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EDGE DISJOINT SPANNING TREES IN RANDOM GRAPHS

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Abstract

We show that almost every $G_{m\text{-out}}$ contains m edge disjoint spanning trees.

Introduction

In this note we consider the maximum number of edge disjoint spanning trees contained in the random graph $G_{m\text{-out}}$. Let a graph $G = (V, E)$ have property A_k if it contains spanning trees $T_1, T_2, \ldots, T_k$ which are pair-wise edge disjoint.

We consider the random graph $G_m = G_{m\text{-out}}$. This has vertex set $V_n = \{1, 2, \ldots, n\}$. Each $v \in V_n$ independently chooses a set out (v) of distinct vertices as neighbours, where each m-subset of V_n-$\{v\}$ is equally likely to be chosen. This produces a random m out-regular diagraph D_m which has been selected uniformly from $\binom{n-1}{m}^n$ distinct possibilities. G_m is obtained by ignoring orientation but *without* coalescing edges. (See [1], [2], [3] for properties of this model.)

Probability statements refer to the probability space of D_m and graph theoretic statements refer to G_m.

THEOREM 1. *Let $m \geq 2$ be a fixed constant. Then*

$$\lim_{n \to \infty} \mathsf{P}(G_m \in \mathsf{A}_m) = 1.$$

[This is clearly best possible.]

The major graph theoretic result underpinning our proof is as follows.

THEOREM 2 (Nash-Williams [5], Tutte [6]).

A graph $G = (V, E)$ has property A_k if and only if for every partition $S_1, S_2, \ldots, S_t$ of V, $2 \leq t \leq |V|$, there at least $k(t-1)$ edges of G joining vertices in different subsets of the partition.

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(The necessity of the condition is obvious. The "meat" is in the sufficiency.)

PROOF of main result. For $S \subseteq V_n$ let $\gamma(S) = |\{vw \in E(D_m)\colon v \in S, w \notin S\}|$.

LEMMA 1. *The following events occur with probability tending to* 1 (as $n \to \infty$).

(i) $S \subseteq V_n$, $1 \leq |S| \leq .49n$ *implies* $\gamma(S) \geq m$

(ii) $S, T \subseteq V_n$, $S \cap T = \emptyset, |S|, |T| \geq .49n$, *implies* $\gamma(S) + \gamma(T) \geq m.$

PROOF. Observe that $\gamma(S) \geq |\{v \in S\colon \text{out}\,(v) \nsubseteq S\}|$. Hence $\gamma(S) \geq m$ for $|S| \leq m$ and

$$\mathsf{P}(\exists S \subseteq V_n\colon m < |S| \leq .49n \text{ and } \gamma(S) < m) \leq$$

$$\leq \sum_{s=m+1}^{\lfloor .49n \rfloor} \binom{n}{s} \binom{s}{s-m+1} \left(\frac{\binom{s-1}{m}}{\binom{n-1}{m}} \right)^{s-m+1} \leq$$

$$\leq \sum_{s=m+1}^{\lfloor .49n \rfloor} \binom{n}{s} s^{n-1} \left(\frac{s}{n}\right)^{m(s-m+1)} = \sum_{s} u_s, \text{ say.}$$

Now

$$\sum_{s=m+1}^{\lfloor n/3 \rfloor} u_s \leq \sum_{s=m+1}^{\lfloor n/3 \rfloor} \left(\frac{ne}{s}\right)^{s} s^{m-1} \left(\frac{s}{n}\right)^{m(s-m+1)} =$$

$$= \sum_{s=m+1}^{\lfloor n/3 \rfloor} e^{s} s^{m-1} \left(\frac{s}{n}\right)^{(m-1)(s-m)} = O(n^{-(m-1)}).$$

Next let $H(\alpha) = \alpha^{\alpha}(1-\alpha)^{1-\alpha}$, then

$$\sum_{s=\lceil n/3 \rceil}^{\lfloor .49n \rfloor} u_s \leq \sum_{s=\lceil n/3 \rceil}^{\lfloor .49n \rfloor} e^{o(n)} H\left(\frac{s}{n}\right)^{-n} \left(\frac{s}{n}\right)^{ms} \leq$$

$$\leq e^{o(n)} \sum_{s=\lceil n/3 \rceil}^{\lfloor .49n \rfloor} \left(\left(\frac{s}{n}\right)^{\frac{s}{n}} \Big\backslash \left(1-\frac{s}{n}\right)^{\left(1-\frac{s}{n}\right)} \right)^{n} = o(1),$$

and (i) follows.

(ii)

$$\mathsf{P}(\exists S, T \subseteq V_n, |S|, |T| \geq .49n,\ S \cap T = \emptyset \text{ and } \gamma(S) + \gamma(T) < m) \leq$$

$$\leq \sum_{s=\lceil .49n \rceil}^{\lfloor .51n \rfloor} \sum_{t=\lceil .49n \rceil}^{n-s} \binom{n}{s} \binom{n-s}{t} \binom{s+t}{s+t-m+1} \left(\frac{\max\{s,t\}}{n}\right)^{m(s+t-m+1)} \leq$$

$$\leq n^2 2^n 2^{.51n} n^{m-1} (.51)^{.98mn-m+1} = o(1).$$

PROOF of Theorem 1. Let $S_1, S_2, \ldots, S_t$ be a partition of V_n where $|S_1| \geq |S_2| \geq \ldots \geq |S_t|$. Now in the graph G_m there precisely $\gamma(S_1) + \gamma(S_2) + + \ldots + \gamma(S_t)$ edges joining different subsets of the partition. But Lemma 1 implies

$$\text{(ii)} \qquad \gamma(S_1) + \gamma(S_2) \geq m$$

and

$$\text{(i)} \qquad \gamma(S_3) + \ldots + \gamma(S_t) \geq (t-2)m$$

and so we can apply Theorem 3. □

We note the following interesting consequence Theorem 1: $G_{2\text{-out}}$ is *super-eulerian* with probability tending to one. (A graph is super-eulerian if it contains a trail which includes every vertex.) This is because every graph in A_2 has this property, Jaeger [4].

*

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