypergraphs?

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