Stationary distribution and cover time of random walks on random digraphs.

Colin Cooper[*] Alan Frieze[†]

November 10, 2011

Abstract

We study properties of a simple random walk on the random digraph $D_{n,p}$ when $np = d \log n,\ d > 1$.

We prove that **whp** the value π_v of the stationary distribution at vertex v is asymptotic to $\deg^-(v)/m$ where $\deg^-(v)$ is the in-degree of v and $m = n(n-1)p$ is the expected number of edges of $D_{n,p}$. If $d = d(n) \to \infty$ with n, the stationary distribution is asymptotically uniform **whp**.

Using this result we prove that, for $d > 1$, **whp** the cover time of $D_{n,p}$ is asymptotic to $d \log(d/(d-1)) n \log n$. If $d = d(n) \to \infty$ with n, then the cover time is asymptotic to $n \log n$.

1 Introduction

Let $D = (V, E)$ be a strongly connected digraph with $|V| = n$, and $|E| = m$. For the simple random walk $\mathcal{W}_v = (\mathcal{W}_v(t),\ t = 0, 1, \ldots)$ on D starting at $v \in V$, let C_v be the expected time taken to visit every vertex of D. The *cover time* C_D of D is defined as $C_D = \max_{v \in V} C_v$.

For connected undirected graphs, the cover time is well understood, and has been extensively studied. It is an old result of Aleliunas, Karp, Lipton, Lovász and Rackoff [2] that $C_G \leq 2m(n-1)$. It was shown by Feige [10], [11], that for any connected graph G, the cover time satisfies $(1 - o(1)) n \log n \leq C_G \leq (1 + o(1)) \frac{4}{27} n^3$, where $\log n$ is the natural logarithm. An example of a graph achieving the lower bound is the complete graph K_n which has cover

[*]Department of Informatics, King's College, University of London, London WC2R 2LS, UK
[†]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA15213, USA. Supported in part by NSF grant CCF0502793.

time determined by the Coupon Collector problem. The *lollipop* graph consisting of a path of length $n/3$ joined to a clique of size $2n/3$ has cover time asymptotic to the upper bound of $(4/27)n^3$.

For directed graphs cover time is less well understood, and there are strongly connected digraphs with cover time exponential in n. An example of this is the digraph consisting of a directed cycle $(1, 2, ..., n, 1)$, and edges $(j, 1)$, from vertices $j = 2, ..., n-1$. Starting from vertex 1, the expected time for a random walk to reach vertex n is $\Omega(2^n)$.

In earlier papers, we investigated the cover time of various classes of (undirected) random graphs, and derived precise results for their cover times. The main results can be summarized as follows:

- [4] If $p = d \log n/n$ and $d > 1$ then **whp** $C_{G_{n,p}} \sim d \log \left(\frac{d}{d-1}\right) n \log n$.
- [7, 8] Let $d > 1$ and let x denote the solution in $(0, 1)$ of $x = 1 - e^{-dx}$. Let X_g be the giant component of $G_{n,p}$, $p = d/n$. Then **whp** $C_{X_g} \sim \frac{dx(2-x)}{4(dx-\log d)} n(\log n)^2$.
- [5] If $r \geq 3$ is a constant and $G_{n,r}$ denotes a random r-regular graph on vertex set $[n]$ with $r \geq 3$ then **whp** $C_{G_{n,r}} \sim \frac{r-1}{r-2} n \log n$.
- [6] If $m \geq 2$ is constant and G_m denotes a *preferential attachment graph* of average degree $2m$ then **whp** $C_{G_m} \sim \frac{2m}{m-1} n \log n$.
- [9] If $k \geq 3$ and $G_{r,k}$ is a random geometric graph in $\Re^k$ of ball size r such that the expected degree of a vertex is asymptotic to $d \log n$, then **whp** $C_{G_{r,k}} \sim d \log \left(\frac{d}{d-1}\right) n \log n$.

A few remarks on notation: We use the notation $a(n) \sim b(n)$ to mean that $a(n)/b(n) \to 1$ as $n \to \infty$. Some inequalities in this paper only hold for large n. We assume henceforth that n is sufficiently large for all claimed inequalities to hold. All **whp** statements in this paper are relative to the class of random digraphs $D_{n,p}$ under discussion, and not the random walk.

In this paper we turn our attention to the cover time of random directed graphs. Let $D_{n,p}$ be the random digraph with vertex set $V = [n]$ where each possible directed edge (i, j), $i \neq j$ is independently included with probability p. It is known that if $np = d \log n = \log n + \gamma$ where $\gamma = (d-1) \log n \to \infty$ then $D_{n,p}$ is strongly connected **whp**. If γ as defined tends to $-\infty$ then **whp** $D_{n,p}$ is not strongly connected. As we do not have a direct reference to this result, we next give a brief proof of this. It is easy to show that if $np = \log n - \gamma$ where $\gamma \to \infty$, there are vertices of in-degree zero **whp**. On the other hand, if $np = \log n + \gamma$ where $\gamma \to \infty$ then [12] shows that the random digraph is Hamiltonian and hence strongly connected. Strong connectivity for $np = \log n + \gamma$ where $\gamma \to \infty$ also follows directly from the proof of (62).

We determine the cover time of $D_{n,p}$ for values of p at or above the threshold for strong connectivity.

Theorem 1. *Let $np = d \log n$ where $d = d(n)$ is such that $\gamma = np - \log n \to \infty$. Then* **whp**

$$C_{D_{n,p}} \sim d \log\left(\frac{d}{d-1}\right) n \log n.$$

Note that if $d = d(n) \to \infty$ with n, then we have $C_{D_{n,p}} \sim n \log n$.

Here $X \sim Y$ **whp** if there are functions ϵ_1, ϵ_2 of n, $\epsilon_1, \epsilon_2 = o(1)$ as $n \to \infty$, such that with probability $1 - \epsilon_1$ we have $X = (1 - \epsilon_2)Y$.

The method we use to find the cover time of $D_{n,p}$ requires us to know the stationary distribution of the random walk. For an undirected graph G, the stationary distribution is $\pi_v = \deg(v)/2m$, where $\deg(v)$ denotes the degree of vertex v, and m is the number of edges in G. For a digraph D, let $\deg^-(v)$ denote the in-degree of vertex v, $\deg^+(v)$ denote the out-degree, and let m be the number of edges in D. For strongly connected digraphs in which each vertex v has in-degree equal to out-degree ($\deg^-(v) = \deg^+(v)$), then $\pi_v = \deg^-(v)/m$. For general digraphs, however, there is no simple formula for the stationary distribution. Indeed, there may not be a unique stationary measure. The main technical task of this paper is to find good estimates for π_v in the case of $D_{n,p}$. Along the way, this implies uniqueness of the stationary measure **whp**.

We summarize our result concerning the stationary distribution in Theorem 2 below. For a given vertex v, define a quantity $\varsigma^*(v)$, which in essence depends on the in-neighbour w of v with minimum out-degree:

$$\varsigma^*(v) = \max_{w \in N^-(v)} \left\{ \frac{\deg^-(w)}{\deg^+(w)} \right\}. \tag{1}$$

Theorem 2. *Let $np = d \log n$ where $d = d(n)$ is such that $np - \log n \to \infty$. Let $m = n(n-1)p$. Then* **whp**, *the stationary distribution π is unique and for all $v \in V$,*

$$\pi_v \sim \frac{deg^-(v) + \varsigma^*(v)}{m}.$$

If $\varsigma^*(v) = o(\deg^-(v))$ for vertex v, the $\varsigma^*(v)$ term can be absorbed into the error term of π_v, in which case $\pi_v \sim \deg^-(v)/m$, where $m \sim n^2p$. We note the following special cases.

Remark 1. *We prove in Lemma 14 that* **whp** *$\varsigma^*(v) = o(deg^-(v))$ for all but $o(n^{1/4})$ vertices v.*

Remark 2. *When $d = 1+\delta$, $\delta > 0$ constant then* **whp** *the maximum in-degree is $O(\log n)$ and the minimum out-degree is $\Omega(\log n)$. In which case, $\pi_v \sim deg^-(v)/m$ for all vertices $v \in V$.*

Remark 3. *It can be shown that if $np - \log n = \omega(\log \log n)$ then* **whp** *$\varsigma^*(v) = o(deg^-(v))$ for all vertices v.*

Remark 4. *If $d = d(n) \to \infty$ with n,* **whp** *the stationary distribution of $D_{n,p}$ is $\pi_v \sim 1/n$.*

2 Outline of the paper

At the heart of our approach to the cover time is the following claim: Suppose that T is a "mixing time" for a simple random walk $\mathcal{W}_u$, and $\boldsymbol{A}_v(t)$ is the event that the walk $\mathcal{W}_u$ does not visit v in steps $T, T+1, \ldots, t$. Then, essentially,

$$\mathbf{Pr}(\boldsymbol{A}_v(t)) \sim e^{-t\pi_v/R_v}. \tag{2}$$

Here R_v is the expected number of visits/returns to v made within T time steps, by a walk $\mathcal{W}_v$, starting from v. The fact that $R_v \geq 1$ follows because the walk starts from v at step $t = 0$, and this is counted as a visit. The proof of (2) is the content of Lemma 3; an established lemma that we have used to prove previous results on this topic. The definition of mixing time T used in Lemma 3 is based on maximum point-wise distance and is given in (4)-(5). Because the walk is on a digraph, we estimate a mixing time $T = o(\log^2 n)$ directly, and this is the topic of Section 7.1. Indeed the proof of Theorem 2 is itself based on an estimate of convergence of the walk to stationarity.

Given (2) we can estimate the cover time from above via

$$C_u \leq t + 1 + \sum_v \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_v(s)).$$

This is (95) and we have used this inequality previously. Here C_u is the expected time for $\mathcal{W}_u$ to visit every vertex. It is valid for arbitrary t and we get our upper bound for C_D by choosing t large enough so that the double sum is $o(t)$.

We estimate the cover time from below by using the Chebyshev inequality. We choose a set of vertices V^{**} that are candidates for taking a long time to visit and estimate the expected size of the set $V^\dagger$ of vertices in V^{**} that have not been visited within our estimate of the cover time. We show that $\mathbf{E}|V^\dagger| \to \infty$. To apply the Chebyshev inequality, we estimate the probability that a given pair of vertices $v, w \in V^{**}$ are unvisited by contracting them to a single vertex γ, and then using (2) to show that $\mathbf{Pr}(\boldsymbol{A}_\gamma(t)) \sim \mathbf{Pr}(\boldsymbol{A}_v(t))\mathbf{Pr}(\boldsymbol{A}_w(t))$.

The main problem for digraphs is that we do not know π_v and much of the paper is devoted to proving that, essentially, **whp**,

$$\pi_v \sim \frac{\deg^-(v) + \varsigma^*(v)}{m} \qquad \text{for all } v \in V. \tag{3}$$

Our proof of this leads easily to a claim that **whp** $T = O(\log^2 n)$ and we will then find that it is easy to prove that $R_v = 1 + o(1)$ for all $v \in V$.

We approximate the stationary distribution $\boldsymbol{\pi}$ using the expression $\boldsymbol{\pi} = \boldsymbol{\pi} P^k$, where P is the transition matrix. For suitable choices of k we find we can bound

$$P_x^{(k)}(y) = \mathbf{Pr}(\mathcal{W}_x(k) = y)$$

from above and below by values independent of x and obtain, essentially,

$$P_x^{(k)}(y) \sim \frac{\deg^-(y) + \varsigma^*(y)}{m}$$

an expression independent of x. Equation (3) follows easily from this.

To estimate $P_x^{(k)}(y)$ from below we proceed as follows: We let $k = 2\ell = \frac{2}{3}\log_{np} n$. We consider two Breadth First Search trees of depth ℓ. T_x^{low} branches out from x to depth ℓ and T_y^{low} branches into y from depth ℓ. Almost all of the walk measure associated with walks of length $2\ell + 1$ from x to y will go from x level by level to the boundary of T_x^{low}, jump across to the boundary of T_y^{low} and then go level by level to y. We analyse such walks and produce a lower bound.

To estimate $P_x^{(k)}(y)$ from above we change the depths of the out-tree from x and the in-tree to y. This eliminates some complexities. In computing the lower bound, we ignored some paths that take more circuitous routes from x to y and we have to show that these do not add much in walk measure.

The structure of the paper is now as follows: Section 3 describes Lemma 3 that we have often used before in the analysis of the cover time. Section 4 establishes many structural properties of $D_{n,p}$. In Section 5 we prove the lower and upper bounds given in Theorem 2. These bounds hold for any digraph with the high probability structures elicited in Section 4. Sections 4 and 5, which form the main body of this paper, are first proved under the assumption that $2 \leq d \leq n^\delta$, for some small $\delta > 0$, an assumption we refer to as **Assumption 1**. In Section 6, we extend the proof of Theorem 2 by removing Assumption 1. Section 7 is short and establishes that the conditions of Lemma 3 hold. To do this, we use a bound on the mixing time, based on results obtained in Sections 5, 6. Finally, in Section 8 we establish the cover time **whp**, as given in Theorem 1.

3 Main Lemma

In this section D denotes a fixed strongly connected digraph with n vertices. A random walk $\mathcal{W}_u$ is started from a vertex u. Let $\mathcal{W}_u(t)$ be the vertex reached at step t, let P be the matrix of transition probabilities of the walk and let $P_u^{(t)}(v) = \mathbf{Pr}(\mathcal{W}_u(t) = v)$. We assume that the random walk $\mathcal{W}_u$ on D is ergodic with stationary distribution π.

Let

$$d(t) = \max_{u,x \in V} |P_u^{(t)}(x) - \pi_x|, \tag{4}$$

and let T be a positive integer such that for $t \geq T$

$$\max_{u,x \in V} |P_u^{(t)}(x) - \pi_x| \leq n^{-3}. \tag{5}$$

Consider the walk $\mathcal{W}_v$, starting at vertex v. Let $r_t = r_t(v) = \mathbf{Pr}(\mathcal{W}_v(t) = v)$ be the probability that this walk returns to v at step $t = 0, 1, \ldots$. Let

$$R_T(z) = \sum_{j=0}^{T-1} r_j z^j \tag{6}$$

and let

$$R_v = R_T(1).$$

The following lemma is used in Section 8 to prove Theorem 1. A proof of Lemma 3 can be found in [7].

Lemma 3. *Fix vertices $u, v \in V$ and for $t \geq T$ let $\boldsymbol{A}_v(t)$ be the event that $\mathcal{W}_u$ does not visit v in steps $T, T+1, \ldots, t$. Suppose that*

(a) *For some constant $\theta > 0$, we have*

$$\min_{|z| \leq 1+\lambda} |R_T(z)| \geq \theta.$$

(b) $T^2 \pi_v = o(1)$ *and* $T\pi_v = \Omega(n^{-2})$.

There exists an absolute constant $K > 0$ and functions $\theta_1, \theta_2 = O(T\pi_v)$ such that if

$$\lambda = \frac{1}{KT}. \tag{7}$$

and

$$p_v = \frac{\pi_v}{R_v(1+\theta_1)}, \tag{8}$$

then we have that for $t \geq T$,

$$\mathbf{Pr}(\boldsymbol{A}_v(t)) = \frac{1+\theta_2}{(1+p_v)^t} + O(T^2 \pi_v e^{-\lambda t/2}). \tag{9}$$

4 Structural Properties of $D_{n,p}$

In this section we gather together some properties of the degree sequence of $D_{n,p}$ which hold **whp**. We stress that throughout this section the probability space is the space $D_{n,p}$, and not the space of walks on a given digraph. These properties are needed for the proof of Theorem 2.

We will make an assumption (Assumption 1) about the average degree which will allow us to split the proofs in this section into two parts, the first part assuming that Assumption 1 holds, later relaxing this assumption in Section 4.4.

Once we complete this section, we next concentrate on estimating the stationary distribution of a digraph with the given properties when Assumption 1 of (22) holds. This is done in Section 5. We then remove Assumption 1 in Section 6.2. For large p outside Assumption 1, a direct proof of the stationary distribution is quite simple, and given separately in Section 6.1.

4.1 Bounds on the Degree Sequence

Chernoff Bounds The following inequalities are used extensively throughout this paper. Let $Z = Z_1 + Z_2 + \cdots Z_N$ be the sum of the independent random variables $0 \leq Z_i \leq 1$, $i = 1, 2, \ldots, N$ with $\mathbf{E}(Z_1 + Z_2 + \cdots + Z_N) = N\mu$. Then for $\epsilon \in (0, 1)$ and any $t, \alpha > 0$,

$$\mathbf{Pr}(|Z - N\mu| \geq \epsilon N\mu) \leq 2e^{-\epsilon^2 N\mu/3}, \tag{10}$$

$$\mathbf{Pr}(Z \geq N\mu + t) \leq e^{-2t^2/N}, \tag{11}$$

$$\mathbf{Pr}(Z \geq \alpha N\mu) \leq (e/\alpha)^{\alpha N\mu}. \tag{12}$$

For proofs see for example Appendix A of Alon and Spencer [3].

The next lemma gives some properties of the degree sequence of $D_{n,p}$. The lemma can be proved by the use of the first and second moment methods (see [4] for very similar calculations). The majority of the properties in Lemma 4(i) are used in Section 8.

Let $np = d \log n$ and let

$$\Delta_0 = C_0 np \textit{ where } C_0 = 30. \tag{13}$$

Lemma 4.

(i) First assume that $np = d \log n$ where $1 < d = O(1)$ and $(d-1)\log n \to \infty$. Let $D(k)$ denote the number of vertices v with $deg^-(v) = k$, and let $\overline{D}(k) = \mathbf{E}D(k)$. Thus

$$\overline{D}(k) = n\binom{n-1}{k}p^k(1-p)^{n-1-k}.$$

Note that

$$\overline{D}(k) \leq \frac{2}{n^{d-1}}\left(\frac{nep}{k}\right)^k. \tag{14}$$

Let

$$\begin{aligned} K_0 &= \{k \in [1,\Delta_0]:\ \overline{D}(k) \le (\log n)^{-2}\}. \\ K_1 &= \{1 \le k \le 15:\ (\log n)^{-2} \le \overline{D}(k) \le \log\log n\}. \\ K_2 &= \{k \in [16,\Delta_0]:\ (\log n)^{-2} \le \overline{D}(k) \le (\log n)^2\}. \\ K_3 &= [1,\Delta_0] \setminus (K_0 \cup K_1 \cup K_2). \end{aligned}$$

For $k \le \Delta_0$, the degree sequence has the following properties.

(a) *If $d-1 \ge (\log n)^{-1/3}$ then*

$$K_1 = \emptyset, \qquad \min\{k \in K_2\} \ge (\log n)^{1/2}, \qquad |K_2| = O(\log\log n).$$

(b) *The following conditions hold* **whp***:*

$$\begin{aligned} &\text{For all } k \in K_0, && D(k) = 0, \\ &\text{For all } k \in K_1, && D(k) \le (\log\log n)^2, && (15) \\ &\text{For all } k \in K_2, && D(k) \le (\log n)^4, && (16) \\ &\text{For all } k \in K_3, && \frac{\overline{D}(k)}{2} \le D(k) \le 2\overline{D}(k). && (17) \end{aligned}$$

(ii) Suppose that $1 < d \le n^\delta$ where δ is a small positive constant. Let $k^* = \lceil (d-1)\log n \rceil$. Let

$$V^* = \left\{v \in V:\ deg^-(v) = k^* \text{ and } deg^+(v) = k^\dagger = \lceil d \log n \rceil\right\}$$

and let $\gamma_d = (d-1)\log\left(\frac{d}{d-1}\right)$. Then **whp**

$$|V^*| \ge \frac{n^{\gamma_d}}{10 d \log n}.$$

(iii) Let $\mathcal{D}$ be the event

$$\left\{\exists v \in V:\ deg^+(v) \ge \Delta_0 \text{ or } deg^-(v) \ge \Delta_0\right\}, \tag{18}$$

then

$$\mathbf{Pr}(\mathcal{D}) \le n^{-10} e^{-10np}. \tag{19}$$

(iv) The number of edges $|E(D_{n,p})| \sim m = n(n-1)p$ **whp***.*

(v) $deg^\pm(v) \sim np$ for all $v \in V$ **whp** *if $d \to \infty$.*

Proof We will only give an outline proof of (ii) as the other claims have (essentially) been proved in [4]. We have

$$\mathbf{E}(|V^*|) = n\binom{n-1}{k^*}\binom{n-1}{k^\dagger} p^{k^*+k^\dagger}(1-p)^{2n-2-k^*-k^\dagger}.$$

Using, $\binom{n}{k} \geq \frac{1}{3k^{1/2}}\left(\frac{ne}{k}\right)^k$, which is obtainable from Robbins' refinement of Stirling's approximation, we see that

$$\mathbf{E}(|V^*|) \geq (1-o(1))\frac{1}{9(d-1)^{1/2}d^{1/2}\log n}\left(\frac{d}{d-1}\right)^{(d-1)\log n}. \tag{20}$$

A separate calculation shows that

$$\mathbf{E}(|V^*|^2) = \mathbf{E}(|V^*|) + (\mathbf{E}(|V^*|))^2\left(1+O\left(\frac{k^*k^\dagger}{n^2p}\right)\right),$$

where $(k^*k^\dagger)/(n^2p) = O(p/n)$. Thus provided we verify that $\mathbf{E}(|V^*|)$ tends to ∞, the Chebyshev inequality will show that $|V^*|$ is concentrated around its mean.

If $d > 2$ then

$$\left(\frac{d}{d-1}\right)^{(d-1)\log n} \geq \exp\left\{(d-1)\log n\left(\frac{1}{d-1} - \frac{1}{2(d-1)^2}\right)\right\} \geq n^{1/2}.$$

Now $d \leq n^\delta$ and so $\mathbf{E}(|V^*|) \to \infty$ follows from (20). If $d = 1+\epsilon \leq 2$ and ϵ is bounded away from zero then so is $\left(\frac{d}{d-1}\right)^{d-1}$ and so $\mathbf{E}(|V^*|) = n^{\Omega(1)}$. So now suppose that $\epsilon = \frac{\omega}{\log n}$ where $\omega = \omega(n) \to \infty$ and $\omega = o(\log n)$. Then, $\left(\frac{d}{d-1}\right)^{(d-1)\log n} \geq \left(\frac{\log n}{\omega}\right)^\omega$. If $\omega \geq \log^{1/2} n$ then $\left(\frac{d}{d-1}\right)^{(d-1)\log n} \geq e^{\log^{1/2} n}$ and if $\omega \leq \log^{1/2} n$ then $\left(\frac{d}{d-1}\right)^{(d-1)\log n} \geq \log^{\omega/2} n$. In either case

$$\mathbf{E}(|V^*|) \geq \log^\theta n \text{ where } \theta = \theta(n) \to \infty. \tag{21}$$

□

4.1.1 Assumption 1: a convenient restriction

We will first carry out the main body of the proof under the following assumption:

$$\textbf{Assumption 1}: \qquad 2 \leq d \leq n^\delta. \tag{22}$$

Here δ is some small fixed positive constant, much less than one. We use the notation $0 < \delta \ll 1$ to express this condition. We note that our choice of the value $d \geq 2$ is somewhat arbitrary, and any constant larger than 1 would suffice. We wait until Section 6 to remove Assumption 1. The proof for $d > n^\delta$ is much simpler and is given separately in Section 6.1. The proof for $1 < d \leq 2$ is given in Section 6.2.

Under Assumption 1, for $d = O(1)$, and with $C_0np = \Delta_0$ given by (13), there is a constant $c_0 > 0$ and interval

$$I = [c_0np, C_0np], \tag{23}$$

such that if $\nu \in [3n/4, n]$ then there exists $\gamma = \gamma(c_0, C_0) > 0$ such that

$$\mathbf{Pr}(Bin(\nu, p) \in I) = 1 - o(n^{-1-\gamma}). \tag{24}$$

When $d \to \infty$ we can take $c_0 = 0.999$ and $C_0 = 1.001$.

Let $\mathcal{E}_S^+$ (resp. $\mathcal{E}_S^-$) be the event that the in-degree (resp. out-degree) of all vertices in $S \subseteq V$ are in the interval I. Thus e.g.

$$\mathcal{E}_S^+ = \left\{ D_{n,p} : \forall v \in S \subseteq V, \deg^+(v) \in [c_0 np, \Delta_0] \right\}. \tag{25}$$

Let $\mathcal{E}_S = \mathcal{E}_S^+ \cap \mathcal{E}_S^-$. Then for any $S \subseteq V$ we have

$$\mathbf{Pr}(\mathcal{E}_S) = 1 - O(n^{-\gamma}). \tag{26}$$

4.2 Properties needed for a lower bound on the stationary distribution

The calculations in this section are made under Assumption 1.

Fix vertices x, y where $x = y$ is allowed. Most short random walks from vertex x to vertex y take the form of a simple directed path, or cycle if $x = y$. We can count such paths (or cycles) with the help of a breadth first out-tree T_x^{low} rooted at x, and a breadth first in-tree T_y^{low} rooted at y. We build these trees to depth ℓ, where

$$\ell = \left\lfloor \frac{2}{3} \log_{np} n \right\rfloor. \tag{27}$$

For a vertex v let $N^-(v)$ be the set of in-neighbours of v and for a set S, let $N^-(S) = \bigcup_{v \in S} N^-(v)$. Define $N^+(v), N^+(S)$ similarly with respect to out-neighbours.

Construction of in-tree T_y^{low}.

For fixed $y \in V$, we build a tree $T_y = T_y^{low}$ rooted at y, by branching backwards in a breadth-first fashion.

Define $Y_0 = \{y\}$, and define $Y_1, \ldots, Y_\ell$ as $Y_{i+1} = N^-(Y_i) \setminus (Y_0 \cup \cdots \cup Y_i)$ for $0 \le i < \ell$. If $w \in Y_{i+1}$ is the out-neighbour of more than one vertex of Y_i, we only keep the edge (w, z) with the label of z as small as possible. Let $Y = \bigcup_{i=0}^{\ell} Y_i$ and let T_y^{low} denote the BFS tree $T_y(\ell)$ constructed by branching back in this manner. For convenience, let $Y_{\le i} = \cup_{j \le i} Y_j$.

Because we need to state the distribution of edges of T_y^{low} rather precisely, we will refine our description of the construction of the tree somewhat. The reason for this is as follows. It may be that, in the fully exposed digraph, a vertex $w \in Y_{i+1}$ has more than one edge pointing to Y_i. However, our construction of T_y^{low} avoids learning this fact.

Let $T_y(i)$ be the tree consisting of the first i levels of the breadth first tree $T_y(\ell)$. Given $T_y(i)$ we construct $T_y(i+1)$ by adding the in-neighbours of Y_i in $V \setminus Y_{\leq i}$. For $v \in Y_i$, let $N_T^-(v)$ be the subset of $Y_{i+1} \cap N^-(v)$ whose edges in the tree $T_y(i+1)$ point to v. The set $N_T^-(v)$ is constructed as follows. We process the vertices of Y_i in increasing order of vertex label. Let this order be $(v_1, v_2, ..., v_{|Y_i|})$. Thus $N_T^-(v_1) = N^-(v_1) \setminus Y_{\leq i}$, and in general $N_T^-(v_k) = N^-(v_k) \setminus (Y_{\leq i} \cup N^-(v_1, ..., v_{k-1}))$.

Let $\deg_T^-(v) = |N_T^-(v)|$ denote the in-degree of $v \in Y$ in T_y^{low}. If $\deg_T^-(v) > 0$ for all $v \in Y_{\leq \ell-1}$, we say *the construction of* T_y^{low} *succeeds*. The construction will fail if, for some i and some $v = v_k$ in Y_i all in-neighbours of v lie in $Y_{\leq i} \cup N^-(v_1, ..., v_{k-1})$.

If $v \in Y_i$ and $w \in Y_{i+1}$ and (w, v) is an edge of $T_y(i+1)$, then $v = v_k$ is the first out-neighbour of w in Y_i in the order $(v_1, ..., v_k)$. Note that we do not know anything about the edges (if any) between w and $(v_{k+1}, ..., v_{|Y_i|})$, because w was removed in our construction. We also remark, that as $w \in Y_{i+1}$, there are no edges from w to $Y_{\leq i-1}$, and thus no edges between w and $Y_{\leq i-1} \cup \{v_1, ..., v_{k-1}\}$.

Associated with this construction of $T_y = T_y^{low}$ is a set of parameters and random variables.

- For $v_j \in Y_i$, let $\sigma(v_j) = |V \setminus [Y_{\leq i} \cup N_T^-(v_1, ..., v_{j-1})]|$. Thus $\sigma(v_j)$ is the number of vertices not in T_y after all in-neighbours of $v_1, ..., v_{j-1}$ have been added to T_y.
- Let $B(v_j) = |N_T^-(v_j)| = \deg_T^-(v_j)$, the in-degree of v_j in T_y. Thus $B(v_j) \sim Bin(\sigma(v_j), p)$.
- Let $\sigma'(v_j) = |V \setminus [Y_{\leq i-1} \cup \{v_1, ..., v_{j-1}\}|$.
- Let $D \sim 1 + Bin(\sigma'(v_j), p)$, and let $D(j, k),\ k = 1, ..., B(v_j)$ be independent copies of D.

The interpretation of the random variable $D(j, k)$ is as follows. If $w_k \in N_T^-(v_j)$ then $D(j, k)$ is the out-degree of w_k in $D_{n,p}$. The 1+ term in the definition of D comes from (w_k, v_j) being the first edge from w_k to Y_i.

Construction of out-tree T_x^{low}. Given the set of vertices Y of T_y^{low}, we define $X_0 = \{x\}, X_1, \ldots, X_\ell$ where $X_{i+1} = N^+(X_i) \setminus (Y \cup X_0 \cup \cdots \cup X_i)$ for $0 \leq i < \ell$. If $w \in X_{i+1}$ is the out-neighbour of more than one vertex of X_i, we only keep the edge (z, w) with the label of z as small as possible, as in the construction of T_y^{low}. Let $X = \bigcup_{i=0}^{\ell} X_i$ and let T_x^{low} denote the BFS tree constructed in this manner. Let $\deg_T^+(v) = |N_T^+(v)|$ denote the out-degree of $v \in X$ in T_x^{low}. Similarly to the construction of T_y^{low}, the value of $\deg_T^+(v)$ is given by a random variable $B(v) \sim Bin(\sigma(v), p)$. If $\deg_X^+(v) > 0$ for all $v \in X_{\leq \ell-1}$, we say the construction of T_x^{low} succeeds. The construction would fail if some vertex $v \in X_i, i \leq \ell - 1$ only had out-neighbours in $X_{\leq i}$.

We gather together a few facts about T_x^{low}, T_y^{low} that we need for the proofs of this section. We say that a sequence of events $\mathcal{A}_n, n \geq 0$ hold 'quite surely'(**qs**) if $\mathbf{Pr}(\mathcal{A}_n) = 1 - O(n^{-K})$ for every constant $K > 0$.

Lemma 5. *Let $\gamma > 0$ be as defined in (23)-(24), then*

(i) *With probability $1 - O(n^{-\gamma})$, the construction of T_x^{low}, T_y^{low} succeeds for all $x, y \in V$.*

(ii) *With probability $1 - O(n^{-\gamma})$,*
For all x, and for all $v \in X_{\leq \ell-1}$, $deg_T^+(v) \in [c_0 np(1 - o(1)), C_0 np]$,
For all y and for all $v \in Y_{\leq \ell-1}$, $deg_T^-(v) \in [c_0 np(1 - o(1)), C_0 np]$.

(iii) *Given $\mathcal{E}_x^+, \mathcal{E}_y^-$, then for $i \leq \ell$, $|X_i| \sim deg^+(x)(np)^{i-1}$, $|Y_i| \sim deg^-(y)(np)^{i-1}$* **qs**.

Proof We give proofs for T_x^{low}, the proofs for T_y^{low} are similar.

Part (i), (ii). Let $X = \{x_0 = x, x_1, \ldots, x_N\}$ where x_i is the i-th vertex added to T_x^{low}. For $x_j \in X$, let $f(x_j) = |N^+(x_j) \cap (Y \cup \{x_0, x_1, \ldots, x_{j-1}\})|$. Thus $\deg^+(v) = \deg_T^+(v) + f(v)$.

We can bound $f(v)$ stochastically by the binomial $Bin(N_X, p)$ where $N_X = |Y| + |X|$. This is true even after constructing T_x^{low}, T_y^{low}, because the out-edges of v counted by $f(v)$ have not been exposed. Assuming $\neg\mathcal{D}$, see (19), we have

$$N_X \leq 2 \sum_{i=1}^{\ell} \Delta_0^i = n^{2/3+o(1)}.$$

Using the Chernoff bound (12), we have with $\omega = \log^{1/2} n$ that

$$\mathbf{Pr}\left(f(v) \geq \frac{np}{\omega}\right) \leq \mathbf{Pr}\left(Bin(n^{2/3+o(1)}, p) \geq \frac{np}{\omega}\right) = O(n^{-10}). \tag{28}$$

The event $\bigcup_{x \in v} \mathcal{E}^+_{X_{\leq \ell-1}} \subseteq \mathcal{E}_V^+$ and the latter holds with probability $1 - O(n^{-\gamma})$. Thus given (28), and $\mathcal{E}_V^+$ we have $\deg_T^+(v) > 0$ for $v \in X_{\leq \ell-1}$ for all $x \in V$. In summary, **whp** the construction of T_x^{low} succeeds for all $x \in V$, and $\deg_T^+(v) \in [c_0 np(1 - o(1)), C_0 np]$ for all $v \in X_{\leq \ell-1}$ in all trees T_x^{low}, $x \in V$.

Part (iii). By construction T_x^{low} was made after T_y^{low}, so $|X_i|$ depends on T_x^{low} and T_y^{low}. Assume $\mathcal{E}_x^+ = \left\{\deg^+(x) \in I = [c_0 np, C_0 np]\right\}$, and that $|Y| \leq n^{2/3+o(1)}$. Strictly speaking we should verify that $|Y| \leq n^{2/3+o(1)}$ before considering T_x^{low}. On the other hand, the proof we give here also applies to T_y^{low}. For $i \geq 1$, $|X_{i+1}|$ is distributed as $Bin(n - o(n), 1 - (1-p)^{|X_i|})$. The number of trials $n - o(n)$ is based on the inductive assumption that $|X_{j+1}| = (1 + o(1))n|X_j|p$ and that $|X_{\leq j}|p = o(1)$. That these assumptions hold **qs**follows from the Chernoff bounds. We thus have **qs**that

$$|X_\ell| \sim \deg^+(x)(np)^{\ell-1}. \tag{29}$$

□

For $u \in X_i$ let P_u denote the path of length i from x to u in T_x^{low} and

$$\alpha_{i,u} = \prod_{\substack{w \in P_u \\ w \neq u}} \frac{1}{\deg^+(w)}.$$

In the event that the construction of T_x^{low} fails to complete to depth ℓ, let $\sum_{u \in X_\ell} \alpha_{\ell,u} = 0$.

Similarly, for $v \in Y_i$ let Q_v denote the path from v to y in T_y^{low} and

$$\beta_{i,v} = \prod_{\substack{w \in Q_v \\ w \neq y}} \frac{1}{\deg^+(w)}. \tag{30}$$

In the event that the construction of T_y^{low} fails to complete to depth ℓ, let $\sum_{v \in Y_\ell} \beta_{\ell,v} = 0$.

Let

$$Z(x,y) = Z^{low}(x,y) = \sum_{\substack{u \in X_\ell \\ v \in Y_\ell}} \alpha_{\ell,u} \beta_{\ell,v} \frac{1_{uv}}{\deg^+(u)} \tag{31}$$

where 1_{uv} is the indicator for the existence of the edge (u,v) and we take $\frac{1_{uv}}{\deg^+(u)} = 0$ if $\deg^+(u) = 0$. Note that $Z(x,y) = 0$ if we fail to construct T_x^{low} or T_y^{low}.

Remark 5. *The importance of the quantity $Z(x,y)$ lies in the fact that it is a lower bound on the probability that $\mathcal{W}_x(2\ell+1) = y$.*

The aim of the next few lemmas is to prove the following statement. Let $I(y,\epsilon)$ denote the interval $[(1-\epsilon)deg^-(y)/m,\ (1+\epsilon)deg^-(y)/m]$, for some $\epsilon = o(1)$. Let $m = n(n-1)p$, then

$$\mathbf{Pr}\left(\exists_{x,y \in V} \text{ such that } Z(x,y) \notin I(y,\epsilon)\right) = O(n^{-\gamma}). \tag{32}$$

The first two lemmas give **whp** *bounds for $\sum_{u \in X_\ell} \alpha_{\ell,u}$, $\sum_{v \in Y_\ell} \beta_{\ell,v}$ respectively, to be used in the third lemma and its corollary.*

Lemma 6. *Let*

$$\mathcal{A}_1(x,y) = \left\{ 1 - \epsilon_X \leq \sum_{u \in X_\ell} \alpha_{\ell,u} \leq 1 \right\}, \tag{33}$$

and let $\epsilon_X = 2/(c\sqrt{\log n})$ for some $c > 0$, then

$$\mathbf{Pr}\left(\neg \mathcal{A}_1(x,y) \cap \mathcal{E}_x^+\right) = o(n^{-10}). \tag{34}$$

Proof

For $u \in X_\ell$ let $xP_u = (u_0 = x, u_1, \ldots, u_\ell = u)$ denote the path from x to u in T_x^{low}. For the random walk on the digraph T_x^{low}, starting at x; X_ℓ is reached with probability $\Phi = 1$ in exactly ℓ steps, after which the walk halts. Thus

$$1 = \Phi = \sum_{u \in X_\ell} \prod_{\substack{v \in P_u \\ v \neq u}} \frac{1}{\deg_T^+(v)} \geq \sum_{u \in X_\ell} \alpha_{\ell,u}. \tag{35}$$

We assume that the construction of T_x^{low} succeeds, and that $\deg_T^+(v) > 0$ for $v \in X_{\leq \ell-1}$, as established in Lemma 5. In the notation of that lemma, $\deg^+(v) = \deg_T^+(v) + f(v)$. Now

$$\begin{aligned}
\Phi &= \sum_{u \in X_\ell} \prod_{v \in P_u} \frac{1}{\deg^+(v) - f(v)} \\
&= \sum_{u \in X_\ell} \left(\prod_{v \in P_u} \frac{1}{\deg^+(v)} \right) \left(\prod_{v \in P_u} \frac{1}{1 - f(v)/\deg^+(v)} \right) \\
&= \sum_{u \in X_\ell} \alpha_{\ell,u} \left(\prod_{v \in P_u} \frac{1}{1 - f(v)/\deg^+(v)} \right).
\end{aligned}$$

Now if

$$\prod_{v \in P_u} \frac{1}{1 - f(v)/\deg^+(v)} \leq 1 + h \qquad \forall u \in X_\ell, \tag{36}$$

then $\sum_{u \in X_\ell} \alpha_{\ell,u} = 1 - o(1)$ provided $h = o(1)$. We next prove we can choose $h = O(1/\sqrt{\log n})$, which determines our value of ϵ_X.

Similar to the proof of (28) of Lemma 5 we have, with $\omega = \sqrt{\log n}$ that

$$\mathbf{Pr}\left(\sum_{v \in xPu} f(v) \geq \frac{np}{\omega} \right) \leq \mathbf{Pr}\left(Bin(n^{2/3+o(1)}, p) \geq \frac{np}{\omega} \right) = O(n^{-10}). \tag{37}$$

Using (28), and (37) it follows that

$$\sum_{v \in P_u} \frac{f(v)}{\deg^+(v) - f(v)} \leq \frac{1}{c_0\omega - 1}.$$

For $0 < x < 1$, $(1-x)^{-1} \leq e^{x/(1-x)}$, and so

$$\begin{aligned}
\prod_{v \in P_u} \frac{1}{1 - f(v)/\deg^+ d(v)} &\leq \exp\left(\sum_{v \in P_u} \left(\frac{f(v)}{\deg^+(v) - f(v)} \right) \right) \\
&\leq \exp\left(\frac{1}{c_0\omega - 1} \right) \leq 1 + \frac{2}{c_0\omega},
\end{aligned} \tag{38}$$

provided $1/(c_0\omega - 1) < 1/2$. There are at most n trees and n paths per tree and so (36), with $\epsilon_X = h = 2/(c\sqrt{\log n})$, follows from (38). This completes the proof of (34). □

The next step is to obtain an estimate of $\sum_{v\in Y_\ell}\beta_{\ell,v}$. The proof is inductive, moving down the tree T_y level by level. For brevity we write $d^+(u)=\deg^+(u)$, $d_T^-(u)=\deg_T^-(u)$ etc.

Let the random variable $W(y,i)$ be defined by

$$W(y,i)=\sum_{u\in N^-(y)}\sum_{v\in Y_i}\prod_{z\in vPu}\frac{1}{d^+(z)},$$

where for $v\in Y_i$ the notation means that the the unique path $vPuy$ from v to y in T_y passes through u, and that vPu is written as $v=z_i,\dots,z_j,\dots,z_1=u$ in the product term.

Note that

$$W(y,\ell)=\sum_{v\in Y_\ell}\beta_{\ell,v}.$$

Define $W^*(y,i)$ by

$$W^*(y,i)=\sum_{u\in N^-(y)}\sum_{v\in Y_i}d_T^-(v)\prod_{z\in vPu}\frac{1}{d^+(z)},$$

where for $v\in Y_\ell$ we define $d_T^-(v)=1$ so that $W(y,\ell)=W^*(y,\ell)$. Note that

$$W^*(y,1)=\sum_{u\in N^-(y)}\frac{d_T^-(u)}{d^+(u)}.$$

We prove the following lemma for a more general value of ℓ, as it is also used in our proof of the upper bound.

Lemma 7. *Let*

$$\mu(y)=\frac{1}{np}\sum_{u\in N^-(y)}\frac{d_T^-(u)}{d^+(u)}.$$

Let

$$\mathcal{A}_2(y)=\left\{D_{n,p}:\sum_{v\in Y_\ell}\beta_{\ell,v}\in[(1-\epsilon)\mu(y),(1+\epsilon)\mu(y)]\right\}. \tag{39}$$

Let $\epsilon=B/\sqrt{\log n}$ for some sufficiently large constant B, and let $\ell=\eta\log_{np}n$ where $0<\eta\le 2/3$. Then under Assumption 1,

$$\mathbf{Pr}(\exists y\in V \text{ such that } \neg\mathcal{A}_2(y))=O(n^{-\gamma}).$$

Proof

The lemma is proved inductively assuming $\mathcal{E}_y^-$ and $\mathcal{E}_{Y\setminus\{y\}}^+$. We prove the induction for $2\le i\le\ell$, where by assumption $(np)^\ell=O(n^{0.67})$.

Let $\mathbf{E}_{[\boldsymbol{d}^+(i)]}W(y,i)$ be the expectation of $W(y,i)$ over $(d^+(v), v \in Y_i)$, conditional on all other degrees $d^+(u) > 0, d_T^-(u), u \in Y_{\leq i-1}$ being fixed such that $|Y_{\leq i-1}| \sim d^-(y)(np)^{i-1} \leq n^{0.67}$ which is true **qs** from Lemma 5.

For $v \in Y_i$, $d^+(v)$ is distributed as $D(v) \sim 1 + Bin(\sigma'(v), p)$, for some $\sigma'(v) \in I_0 = [n - O(n^{0.67}), n]$. Given the values $\sigma'(v)$ for $v \in Y_i$, the $D(v)$ are independent random variables.

For $v \in Y_i$, let vPu be written $vwPu$, where $(v,w) \in T_y$. Then

$$\mathbf{E}_{[\boldsymbol{d}^+(i)]}\left(\prod_{z \in vPu} \frac{1}{d^+(z)}\right) = \mathbf{E}\left(\frac{1}{d^+(v)}\right)\left(\prod_{z \in wPu} \frac{1}{d^+(z)}\right),$$

where given $\mathcal{E}^+_{Y\setminus\{y\}}$, and $\delta = \max(n^{-0.33}, n^{-\gamma})$,

$$\mathbf{E}\left(\frac{1}{D(v)}\right) = (1+O(\delta))\frac{1}{np}.$$

This follows from the identity

$$\sum_{j=0}^{N} \frac{1}{j+1}\binom{N}{j} p^j q^{N-j} x^{j+1} = \frac{1}{(N+1)p}(q+px)^{N+1},$$

obtained by integrating $(q+px)^N$; and from $\mathbf{Pr}(\neg\mathcal{E}^+_v) = O(n^{-1-\gamma})$. Thus

$$\begin{aligned}
\mathbf{E}_{[\boldsymbol{d}^+(i)]}W(y,i) &= (1+O(\delta))\frac{1}{np}\sum_{w \in Y_{i-1}}\sum_{v \in N^-_T(w)}\left(\prod_{z \in wPu}\frac{1}{d^+(z)}\right) \\
&= (1+O(\delta))\frac{1}{np}\sum_{w \in Y_{i-1}} d^-_T(w)\left(\prod_{z \in wPu}\frac{1}{d^+(z)}\right) \\
&= (1+O(\delta))\frac{1}{np}W^*(y, i-1).
\end{aligned}$$

To obtain a concentration result, let $U(i) = W(y,i)\cdot((1-o(1))c_0 np)^i$, we can write $U(i) = \sum_{v \in Y_i} U_v$, where U_v are independent random variables. Assuming $\mathcal{E}^+_{Y\setminus\{y\}}$ and that Lemma 5(ii) holds we have $(c_0(1-o(1))/C_0)^i \leq U_v \leq 1$.

Let $\epsilon_i = \sqrt{3K\log n/(\mathbf{E}U)}$ for some large constant K. Then

$$\mathbf{Pr}(|U(i) - \mathbf{E}U| \geq \epsilon\mathbf{E}U) \leq 2e^{-\frac{\epsilon^2}{3}\mathbf{E}U} = O(n^{-K}),$$

and so

$$\mathbf{Pr}(|W(y,i) - \mathbf{E}W| \geq \epsilon\mathbf{E}W) = O(n^{-K}).$$

Note that $\mathbf{E}U \geq |Y_i|(c_0(1-o(1))/C_0)^i \geq (c_0/2)(c_0np/2C_0)^i$. Thus $\epsilon_i \leq 1/\sqrt{(A\log n)^{i-1}}$ for some $A > 0$ constant. For $i \geq 2$, $\epsilon_i = O(1/\sqrt{\log n})$, and thus $\epsilon_i = o(1)$.

In summary, with probability $1 - O(n^{-K})$,

$$W(y,i) = (1 + O(\delta) + O(\epsilon_i))\frac{1}{np}W^*(y,i-1).$$

Continuing in this vein, let $\mathbf{E}_{[\boldsymbol{d}_T^-(i-1)]}W^*(y,i-1)$ be the expectation of $W^*(y,i-1)$ over $(d_T^-(v), v \in Y_{i-1})$, conditional on all other degrees $(d^+(u), d_T^-(u), u \in Y_{\leq i-2})$ being fixed. For $v \in Y_{i-1}$, $d_T^-(v)$ is distributed as $B(v) \sim Bin(\sigma(v), p)$ conditional on $\mathcal{E}^-_{Y\setminus Y_\ell}$. Let $1_{\mathcal{X}}$ denote the indicator for an event $\mathcal{X}$, then

$$\mathbf{E}B(v) = \mathbf{E}(B(v)\cdot 1_{\mathcal{E}^-_{Y\setminus Y_\ell}}) + \mathbf{E}(B(v)\cdot 1_{\neg\mathcal{E}^-_{Y\setminus Y_\ell}}),$$

and, splitting the second event on $\mathcal{D}$ gives

$$\mathbf{E}(B(v)\cdot 1_{\neg\mathcal{E}^-_{Y\setminus Y_\ell}}) = O(\Delta_0 n^{-\gamma}) + O(nn^{-10}).$$

Thus, given $\mathcal{E}^-_{Y\setminus Y_\ell}$ we have $\mathbf{E}d_T^-(v) = (1+O(\delta))np$.

Thus

$$\mathbf{E}_{[\boldsymbol{d}_T^-(i-1)]}\left(d_T^-(v)\prod_{z\in vPu}\frac{1}{d^+(z)}\right) = (\mathbf{E}d_T^-(v))\left(\prod_{z\in vPu}\frac{1}{d^+(z)}\right),$$

and

$$\mathbf{E}_{[\boldsymbol{d}_T^-(i-1)]}W^*(y,i-1) = (1+O(\delta))np\, W(y,i-1).$$

Using Lemma 5 (ii) and arguments similar to above, for $i \geq 3$ with probability $1 - O(n^{-K})$

$$W^*(y,i-1) = (1+O(\delta)+O(\epsilon_{i-1}))np\, W(y,i-1)$$

completing the induction for $i \geq 3$.

The final step is to use

$$W(y,2) = (1+O(\delta)+O(\epsilon_2))\frac{1}{np}\, W^*(y,1),$$

and thus **whp**

$$\begin{aligned} W(y,\ell) &= \prod_{i=2}^{\ell}(1+O(\delta)+O(\epsilon_i))^2\frac{1}{np}\, W^*(y,1) \\ &= \left(1+O\left(\frac{1}{\sqrt{\log n}}\right)\right)\frac{1}{np}\sum_{u\in N^-(y)}\frac{d_T^-(u)}{d^+(u)}. \end{aligned}$$

Thus from (24)

$$\mathbf{Pr}(\exists y \in V \text{ such that } \neg\mathcal{A}_2(y)) = O(\mathbf{Pr}(\exists v \in V : \deg^{\pm}(v) \notin I)) = O(n^{-\gamma}).$$

□

Corollary 8. *Provided Assumption 1 holds, let*

$$\mathcal{A}_2(y) = \left\{ \sum_{v \in Y_\ell} \beta_{\ell,v} \in \left[(1-\epsilon)\frac{deg^-(y)}{np},\ (1+\epsilon)\frac{deg^-(y)}{np} \right] \right\}, \tag{40}$$

where $\epsilon = B/\sqrt{\log n}$*, then*

$$\mathbf{Pr}\left(\exists y \in V \ \textit{such that}\ \neg\mathcal{A}_2(y)\right) = O(n^{-\gamma}). \tag{41}$$

Proof Referring to (39), under Assumption 1 and $\neg\mathcal{D}$, then $d_T^-(u) = \deg^-(u)(1-o(1))$ simultaneously for all $u \in N^-(y)$ with probability $1 - O(n^{-1-\gamma})$. Let $\zeta = 1/\log\log\log n$. A vertex is *normal* if at most $\zeta_0 = \lceil 4/(\zeta^3 d) \rceil$ of its in-neighbours have out-degrees which are not in the range $[(1-\zeta)np, (1+\zeta)np]$, and similarly for in-degrees. Let $\mathcal{N}(y)$ be the event y is normal. We observe that

$$\mathbf{Pr}(\neg\mathcal{N}(y) \mid \mathcal{E}_y^-) \le 2 \sum_{s=c_0 np}^{C_0 np} \binom{s}{\zeta_0} (2e^{-\zeta^2 np/3})^{\zeta_0} = O(n^{-\Omega(\log\log\log n)}),$$

where $\mathcal{E}_y^-$ is given by (25), and thus (see (24))

$$\mathbf{Pr}(\neg(\mathcal{N}(y) \cap \mathcal{E}_y^-)) = O(n^{-1-\gamma}). \tag{42}$$

Now if y is normal then

$$\deg^-(y)\frac{1-\zeta}{1+\zeta} - O(\zeta_0) \le \sum_{u \in N^-(y)} \frac{\deg^-(u)}{\deg^+(u)} \le \deg^-(y)\frac{1+\zeta}{1-\zeta} + O(\zeta_0).$$

□

Recall the definition of $Z(x, y)$,

$$Z(x, y) = \sum_{\substack{u \in X_\ell \\ v \in Y_\ell}} \alpha_{\ell,u}\beta_{\ell,v}\frac{1_{uv}}{\deg^+(u)}, \tag{43}$$

where 1_{uv} is the indicator for the existence of the edge (u, v) and we take $\frac{1_{uv}}{\deg^+(u)} = 0$ if $\deg^+(u) = 0$. The next lemma gives a high probability bound for $Z(x, y)$.

Lemma 9. *Let*

$$\mathcal{A}_3(x,y) = \left\{ Z(x,y) \in \left[(1-\epsilon_Z)\frac{deg^-(y)}{m},\ (1+\epsilon_Z)\frac{deg^-(y)}{m} \right\} \right],$$

where $\epsilon_Z = B/(\sqrt{\log n})$*, for some constant* $B > 0$*. Then given Assumption 1,*

$$\mathbf{Pr}(\exists x,y : \neg\mathcal{A}_3(x,y)) = O(n^{-\gamma}). \tag{44}$$

Proof

Let

$$\mathcal{B} = \mathcal{B}(x,y) = (\mathcal{E}^+_{X\setminus X_\ell} \cap \mathcal{E}^-_{Y\setminus Y_\ell} \cap \mathcal{A}_1(x,y) \cap \mathcal{A}_2(y) \cap \mathcal{L}),$$

where $\mathcal{E}$ is given by (25), $\mathcal{A}_1, \mathcal{A}_2$ by (33), (40), and $\mathcal{L}$ is the event that Lemma 5 holds.

Let $u \in X_\ell$ and let $w \in Y \setminus Y_\ell$. As $X_\ell \cap Y = \emptyset$, we know that u is not an in-neighbour of w. Other out-edges of u are unconditioned by the construction of T_x^{low}, T_y^{low}. Given $Y \setminus Y_\ell \le n^{2/3+o(1)}$, the distribution of $\deg^+(u)$ is $Bin(\nu, p)$ for some $n - n^{0.67} \le \nu \le n-1$. Thus

$$\mathbf{E}\left(\frac{1_{uv}}{\deg^+(u)} \,\middle|\, \mathcal{B} \right) = \sum_{k=1}^{\nu} \binom{\nu}{k} p^k (1-p)^{\nu-k} \frac{k}{\nu}\frac{1}{k} = \frac{1}{n}\left(1 + O(n^{-0.33})\right). \tag{45}$$

Here k/ν is the conditional probability that edge (u,v) is present, given that u has k out-neighbours.

We use the notation $\mathbf{Pr}_{\mathcal{C}}(\cdot) = \mathbf{Pr}(\cdot \mid \mathcal{C})$ etc, for any event $\mathcal{C}$. From (33), (40), (45),

$$\mathbf{E}_{\mathcal{B}}(Z) = (1 + O(\epsilon_Z))\frac{\deg^-(y)}{m}. \tag{46}$$

Conditional on $\mathcal{B}$, $|Y_\ell| \le n^{2/3+o(1)}$ by construction, and as the edges from u to Y_ℓ are unexposed,

$$\mathbf{Pr}_{\mathcal{B}}\left(|N^+(u) \cap Y_\ell| \ge 1000\right) \le \mathbf{Pr}(Bin(n^{2/3+o(1)}, p) \ge 1000) \le n^{-10}. \tag{47}$$

Let

$$\mathcal{F} = \mathcal{F}(x,y) = \left\{ |N^+(u) \cap Y_\ell| < 1000,\ \forall u \in X_\ell \right\},$$

and let $\mathcal{G}(x,y) = \mathcal{B}(x,y) \cap \mathcal{F}(x,y) \cap \mathcal{E}^+_{X_\ell}$. The quantity of interest to us is the value of $Z(x,y)$ conditional on $\mathcal{G}(x,y)$. We first obtain $\mathbf{E}_{\mathcal{G}}(Z)$ from $\mathbf{E}_{\mathcal{B}}(Z)$ using

$$\mathbf{E}_{\mathcal{B}}(Z) = \mathbf{E}_{\mathcal{B}}(Z \cdot 1_{\mathcal{F}(x,y)\cap\mathcal{E}^+_{X_\ell}}) + \mathbf{E}_{\mathcal{B}}(Z \cdot 1_{\neg[\mathcal{F}(x,y)\cap\mathcal{E}^+_{X_\ell}]}). \tag{48}$$

The event $\neg[\mathcal{F}(x,y) \cap \mathcal{E}^+_{X_\ell}] \subseteq \left[\mathcal{F}(x,y) \cap \neg\mathcal{E}^+_{X_\ell}\right] \cup [\neg\mathcal{F}(x,y)]$. Using (43), we obtain

$$\begin{aligned}
&\mathbf{E}_{\mathcal{B}}(Z \cdot 1_{\neg[\mathcal{F}(x,y)\cap\mathcal{E}^+_{X_\ell}]}) \\
&= O(\mathbf{E}_{\mathcal{B}}(Z)n^{-\gamma}) + O\left(\frac{1000}{(c_0 np)^{2\ell}}\right)|X_\ell|\left(O(n^{-(1+\gamma)}) + O(n^{-10})\right) + O(|X_\ell||Y_\ell|)O(n^{-10}) \qquad (49) \\
&= \mathbf{E}_{\mathcal{B}}(Z)\ O(n^{-\gamma}).
\end{aligned}$$

To see this, partition the vertices of X_ℓ into sets R, S, where vertices in R have out-degree in $[c_0np, C_0np]$, and vertices of S do not. The first term in (49) is the contribution to the first term in the RHS of (48) from the vertices in R, multiplied by the probability of $\neg\mathcal{E}_{X_\ell}$. Assuming $\mathcal{F}(x, y)$ holds, the second term in the RHS of (49) is the contribution to the first term in the RHS of (48) from the vertices in S. The last term in the RHS of (49) is the contribution to the first term in the RHS of (48) in the case where $\neg\mathcal{F}(x, y)$ holds.

Thus

$$\mathbf{E}_{\mathcal{G}}(Z) = \frac{\mathbf{E}(Z \cdot 1_{\mathcal{B}} \cdot 1_{\mathcal{F}(x,y)\cap\mathcal{E}^+_{X_\ell}})}{\mathbf{Pr}(\mathcal{G})} = \frac{\mathbf{E}_{\mathcal{B}}(Z.1_{\mathcal{F}(x,y)\cap\mathcal{E}^+_{X_\ell}})\mathbf{Pr}(\mathcal{B})}{\mathbf{Pr}(\mathcal{G})},$$

and so

$$\mathbf{E}_{\mathcal{G}}(Z) = \mathbf{E}_{\mathcal{B}}(Z)(1 + O(n^{-\gamma})) = (1 + O(\epsilon_Z))\frac{\deg^-(y)}{np}\frac{1}{n}. \tag{50}$$

We now examine the concentration of $(Z \mid \mathcal{G})$. Let $A = 1000/((1 - o(1))c_0np)^{2\ell+1}$. It follows from Lemma 5(ii) that given $\mathcal{G}$ we have $Z_u \leq A$. Let $\widehat{Z}_u = Z_u/A$, then for $u \in X_\ell$, the $\widehat{Z}_u$ are independent random variables, and $0 \leq \widehat{Z}_u \leq 1$. Let $\widehat{Z} = \sum_{u\in X_\ell} \widehat{Z}_u$ and let $\widehat{\mu} = \mathbf{E}_{\mathcal{G}}(\widehat{Z})$. Thus

$$\widehat{\mu} = n^{1/3+o(1)}. \tag{51}$$

It follows from (10) that if $0 \leq \theta \leq 1$,

$$\mathbf{Pr}_{\mathcal{G}}(|\widehat{Z} - \widehat{\mu}| \geq \theta\widehat{\mu}) \leq 2e^{-\theta^2\widehat{\mu}/3}.$$

With $\theta = 4(np/\widehat{\mu})^{1/2}$ we find that,

$$\mathbf{Pr}_{\mathcal{G}}(|\widehat{Z} - \widehat{\mu}| \geq 4(np\widehat{\mu})^{1/2}) = o(n^{-4}),$$

and hence that

$$\mathbf{Pr}_{\mathcal{G}}(|Z - \mathbf{E}_{\mathcal{G}}Z| \geq 4A(np\widehat{\mu})^{1/2}) = o(n^{-4}).$$

Using (51) we have $4A(np\,\widehat{\mu})^{1/2} = O(n^{-7/6+o(1)})$, and so

$$\mathbf{Pr}_{\mathcal{G}}\left(|Z - \mathbf{E}_{\mathcal{G}}(Z)| = O\left(\frac{1}{n^{7/6+o(1)}}\right)\right) = 1 - o(n^{-4}).$$

We see from (50) that $\mathbf{E}_{\mathcal{G}}(Z) = (1 + O(\epsilon_Z))\frac{\deg^-(y)}{m}$. Thus

$$\mathbf{Pr}_{\mathcal{G}}\left(Z(x, y) \neq (1 + O(\epsilon_Z))\frac{\deg^-(y)}{m}\right) = o(n^{-4}). \tag{52}$$

Using (26), (34), (41) and (47),

$$\begin{aligned}
&\mathbf{Pr}\left(\bigcup_{x,y}\neg\mathcal{G}(x, y)\right)\\
&\leq \mathbf{Pr}(\neg\mathcal{E}_V) + \mathbf{Pr}(\neg\mathcal{L}) + \mathbf{Pr}\left(\bigcup_{x,y}\neg\mathcal{F}(x, y)\right) + \mathbf{Pr}\left(\bigcup_{x,y}\neg\mathcal{A}_1(x, y)\right) + \mathbf{Pr}\left(\bigcup_{y}\neg\mathcal{A}_2(y)\right)\\
&= O(n^{-\gamma}).
\end{aligned} \tag{53}$$

Thus finally, from (52) and (53)

$$\mathbf{Pr}(\exists x, y : \neg \mathcal{A}_3(x, y)) = O(n^{-\gamma}). \tag{54}$$

□

4.3 Properties needed for an upper bound on the stationary distribution

We remind the reader that $np = d \log n$ where $d \leq n^{\delta}$, where δ is some small positive constant. Let

$$\Lambda = \log_{np} n.$$

We will use the following values of ℓ in our proofs:

$$\ell_0 = (1 + \eta)\Lambda, \quad \ell_1 = (1 - 10\eta)\Lambda, \quad \ell_2 = 11\eta\Lambda.$$

We first show that small sets of vertices are sparse **whp**.

Lemma 10. *Let ζ be a positive constant satisfying $2\delta < \zeta < 1/2$, and let $s_0 = (1 - 2\zeta)\Lambda$.* **Whp** *for all $S \subseteq V$, $|S| \leq s_0$, the set S contains at most $|S|$ edges.*

Proof The expected number of sets S with more than $|S|$ edges can be bounded by

$$\begin{aligned} \sum_{s=3}^{s_0} \binom{n}{s} \binom{s^2}{s+1} p^{s+1} &\leq \sum_{s=3}^{s_0} (e^2 np)^s sep \\ &\leq \exp\left(-\zeta \log n + \log np\right) = o(n^{-\zeta/2}). \end{aligned}$$

□

For the upper bound we need to slightly alter our definition of breadth-first trees and call them T_x^{up}, T_y^{up}. This time we grow T_x^{up} to a depth ℓ_1 and T_y^{up} to a relatively small depth ℓ_2. With this choice, Lemma 10 implies that Y will contain no more than $|Y|$ edges **whp**. This reduces the complexity of the argument. We fix x, y and grow T_x^{up} from x to a depth ℓ_1, and T_y^{up} into y to a depth ℓ_2. The definition of T_x^{up} is slightly different from T^{low}, but we retain some of the notation.

Construction of T_x^{up}. We build a tree T_x^{up}, much as in Section 4.2, by growing a breadth-first out-tree from x to depth ℓ. The difference is that we construct T_x^{up} before T_y^{up}, so that T_x^{up} is not disjoint from Y. As before, let $X_0 = \{x\}$, and $X_i,\ i \geq 1$ be the i-th level set of the tree. Let $T_x^{up}(i)$ denote the BFS tree up to and including level i, and let $T_x^{up} = T_x^{up}(\ell_1)$. Let $X_{\leq i} = \cup_{j \leq i} X_j$, and let $X = X_{\leq \ell_1}$. In Section 5.2 below we will need to consider a larger set $X_{\leq \ell_3}$ where $\ell_3 = (1 - \eta/10)\Lambda$.

Construction of T_y^{up}. Our upper bound construction of T_y^{up} is the same as for the lower bound, except that we only grow it to depth ℓ_2.

Our aim is to prove an upper bound similar to the lower bound proved in Lemma 9. For $u \in X_{\ell_1}$ we let

$$\alpha_{\ell_1,u} = \mathbf{Pr}(\mathcal{W}_x(\ell_1) = u)$$

where

$$\sum_{u \in X} \alpha_{\ell_1,u} \leq 1. \tag{55}$$

The LHS of (55) is one, except when we fail to construct T_y^{up} to level ℓ_2.

This is the only place where we write a structural property of $D_{n,p}$ in terms of a walk probability. This is of course valid, since $\alpha_{\ell_1,u}$ is the sum over walks of length ℓ_1 from x to u of the product of reciprocals of out-degrees. Fortunately, all we need is (55).

We also define the $\beta_{i,v}$ as we did in (30) and now we let

$$Z(x,y) = Z^{up}(x,y) = \sum_{\substack{u \in X \\ v \in Y_{\ell_2} \setminus X}} \alpha_{\ell_1,u} \beta_{\ell_2,v} \frac{1_{uv}}{\deg^+(u)}. \tag{56}$$

The next lemma follows from Corollary 8.

Lemma 11. *Let ℓ be as in (27). If $2 \leq k \leq \ell$ then for some $\epsilon_Y = o(1)$ we have*

$$\mathbf{Pr}\left(\sum_{v \in Y_k} \beta_{k,v} \geq (1+\epsilon_Y) \frac{deg^-(y)}{np}\right) = o(n^{-1-\gamma/2})$$

where γ is as in (24).

It follows by an argument similar to that for Lemma 9 that

Lemma 12. *For some $\epsilon_Y = o(1)$ we have that*

$$\mathbf{Pr}\left(\exists x, y : \ Z(x,y) \geq (1+\epsilon_Y) \frac{deg^-(y)}{m}\right) = O(n^{-\gamma/2}). \tag{57}$$

In computing the expectation of Z, some of the vertices in X of T_x^{up} may be inspected in our construction of T_y^{up}, or of T_x^{up} up to level ℓ_1. Thus $\mathbf{E}(1_{uv}/\deg^+(u)) \leq (1/n)(1+o(1))$, (see (45)).

Remark 6. *The upper bound for $Z(x,y)$ obtained above is parameterized by $\ell_0 = (1+\eta)\Lambda$. Provided $\eta > 0$ constant, so that Lemma 12 holds, we can apply this argument simultaneously for $n^{\gamma/3}$ different values of η.*

We next prove a lemma about non-tree edges inside X, and edges from X to $Y \setminus Y_{\ell_2}$.

Lemma 13.

(a) *Let* $\ell_3 = (1-\eta/10)\Lambda$ *and*

$$\mathcal{L}_a(\ell_3) = \{\forall\ z \in X_{\leq \ell_3} :\ z \ \ \text{has}\ \leq 100/\eta \text{ in-neighbours in } X_{\leq \ell_3}\}.$$

Then $\mathbf{Pr}(\neg\mathcal{L}_a(\ell_3)) = O(n^{-9})$.

(b) *Let* $X_\ell^\circ = \{v \in X_\ell : N^+(v) \cap X_{\leq \ell} \neq \emptyset\}$ *and*

$$\mathcal{L}_b(\ell) = \left\{|X_\ell^\circ| \leq 18{\Delta_0}^{2\ell} p + \log^2 n\right\}.$$

Then $\mathbf{Pr}(\neg\mathcal{L}_b(\ell)) = O(n^{-10})$ *for* $\ell \leq \ell_3$.

(c) *Let*

$$t_0 = \left\lceil \frac{K\Lambda}{\log np} \right\rceil \qquad \text{where } K = 2\log(100C_0/\eta c_0). \tag{58}$$

Fix $t \leq t_0$ *and* $i > 2\eta\Lambda$ *and let*

$$S_{i,t}^\circ = \{z \in X :\ z \text{ is reachable from } X_i^\circ \text{ in at most } t \text{ steps}\}.$$

Then let $A^\circ = A^\circ(x,y,t)$ *be the number of edges from* $S_{i,t}^\circ \cap X$ *to* $Y_{\ell_2 - t}$*. Then,*

$$\mathbf{Pr}(\exists x,y,t :\ A^\circ \geq \log n) = O(n^{-10}).$$

(d) *Let* $A = A(x,y,t)$ *be the number of edges between* X_{ℓ_1} *and* $Y_{\ell_2-t} \setminus X$*, where* $t_0 < t \leq \ell_2 - 1$*.*

$$\mathbf{Pr}(\exists x,y,t :\ A \geq 9|X_{\ell_1}||Y_{\ell_2-t}|p + \log^2 n) = o(n^{-10}).$$

Proof
(a) Let $z \in X(\ell_3)$. Let ζ be the number of in-neighbours of z in $X_{\leq \ell_3}$. In the construction of $T_x^{up}(\ell_3)$, we only exposed one in-neighbour of z. Thus ζ is distributed as $1 + Bin(|X_{\leq \ell_3}|, p) \leq 1 + Bin(\Delta_0^{\ell_3}, p) + n\mathbf{Pr}(\mathcal{D})$. We apply (12) and (19) to deal with the binomial. Hence if $r + 1 = 100/\eta$,

$$\mathbf{Pr}(\zeta \geq r+1) \leq \Delta_0^{r\ell_3} p^r + n^{-10} e^{-10np} \leq 2n^{r(\delta - \eta/10)} + n^{-10}e^{-10np} = O(n^{-9}).$$

Part (a) of the lemma follows.

(b) For $v \in X_\ell$ the out edges of v are unconditioned during the construction of $T_x^{up}(\ell)$. The number of out edges of v to $X_{\leq \ell}$ is $Bin(|X_{\leq \ell}|, p)$. Unless $\mathcal{D}$ occurs, $|X_{\leq \ell}| \leq 2{\Delta_0}^\ell$ and

$$\mathbf{Pr}(|N^+(v) \cap X_{\leq \ell}| > 0 \ \mid \neg\mathcal{D}) \leq 1 - (1-p)^{2{\Delta_0}^\ell} \leq 2{\Delta_0}^\ell p,$$

and

$$\mathbf{E}(|X_\ell^\circ| \ \mid \neg\mathcal{D}) \le 2{\Delta_0}^{2\ell}p.$$

By (12)

$$\mathbf{Pr}(|X_\ell^\circ| \ge 18{\Delta_0}^{2\ell}p + \log^2 n) = O(n^{-10}) + \mathbf{Pr}(\mathcal{D}) = O(n^{-10}).$$

(c) Let $S(u,t')$ be the set of vertices in X that a walk starting from $u \in X_i$ can reach in $\ell_1 - i + t - t'$ steps. Thus unless $\mathcal{D}$ occurs, $|S(u,t')| \le {\Delta_0}^{\ell_1-i+t-t'}$. So, given $\neg\mathcal{D}$,

$$|S_{i,t}^\circ| \le 2|X_i^\circ|{\Delta_0}^{\ell_1-i+t}. \tag{59}$$

We can assume that, after constructing T_x^{up} we construct T_y^{up} to level Y_{ℓ_2-t}, and then inspect the edges from $S_{i,t}^\circ$ to $Y_{\ell_2-t} \setminus X$. These edges are unconditioned at this point and their number A is stochastically dominated by $Bin(|S_{i,t}^\circ|\ |Y_{\ell_2-t}|, p)$. Given $\mathcal{L}_b(i)$ of part (b) of this lemma,

$$|X_i^\circ| \le 18{\Delta_0}^{2i}p + \log^2 n. \tag{60}$$

Let $i = a\Lambda$, where $2\eta \le a \le 1 - 10\eta$.

Case $2\eta \le a \le (1+\epsilon)/2$ for some small $\epsilon > 0$ constant.
Using (59), (60) and $|Y_{\ell_2-t}| \le {\Delta_0}^{\ell_2-t}$ gives

$$\begin{aligned}
\mathbf{E}A^\circ &\le (18{\Delta_0}^{2i}p + \log^2 n)2{\Delta_0}^{\ell_1-i+t}{\Delta_0}^{\ell_2-t}p + n^2(\mathbf{Pr}(\mathcal{D}) + \mathbf{Pr}(\mathcal{L}_b(i)))\\
&\le 36C_0^{\ell_0+\ell_1}\ p^2\ (np)^{\ell_0+i} + 2C_0^{\ell_0}(\log^2 n)\ p\ (np)^{\ell_0-i} + O(n^{-9})\\
&\le 36C_0^{\ell_0+\ell_1}(np)^2 n^{-\frac{1}{2}+\eta+\epsilon} + 2C_0^{\ell_0}\log^2 n(np)n^{-\eta} + O(n^{-9})\\
&= O(n^{-\eta/2}).
\end{aligned}$$

Case $(1+\epsilon)/2 \le a \le 1 - 10\eta$.
For $i \ge (1+\epsilon)/2\Lambda$, $|X_i^\circ| \le 20{\Delta_0}^{2i}p$. Thus

$$\begin{aligned}
\mathbf{E}A^\circ &\le 20{\Delta_0}^{2i}p \cdot 2{\Delta_0}^{\ell_1-i+t}{\Delta_0}^{\ell_2-t}p + n^2(\mathbf{Pr}(\mathcal{D}) + \mathbf{Pr}(\mathcal{L}_b(i)))\\
&\le 40C_0^{\ell_0+\ell_1}\ p^2\ (np)^{\ell_0+i} + O(n^{-9})\\
&\le 40C_0^{\ell_0+\ell_1}(np)^2 n^{-9\eta} + O(n^{-9})\\
&= O(n^{-\eta/2}).
\end{aligned}$$

In either case, with probability $1 - o(n^{-10})$, $A \le \log n$.

(d) After growing T_x^{up} to level ℓ_1, we grow T_y^{up} to level $\ell_2 - t$. Then $A(t)$ has a binomial distribution and $\mathbf{E}A(t) \le |X_{\ell_1}||Y_{\ell_2-t}|p$. The result follows from the Chernoff inequality. □

4.4 Small average degree: $1 + o(1) \le d \le 2$

This section contains further lemmas needed for the case $1 + o(1) \le d \le 2$.

We will assume now that

$$1 + o(1) \leq d \leq 2.$$

Let a vertex be *small* if it has in-degree or out-degree at most $np/20$ and *large* otherwise. Let *weak distance* refer to distance in the underlying undirected graph of $D_{n,p}$.

Lemma 14.

(a) **Whp** *there are fewer than* $n^{1/5}$ *small vertices.*

(b) *If* $np \geq 2\log n$ *then* **whp** *there are no small vertices.*

(c) **Whp** *every pair of small vertices are at weak distance at least*

$$\ell_{10} = \frac{\log n}{10 \log\log n}$$

apart.

(d) **Whp** *there does not exist a vertex* v *with* $\max\left\{deg^+(v), deg^-(v)\right\} \leq \log n/20$.

(e) *Let* $\varsigma^*(v)$ *be given by* (1). **Whp** *for all vertices* y,

$$\sum_{u \in N^-(y)} \frac{deg^-(u)}{deg^+(u)} = (1 + o(1))(deg^-(y) + \varsigma^*(y)).$$

Proof
(a) The expected number of small vertices is at most

$$n \sum_{k=0}^{\log n/20} \binom{n-1}{k} p^k q^{n-1-k} = O(n^{.1998}). \tag{61}$$

Part (a) now follows from the Markov inequality.

(b) For $np \geq 2\log n$ the RHS of (61) is $o(1)$.

(c) The expected number of pairs of small vertices at distance ℓ_{10} or less is at most

$$n^2 \sum_{k=0}^{\ell_{10}} 2^k n^k p^{k+1} \left(2 \sum_{l=0}^{\log n/20} \binom{n-1}{l} p^l q^{n-1-l} \right)^2 =$$
$$O(n\ell_{10}(2d\log n)^{\ell_{10}+1}(20ed)^{\log n/10} n^{-2d}) = O(n \cdot n^{1/10+o(1)} \cdot n^{1/2} \cdot n^{-2}) = o(1).$$

(d) The expected number of vertices with small out- and in-degree is $O(n^{1-2\times .8002}) = o(1)$.

(e) For $1 \leq k \leq \Delta_0$ let

$$\lambda_k = \begin{cases} 1 & 1 \leq k \leq \frac{\log n}{(\log\log n)^4} \\ (\log\log n)^4 & \frac{\log n}{(\log\log n)^4} \leq k \leq \Delta_0 \end{cases}.$$

Let $\epsilon = \frac{1}{\log\log n}$. The probability that there exists a vertex of in-degree $k \in [1, \Delta_0]$ with λ_k in-neighbours of in or out-degree outside $(1 \pm \epsilon)np$, is bounded by

$$\sum_{k=1}^{\Delta_0} n \binom{n-1}{k} p^k q^{n-1-k} \binom{k}{\lambda_k} (4e^{-\epsilon^2 np/3})^{\lambda_k} \leq \sum_{k=1}^{\Delta_0} 2n^{1-d} \left(\frac{nep}{k} \cdot 2 \cdot n^{-\epsilon^2 d \lambda_k/(4k)} \right)^k = o(1).$$

Now assume that there are fewer than λ_k neighbours of v of in or out-degree outside $(1 \pm \epsilon)np$. Assuming at most one neighbour w of y is small,

$$\sum_{u \in N^-(y) \setminus \{w\}} \frac{\deg^-(u)}{\deg^+(u)} = \begin{cases} (1 + O(\epsilon))k & 1 \leq k \leq \frac{\log n}{(\log\log n)^4} \\ (1 + O(\epsilon))(k - \lambda_k) + O(\lambda_k) & \frac{\log n}{(\log\log n)^4} \leq k \leq \Delta_0 \end{cases}.$$

This completes the proof of the lemma. □

Let *weak* distance refer to distance in the underlying graph of $D_{n,p}$, and let a cycle in the underlying graph be called a *weak* cycle.

Lemma 15. Whp *there does not exist a small vertex that is within weak distance ℓ_{10} of a weak cycle C of length at most ℓ_{10}.*

Proof Let v, C be such a pair. Let $|C| = i$ and j be the weak distance of v from C. The probability that such a pair exists is at most

$$\sum_{i=3}^{\ell_{10}} (2np)^i i \sum_{j=0}^{\ell_{10}} (2np)^j \sum_{l=0}^{\log n/20} 2 \binom{n-1}{l} p^l q^{n-1-l}$$
$$= O(n^{1/10+o(1)} \cdot n^{1/10+o(1)} \cdot n^{-4/5+o(1)}) = o(1).$$

□

5 Analysis of the random walk: Estimating the stationary distribution

In this section we keep Assumption 1 and assume that we are dealing with a digraph which has all of the high probability properties of Section 4.3.

5.1 Lower Bound on the stationary distribution

We use the properties described in Section 4.2. We derive a lower bound on $P_x^{2\ell+1}(y)$. For this lower bound we only consider (x, y)-paths of length $2\ell+1$ consisting of a T_x^{low} path from x to X_ℓ followed by an edge from X_ℓ to Y_ℓ and then a T_y^{low} path to y. The probability of following such a path is $Z(x, y)$, see (31). Lemma 9 implies that

$$P_x^{(2\ell+1)}(y) \geq (1-o(1))\frac{\deg^-(y)}{m} \text{ for all } v \in V. \tag{62}$$

Lemma 16. *For all $y \in V$,*

$$\pi_y \geq (1-o(1))\frac{deg^-(y)}{m}.$$

Proof It follows from (62) that for any $y \in V$,

$$\pi_y = \sum_{x \in V} \pi_x P_x^{(2\ell+1)}(y) \geq (1-o(1))\frac{\deg^-(y)}{m} \sum_{x \in V} \pi_x = (1-o(1))\frac{\deg^-(y)}{m}. \tag{63}$$

□

5.2 Upper Bound on the stationary distribution

Lemma 16 above proves that the expression in Theorem 2 is a lower bound on the stationary distribution. As $\sum \pi_y = 1$, this can be used to derive an upper bound of $\pi_y \leq (1+o(1))\frac{\deg^-(y)}{m}$ which holds for all but $o(n)$ vertices y. In this section we extend this upper bound *to all* $y \in V$.

We use the properties described in Section 4.3. We now consider the probability of various types of walks of length ℓ_0+1 from x to y. Some of these walks are simple directed paths in BFS trees constructed in a similar way to the lower bound, and some use back edges of these BFS trees, or contain cycles etc. We will upper bound $P_x^{\ell_0+1}(y)$ as a sum

$$P_x^{\ell_0+1}(y) \leq Z_x^{\ell_0+1}(y) + S_x^{\ell_0+1}(y) + Q_x^{\ell_0+1}(y) + R_x^{\ell_0+1}(y), \tag{64}$$

where the definitions of the probabilities on the right hand side are described below.

$\boldsymbol{Z_x^{\ell_0+1}(y)}$. This is the probability that $\mathcal{W}_x(\ell_0+1) = y$ and the (ℓ_1+1)th edge (u, v) is such that $u \in X$ and $v \in Y_{\ell_2} \setminus X$, and the last ℓ_2 steps of the walk use edges of the tree T_y^{up}. These are the simplest walks to describe. They go through T_x^{up} for ℓ_1 steps and then level by level through T_y^{up}. They make up almost all of the walk probability.

$\boldsymbol{S_x^{\ell_0+1}(y)}$**.** This is the probability that $\mathcal{W}_x(\ell_0+1)=y$ goes from x to y without leaving X. This includes any special cases such as, for example, a walk $xyxy...xy$ based on the existence of a cycle $(x,y),(y,x)$ in the digraph.

$\boldsymbol{Q_x^{\ell_0+1}(y)}$**.** This is the probability that $\mathcal{W}_x(\ell_0+1)=y$ and the (ℓ_1+1)th edge (u,v) is such that $v\in Y_{\ell_2}\cap X$ and the last ℓ_2 steps of the walk use edges of the tree T_y^{up}. We exclude walks within X that are counted in $\boldsymbol{S_x^{\ell_0+1}(y)}$.

$\boldsymbol{R_x^{\ell_0+1}(y)}$**.** This is the probability that $\mathcal{W}_x(\ell_0+1)=y$ and during the last ℓ_2 steps, the walk uses some edge which is a back or cross edge with respect to the tree T_y^{up}.

Upper bound for $Z_x^{\ell_0+1}(y)$.

It follows from (57) that

$$Z_x^{\ell_0+1}(y)\leq(1+o(1))\frac{\deg^- y}{m}. \tag{65}$$

Upper bound for $S_x^{\ell_0+1}(y)$.

Let $\mathcal{W}_x$ be a walk of length t in X, and let $\mathcal{W}_x(t)=v$. Let $d^-_{\max}=\max_{w\in X}|N^-(w)\cap X|$. Tracing back from v for t steps, the number of walks length t in X terminating at v is at most $(d^-_{\max})^t$; so this serves as an upper bound on the number of walks from x to v of this length. By Lemma 13(a), we may assume that $d^-_{\max}\leq 100/\eta$.

Applying this description, there can be at most $(100/\eta)^{\ell_0+1}$ walks of length ℓ_0+1 from x to y, which do not exit from X. We conclude that

$$S_x^{\ell_0+1}(y)\leq\left(\frac{(100/\eta)}{c_0np}\right)^{\ell_0+1}=o\left(\frac{1}{n^{1+\eta/2}}\right). \tag{66}$$

Upper bound for $Q_x^{\ell_0+1}(y)$.

We say that a walk $\mathcal{W}_x$ *delays for t steps*, if $\mathcal{W}_x$ *exits X for the first time* at step ℓ_1+t. A walk *delays* at level i, if the walk takes a cross edge (to the same level i) or a back edge (to a level $j<i$) i.e. a *non-tree edge* $e=(u,v)$ contained in X that is not part of T_x^{up}.

Lemma 17. *Let* $t_0=\left\lceil\frac{K\Lambda}{\log np}\right\rceil$ *where* $K=2\log(100C_0/\eta c_0)$*, then*

$$\mathbf{Pr}(\mathcal{W}_x(\ell_0+1)=y \text{ and } \mathcal{W}_x \text{ delays for } t_0 \text{ or more steps})=o(1/n).$$

Proof The only way for a walk to exit from X is via X_{ℓ_1} (recall that edges oriented out from X_i end in X_{i+1}). Let $\mathcal{W}_x$ be an (x,y)-walk which delays for t steps, and then takes edge $e=(u,v)$ between X_{ℓ_1} and $Y_{\ell_2-t}\setminus X$. There are at most $(100/\eta)^{\ell_1+t}$ walks of length ℓ_1+t from x to u within X. After reaching vertex v, $\mathcal{W}_x$ follows the unique path from v to y in T_Y.

Applying Lemma 13(d) we see that the total probability $P^\dagger(t)$ of such (x,y)-walks of length ℓ_0+1 and delay t is

$$\begin{aligned}
P^\dagger(t) &\leq \frac{(100/\eta)^{\ell_1+t}\left(9|X_{\ell_1}||Y_{\ell_2-t}|p+\log^2 n\right)}{(c_0np)^{\ell_0+1}} \\
&\leq \frac{(100/\eta)^{\ell_1+t}\left(9C_0^{\ell_0}(np)^{\ell_0-t}p+\log^2 n\right)}{(c_0np)^{\ell_0+1}} \\
&= O\left(\frac{1}{n}\left(\frac{(100/\eta)C_0}{c_0}\right)^{\ell_0}\left(\frac{1}{(np)^t}+\frac{\log^2 n}{n^\eta}\right)\right).
\end{aligned}$$

So,

$$P^\dagger(\geq t_0)=\sum_{t\geq t_0}P^\dagger(t)=O\left(\frac{A^\Lambda}{n}\left(\frac{1}{(np)^{t_0}}+\frac{\log^2 n}{n^\eta}\right)\right), \tag{67}$$

where $A=(100C_0/\eta c_0)^{1+\eta}$.

Now $A^\Lambda=n^{o(1)}$. Also $A^\Lambda/np=o(1)$ if $\log^2 np\geq 2(\log A)(\log n)$ in which case the RHS of (67) is $o(1/n)$, which is what we need to show. So assume now that $\log^2 np\leq 2(\log A)(\log n)$. This means that $\Lambda\to\infty$ and then

$$\frac{A^\Lambda}{(np)^{t_0}}\leq\left(\frac{A}{e^K}\right)^\Lambda\to 0.$$

Thus in both cases

$$P^\dagger(\geq t_0)=o(1/n). \tag{68}$$

□

We can now focus on walks with delay t, where $1\leq t<t_0$. A *non-tree* edge of X is an edge induced by X which is not an edge of T_x^{up}. For $4i\leq(1-\epsilon)\Lambda$, Lemma 10 implies that **whp** the set $U=X_{\leq i}$ contains at most $|S|$ edges. For, if U contained more than $|U|+1$ edges then it would contain two distinct cycles C_1,C_2. In which case, C_1,C_2 and the shortest undirected path in U joining them would form a set S which satisfies the conditions of Lemma 10. Thus there is at most one *non-tree edge* $e=(u,v)$ contained in $X_{\leq(1-\epsilon)\Lambda/4}$.

Let $\theta=2\eta\Lambda$. We classify walks into two types.

Type 1 Walks. These have a delay caused by using a non-tree edge of $X_{\leq\theta}$, but no delay arising at any level $i>\theta$. Thus, once the walk finally exits X_θ to $X_{\theta+1}$ it moves forward at each step towards X_{ℓ_1}, and then exits to $Y_{\ell_2-t}\setminus X$.

Type 2 Walks. These have a delay arising at some level X_i, $i>\theta$. We do not exclude previous delays occurring in $X_{\leq\theta}$, or subsequent delays at any level.

Type 1 Walks. We can assume that $X_{\leq\theta}$ induces exactly one non-tree edge $e = (u, v)$. Let $u \in X_i$ then $v \in X_j,\ j \leq i$. There are two cases.

(a) e **is a cross edge, or back edge not inducing a directed cycle**.

Here the delay is $t = i + 1 - j$ and this is less than t_0 by assumption. Then, as we will see,

$$\mathbf{Pr}(\text{Type 1(a) walk}) \leq \frac{1}{(c_0np)^2}\frac{1}{\deg^+(w)}\sum_{w\in N^+(v)} Z_w^{(\ell_0-i+j+1)}(y) = O\left(\frac{1}{n(np)^2}\right). \tag{69}$$

The term $1/(c_0np)^2$ arises from the walk having to take the out-neighbour of x that leads to u in T_x^{up} and then having to take the edge (u, v). The next step of the walk is to choose $w \in N^+(v)$ and it must then follow a path to y level by level through the two trees. The value of $Z_w^{\ell_0-i+j+1}(y)$ can be obtained as follows. Let $\ell_0' = \ell_0 - (i - j) - 1$, then as $t < t_0 = o(\Lambda)$ we have that $\ell_0' \sim \ell_0$. For $w \in N^+(v)$ replace ℓ_0 by ℓ_0' in (57) above, to obtain $Z(w, y) = O(1/n)$, see Remark 6. This verifies (69).

(b) e **is a back edge inducing a directed cycle**.

Let xPu be the path from x to u in $T_x^{up}(\theta)$. As v is a vertex of xPu, we can write $xPu = xPv, vPu$ and cycle $C = vCv = vPu, (u, v)$. Let $\sigma \geq 2$ be the length of C. For some w in vPu the walk is of the form $xPv, vPw, (wCw)^k, wPz$, where wCw is C started at w, the walk goes round wCw, k times and exits at w to $u' \in N^+(w) \setminus C$ and then moves forward along wPz to $z \in X_{\ell_1}$ and then onto y. The delay is $t = k\sigma$ and this is less than t_0 by assumption

We claim that

$$\mathbf{Pr}(\text{Type 1(b) walk}) \leq \sum_{w\in C}\sum_{k\geq 1}(c_0np)^{-k\sigma}\frac{1}{\deg^+(w)-1}\sum_{u'\in N^+(w)\setminus C} Z_{u'}^{\ell_0-k\sigma+1}(y) = O\left(\frac{1}{n(np)^2}\right). \tag{70}$$

The term $(c_0np)^{-k\sigma}$ accounts for having to go round C k times and we can argue that $Z_{u'}^{\ell_0-k\sigma+1}(y) = O(1/n)$ as we did for Type 1(a) walks.

So from (69) and (70) we have that

$$\mathbf{Pr}(\text{Type 1 walk}) = O\left(\frac{1}{n(np)^2}\right). \tag{71}$$

Type 2 Walks. Suppose $\mathcal{W}_x$ is a walk which exits X at step $\ell_1 + t$ and is delayed at some level $i > \theta$ by using an edge (u, v). The walk arrives at vertex $u \in X_i$ for the first time at some step $i + t'$ and traverses a cross or back edge to $v \in X_j,\ j \leq i$.

A contributing walk will have to use one of the $A^\circ(x, y, t) \leq \log n$ edges described in Lemma 13(c). By Lemma 13(a) there are at most $(100/\eta)^{\ell_1+t}\log n$ from x to $u \in X_i^\circ$. Once the walk reaches $w \in Y_{\ell_2 - t}$ there is (by assumption) a unique path in T_y^{up} from w to y. Let $P(i,t)$ be the probability of these Type 2 walks, then

$$P(i,t) \leq \frac{(100/\eta)^{\ell_1+t} \log n}{(c_0 np)^{\ell_0+1}} = O\left(\frac{1}{n^{1+\eta/2}}\right). \tag{72}$$

Thus finally from (68), (71), (72)

$$Q_x^{\ell_0+1}(y) = P^\dagger(\geq t_0) + \mathbf{Pr}(\text{Type 1 walk}) + \sum_{1\leq t\leq t_0} \sum_{\theta \leq i \leq \ell_i} P(i,t) = O\left(\frac{1}{n}\frac{1}{(np)^2}\right). \tag{73}$$

Upper bound for $R_x^{\ell_0+1}(y)$.

Let $Y = Y_{\leq \ell_2}$ be the vertex set of $T_y^{up}(\ell_2)$. We assume that Y induces a unique edge $e = (u,v)$ which is not in T_y^{up}. Note that the condition that $|Y|$ induces at most $|Y|$ edges holds, even if we replace ℓ_2 with $2\ell_2$ based on the construction of $T_Y(2\ell_2)$ to depth $2\ell_2$, by branching backwards from y. We consider two cases.

(i) e **is a cross or forward edge, or back edge not inducing a directed cycle.**

We have $u \in Y_i, v \in Y_j$ for some $i \leq j \leq \ell_2$. We suppose the (x,y)-walk is of the form $xWu, (u,v), vWy$ where $u \notin vWy$, so that vWy is a unique path in T_y^{up}.

Case 1: $i > (4\eta/5)\Lambda$.

Let $\ell_3 = (1-\eta/10)\Lambda$. The length of the path $(u,v), vWy$ is j, so the length of the walk xWu is $\ell_0 - j + 1$. Let h be the distance from u to $X_{\leq \ell_3}$ in T_y^{up}. Then

$$h = \max\{0, \ell_0 - \ell_3 - j + 1\} \leq \max\{0, \ell_0 - \ell_3 - i + 1\}.$$

Let $w \in X_{\leq \ell_3}$. By Lemma 13, the number of (x,w)-walks of length $\ell \leq \ell_3$ in $X_{\leq \ell_3}$ passing through w at step ℓ is bounded by $(100/\eta)^{\ell_3}$. The the number of walks length h from u to $X_{\leq \ell_3}$ is at most ${\Delta_0}^h$. Thus, the number of (x,y)-walks passing through $e = (u,v)$ is bounded by $(100/\eta)^{\ell_3}{\Delta_0}^h$. Thus

$$R_x^{\ell_0+1}(y) = O\left(\frac{(100/\eta)^{\ell_3}{\Delta_0}^{9\eta\Lambda/10}}{(c_0 np)^{\ell_0+1}}\right) = O(n^{-1-\eta/20}). \tag{74}$$

Case 2: $0 < i \leq (4\eta/5)\Lambda$.

Let $i = a\Lambda$. Let $\eta' = \eta(1-a)$, $\ell_1' = (1-10\eta')\Lambda$, $\ell_2' = 11\eta'\Lambda$, and let $\ell_0' = \ell_1' + \ell_2'$. As observed above, the vertex set U of the tree T_U of height ℓ_2' above u induces no extra edges, so we can apply the upper bound result for walks of length $\ell_0'+1$ from x to u based on the assumption $R_x^{\ell_0'+1}(u)=0$. Thus

$$P_x^{\ell_0'+1}(u) \leq (1+o(1))\frac{\deg^-(u)}{m}.$$

The probability the walk then follows the path $(u,v), vPy$ is $O(1/(np)^2)$. Thus

$$R_x^{\ell_0+1}(y) = O\left(\frac{\deg^-(u)}{m(np)^2}\right). \tag{75}$$

(ii) e **is a back edge inducing a directed cycle.**

In this case, there is an edge $e=(u,v)$ where $u \in Y_i, v \in Y_j$ and $j > i$. Let vPu denote the path from v to u in T_y^{up}, and C the cycle $vPu,(u,v)$. There is some $k \geq 1$ such that the walk is $P_0 = xPu,(uCu)^k, uPy$. Let σ be the length of C, let τ be the distance from u to y in T_y^{up}, and let $s=\tau+k\sigma$. Let $\ell=\ell_0-s$. Then $\ell+1$ is the length of the walk xPu from x to u prior to the final s steps.

Either $\ell < (1+4\eta/5)\Lambda$ and the argument in Case 1 ($i \geq 4\eta/5)\Lambda$) above can be applied, giving us the bound

$$R_x^{\ell_0+1}(y) = O\left(\frac{(100/\eta)^{\ell_3}{\Delta_0}^{9\eta\Lambda/10}}{(c_0np)^{\ell_0+1}}\right) = O(n^{-1-\eta/20}). \tag{76}$$

Or $\ell \geq (1+4\eta/5)\Lambda$ and we adapt Case 2. Let w be the predecessor of u on P_0. We can use Remark 6 as above to obtain $P_x^{\ell}(w) \leq (1+o(1))\deg^-(w)/m$. As $k\sigma \geq 2, \tau \geq 0$, (the worst case is $u=y, w \in N^-(y)$), we obtain

$$R_x^{\ell_0+1}(y) = O\left(\frac{\deg^-(w)}{m(np)^2}\right). \tag{77}$$

Thus, using (74), (75), (76), (77) we have

$$R_x^{\ell_0+1}(y) = O\left(\frac{1}{n}\cdot\frac{1}{(np)^2}\right). \tag{78}$$

We have therefore shown that $S_x^{\ell_0+1}(y)+Q_x^{\ell_0+1}(y)+R_x^{\ell_0+1}(y)=o(1/n)$ completing the proof that

$$P_x^{\ell_0+1}(y) \leq (1+o(1))\frac{\deg^-(y)}{m} \tag{79}$$

Lemma 18. *For all $y \in V$,*

$$\pi_y = (1+o(1))\frac{deg^-(y)}{m}.$$

Proof It follows from (62) that for any $y \in V$,

$$\pi_y = \sum_{x \in V} \pi_x P_x^{(\ell_0+1)}(y) \leq (1+o(1)) \frac{\deg^-(y)}{m} \sum_{x \in V} \pi_x = (1+o(1)) \frac{\deg^-(y)}{m}. \tag{80}$$

The lemma now follows from Lemma 16. □

6 Stationary distribution: Removing Assumption 1

6.1 Large average degree case

6.1.1 $np \geq n^\delta$.

We can deal with this case by using a concentration inequality (81) from Kim and Vu [13]: Let $\Upsilon = (W, E)$ be a hypergraph where $e \in E$ implies that $|e| \leq s$. Let

$$Z = \sum_{e \in E} w_e \prod_{i \in e} z_i$$

where the $w_e, e \in E$ are positive reals and the $z_i, i \in W$ are independent random variables taking values in $[0, 1]$. For $A \subseteq W, |A| \leq s$ let

$$Z_A = \sum_{\substack{e \in E \\ e \supseteq A}} w_e \prod_{i \in e \setminus A} z_i.$$

Let $M_A = \mathbf{E}(Z_A)$ and $M_j(Z) = \max_{A,|A| \geq j} M_A$ for $j \geq 0$. There exist positive constants a and b such that for any $\lambda > 0$,

$$\mathbf{Pr}(|Z - \mathbf{E}(Z)| \geq a\lambda^s \sqrt{M_0 M_1}) \leq b|W|^{s-1} e^{-\lambda}. \tag{81}$$

For us, W will be the set of edges of $\vec{K}_n$ the complete digraph on n vertices. z_i will be the indicator variable for the presence of the ith edge of $\vec{K}_n$. E will be the set of sets of edges in walks of length $s = \lceil 2/\delta \rceil$ between two fixed vertices x and y in $\vec{K}_n$, and $w_e = 1$. Z will be the number of walks of length s that are in $D_{n,p}$. In which case we have

$$\mathbf{E}(Z) = (1+o(1)) n^{s-1} p^s$$
$$M_j \leq (1+o(1)) n^{s-j-1} p^{s-j} \leq (1+o(1)) \mathbf{E}(Z)/np \qquad for\ j \geq 1.$$

So $M_0 = \mathbf{E}(Z)$ and applying (81) with $\lambda = (\log n)^2$ we see that for any x, y we have

$$\mathbf{Pr}(|Z - \mathbf{E}(Z)| = O(\mathbf{E}(Z) n^{-\delta/2} \log^{O(1)} n)) = 1 - O(n^{-3}).$$

Thus **whp**

$$P_x^s(y) = (1+o(1)) \frac{n^{s-1} p^s}{((1-\epsilon_1)np)^s} \sim \frac{1}{n}, \qquad \forall x, y \in V.$$

We now finish with the arguments of Lemmas 16 and 18. □

6.2 Small average degree case

6.2.1 Lower bound on stationary distribution

A vertex is *small* if it has in-degree or out-degree at most $np/20$ and *large* otherwise. In the proofs of Section 4.2 we assumed x, y were large. We proceed as in Section 5.1 but initially restrict our analysis to large x, y. Also, with the exception of Y_1 we do not include small vertices when creating the X_i, Y_i. Avoiding the $\leq n^{1/5}$ small vertices (see Lemma 14(a)) is easily incorporated because in the proof we have allowed for the avoidance of $n^{2/3+o(1)}$ vertices from $\bigcup_i X_i$ etc. Provided there are no small vertices in $N^-(y)$, our previous lower bound analysis holds. In this way, we show for all large x, y that,

$$P_x^{(2\ell+1)}(y) \geq (1-o(1))\frac{\deg^-(y)}{m}. \tag{82}$$

If x is small, then it will only have large out-neighbours (see Lemma 14(c)) and so if y is large then

$$P_x^{(2\ell+2)}(y) = \frac{1}{\deg^+(x)} \sum_{z \in N^+(x)} P_z^{(2\ell+1)}(y) \geq (1-o(1))\frac{\deg^-(y)}{m}. \tag{83}$$

A similar argument deals with small y and x arbitrary i.e.

$$P_x^{(2\ell+2)}(y) = \sum_{z \in N^-(y)} \frac{P_x^{(2\ell+1)}(z)}{\deg^+(z)} \geq (1-o(1)) \sum_{z \in N^-(y)} \frac{\deg^-(z)}{m}\frac{1}{\deg^+(z)} \geq (1-o(1))\frac{\deg^-(y)}{m}. \tag{84}$$

We have used Lemma 14(e) to justify the last inequality.

In the case that some $u \in N^-(y)$ has small out-degree, then by Lemma 14(c) there is at most one such u **whp**. For $z \in N^-(y)$, we repeat the argument above for each factor $P_x^{2\ell+1}(z)$. The extra term $\varsigma^*(y)$ now arises from $\deg^-(u)/\deg^+(u)$ and

$$P_x^{(2\ell+2)}(y) = \sum_{z \in N^-(y)} \frac{P_x^{(2\ell+1)}(z)}{\deg^+(z)} \geq (1-o(1))\frac{1}{m} \sum_{z \in N^-(y)} \frac{\deg^-(z)}{\deg^+(z)} \geq (1-o(1))\frac{\deg^-(y)+\varsigma^*(y)}{m}.$$

We can now proceed as in (63).

6.2.2 Upper bound on stationary distribution

We first explain how the upper bound proof in Section 5.2 alters if Assumption 1 is removed. The assumption that the minimum degree was at least $c_0 np$ was used in the following places:

1. We assumed in Section 4.2 that $\deg^+(x), \deg^-(y) \geq c_0 np$. These assumptions can be circumvented by using Lemma 14(c) with the methods used in the lower bound case.

2. In (66), (72), (74). In these cases we used $(c_0np)^{\ell_0}$ as a lower bound on the product of out-degrees on a path of length λ for some $\lambda \geq \ell_1$. Using Lemmas 14 and 15, we see that small vertices are at weak distance at least ℓ_{10} and therefore there can be at most 11 such vertices on any walk length $\ell_0 + 1$. Thus, after dropping Assumption 1, we replace this lower bound by $(c_0np)^{\lambda-11}$, and the proof continues essentially unchanged.

3. In the proof of Lemma 9 we made a re-scaling $B = 1000/(c_0np)^{2\ell+1}$. The exponent $2\ell + 1$ was replaced by $\ell_0 + 1$ in the proof of (57) in Lemma 12. We now replace $\ell_0 + 1$ by $\ell_0 - 10$.

4. In the proof of Lemma 7 we made a re-scaling $U(i) = W(y, i) \cdot (c_0np)^i$ at each level $3 \leq i \leq \ell$. Assume that $2\ell_2 < \ell_{10}$ i.e. $\eta \leq 1/250$ so that there is at most one small vertex u in Y. If we replace $(c_0np)^i$ by $(c_0np)^{i-1}$ does not affect our concentration results, provided $i \geq 3$. The bounds on U_v are now $(c_0/np)(c_0/C_0)^i \leq U_v \leq 1$, and $\epsilon_i = 1/\sqrt{(A \log n)^{i-2}}$. If the small vertex $u \in N^-(y)$ then the direct calculations used in the lower bound hold. If the small vertex u is in levels $i = 2, 3$ this adds an extra term of $O(\deg^-(u)/(m(np)^{i-1}))$ to our estimate of $Z_x^{\ell_0+1}(y)$ in Section 5.2.

5. In (70), (75), (77). It follows from Lemma 15, that if e.g. T_y^{up} contains a non-tree edge, then no vertex of T_y^{up} is small, and the calculations in the proof are unaltered.

Thus the proof as is works perfectly well if we assume that y is large and if it has no small in-neighbours and there is no small vertex in Y. We call such a vertex y *ordinary.*

If y is small then from Lemmas 14 and 15 we can assume that all of its in-neighbours are ordinary. This is under the assumption that $2\ell_2 < \ell_{10}$ e.g. if $\eta \leq 1/250$. So in this case we can use Lemma 14(e) and obtain

$$P_x^{(\ell_0+2)}(y) = \sum_{\xi \in N^-(y)} \frac{P_x^{(\ell_0+1)}(\xi)}{\deg^+(\xi)} \leq \frac{1 + o(1)}{m} \sum_{\xi \in N^-(y)} \frac{\deg^-(\xi)}{\deg^+(\xi)} = (1 + o(1)) \frac{\deg^-(y)}{m}.$$

Suppose now that y is large and that there is a small vertex $u \in Y$. We can assume from Lemma 15 that Y does not contain any edge not in T_y^{up}. Either $u \in N^-(y)$ or, if not, from point 4. of the discussion above, an extra $O(\deg^-(u)/(m(np)))$ is added to $Z_x^{\ell_0+1}(y)$ for the probability of the (x, y)-walk going via u.

In the case where $u \in N^-(y)$ then as in the lower bound

$$\begin{aligned} P_x^{(\ell_0+1)}(y) &\leq \frac{1 + o(1)}{m} \left(\frac{\deg^-(u)}{\deg^+(u)} + \sum_{u \in N^-(y) \setminus w} \frac{\deg^-(u)}{\deg^+(u)} \right) \\ &= \frac{(1 + o(1))}{m} \left(\deg^-(y) + \varsigma^*(y) \right). \end{aligned}$$

We have now completed the proof of the asymptotic steady state without Assumption 1.

7 Mixing time and the conditions of Lemma 3

7.1 Upper Bound on Mixing time

Let T be a mixing time as defined in (5) and let $\ell = O(\log_{np} n)$ be given by (27). We prove that (**whp**) T satisfies

$$T = o(\ell \log n) = o((\log n)^2). \tag{85}$$

The total variation distance $\|\theta_1 - \theta_2\|$ between two distributions θ_1, θ_2 on a set V is defined as

$$\|\theta_1 - \theta_2\| = \frac{1}{2} \sum_{v \in V} |\theta_1(v) - \theta_2(v)|.$$

Let $P_x^{(t)}$ denote the t-step distribution of the walk, started from x and let

$$\bar{d}(t) = \max_{x,x' \in V} \|P_x^{(t)} - P_{x'}^{(t)}\| \tag{86}$$

be the maximum over x, x' of the variation distance between $P_x^{(t)}$ and $P_{x'}^{(t)}$. It is proved in Lemma 20 of Chapter 2 of Aldous and Fill [1] that

$$\bar{d}(s+t) \leq \bar{d}(s)\bar{d}(t) \text{ and } \max_x \|P_x^{(t)} - \pi_x\| \leq \bar{d}(t). \tag{87}$$

Equation (44) implies that whp

$$\bar{d}(2\ell + 1) = O\left(\frac{1}{\sqrt{\log n}}\right), \tag{88}$$

and using (87) and (88), we can choose T as in (85) so that $\bar{d}(T) = O(n^{-K})$, for any $K > 0$, thus satisfying condition (5).

7.2 Conditions of Lemma 3

We see immediately from (85) that Condition (b) of Lemma 3 is satisfied.

We show below that **whp** for all $v \in V$

$$R_T(1) = 1 + o(1). \tag{89}$$

Using (89), the proof that Condition (a) of Lemma 3 is satisfied, is as follows. Let $\lambda = 1/KT$ as in (7). For $|z| \leq 1 + \lambda$, we have

$$R_T(z) \geq 1 - \sum_{t=1}^{T} r_t |z|^t \geq 1 - (1+\lambda)^T \sum_{t=1}^{T} r_t = 1 - o(1).$$

Thus for $v \in V$, the value of p_v in (8) is given by

$$p_v = (1 + o(1))\frac{\deg^-(v)}{m}. \tag{90}$$

Proof of (89)**:** If $d \geq (\log n)^2$, then the minimum out-degree of $D_{n,p}$ is $\Omega(d \log n)$. In which case we have for any x, y

$$\mathbf{Pr}(\mathcal{W}_v(t) = y \mid \mathcal{W}_v(t-1) = x) = O\left(\frac{1}{d \log n}\right). \tag{91}$$

The expected number of returns to $v \in V$ by $\mathcal{W}_v$ during T steps, is therefore $O(T/d\log n)$= $o(1)$.

Now assume that $d \leq (\log n)^2$.

(i) Lemma 10 implies that if H is the subgraph of $D_{n,p}$ induced by vertices at weak distance at most $\Lambda/20$ from v then H contains at most $|V(H)|$ edges.

(ii) Lemma 14 implies that there is at most one small vertex in H.

(iii) Lemma 15 implies that there is no small vertex within weak distance 10 of a weak cycle of length ≤ 10.

Assume that conditions (i), (ii), (iii) hold. Let A_4 denote the set of vertices $u \neq v$ such that $D_{n,p}$ has a path of length at most 4 from u to v. We show next that:

$$\text{With probability } 1 - O(1/(np)^2),\ \mathcal{W}_v(i) \notin A_4,\ 1 \leq i \leq 4. \tag{92}$$

For this to happen, there has to be a cycle C of length at most 8 containing v. If such a cycle exists then all vertices within weak distance 10 of v have degree at least $np/20$. Furthermore, the only way that the walk can reach A_4 in 4 or less steps is via this cycle. This verifies (92). Assume then that $\mathcal{W}_v(i) \notin A_4$, $1 \leq i \leq 4$.

Suppose next that there is a time $T_1 \leq T$ such that $\mathcal{W}_v(T_1) = v$. Let $T_2 = \min\{\tau \leq T_1 : \mathcal{W}_v(t) \in A_4, \tau \leq t \leq T_1\}$. It must be the case that $d(T_2) = 4$ where $d(t)$ is the distance from $\mathcal{W}_v(t)$ to v.

If A_4 does not contain a small weak cycle then the walk must proceed directly to v in 4 steps. The probability of this is $O(1/(np)^3)$, since at most one vertex on the path of length 4 from $x = \mathcal{W}(T_2)$ to v will be of degree at most $np/20$.

If there is a small weak cycle C then there is an edge e of C whose removal leaves an in-branching of depth 4 into v. There are now 2 paths that $\mathcal{W}$ can follow from x to v. One uses e and one does not. Each path has a probability of $O(1/(np)^4)$ of being followed. Putting this altogether we see that the expected number of returns to v is $O(1/(np)^2 + T/(np)^3) = o(1)$. This completes the proof of (89).

8 The Cover Time of $D_{n,p}$

8.1 Upper Bound on the Cover Time

For $np = d\log n$, d constant, let $t_0 = (1+\epsilon)\left(d\log\left(\frac{d}{d-1}\right)\right) n\log n$. For $np = d\log n$ $d = d(n) \to \infty$ let $t_0 = (1+\epsilon)n\log n$. In both cases we assume $\epsilon \to 0$ sufficiently slowly to ensure that all inequalities below are valid.

Let $T_D(u)$ be the time taken by the random walk $\mathcal{W}_u$ to visit every vertex of D. Let U_t be the number of vertices of D which have not been visited by $\mathcal{W}_u$ at step t. We note the following:

$$C_u = \mathbf{E}(T_D(u)) = \sum_{t>0} \mathbf{Pr}(T_D(u) \geq t), \tag{93}$$

$$\mathbf{Pr}(T_D(u) \geq t) = \mathbf{Pr}(T_D(u) > t-1) = \mathbf{Pr}(U_{t-1} > 0) \leq \min\{1, \mathbf{E}(U_{t-1})\}. \tag{94}$$

Recall that $\boldsymbol{A}_v(t)$ denotes the event that $\mathcal{W}_u(t)$ did not visit vertex v in the interval $[T,t]$. It follows from (93), (94) that for any $t \geq T$,

$$C_u \leq t + 1 + \sum_{s\geq t} \mathbf{E}(U_s) \leq t + 1 + \sum_{v}\sum_{s\geq t} \mathbf{Pr}(\boldsymbol{A}_v(s)). \tag{95}$$

Assume first that $d(n) \to \infty$. If $s/T \to \infty$ then (9) of Lemma 3 together with the value of p_v given by (90), and concentration of in-degrees implies that

$$\mathbf{Pr}(\boldsymbol{A}_v(s)) \leq (1+o(1)) \exp\left\{-\frac{(1-o(1))s}{n}\right\} + O(e^{-\Omega(s/T)}). \tag{96}$$

Plugging (96) into (95) we get

$$\begin{aligned} C_u &\leq t_0 + 1 + 2n \sum_{s\geq t_0} \left(\exp\left\{-\frac{(1-o(1))s}{n}\right\} + O(e^{-\Omega(s/T)})\right) \\ &\leq t_0 + 1 + 3n^2 \exp\left\{-\frac{(1-o(1))t_0}{n}\right\} + O(nTe^{-\Omega(t_0/T)}) \\ &= (1+o(1))t_0. \end{aligned} \tag{97}$$

We now assume that d is bounded as $n \to \infty$, and the conditions of Lemma 4 hold. For $v \in V$ we have

$$\mathbf{Pr}(\boldsymbol{A}_v(s)) = (1+o(1)) \exp\{-(1+o(1/\log n))\pi_v s\} + O(e^{-\Omega(s/T)})$$

where, by Lemma 16,

$$\pi_v \geq (1-o(1)) \frac{\deg^-(v)}{m}.$$

In place of (97) we use the bounds on the number of vertices of degree k given in Lemma 4, in terms of the sets $K_i,\ i = 0, 1, 2, 3$. Thus

$$C_u \leq t_0 + 1 + o(1) + \sum_{i=0}^{3} S_i \tag{98}$$

where

$$\begin{aligned} S_i &= \sum_{k\in K_i} D(k) \sum_{s\geq t_0} \exp\left\{-\frac{(1-o(1))ks}{m}\right\} \\ &\leq 2m \sum_{k\in K_i} \frac{D(k)}{k} e^{-(1-o(1))kt_0/m} \\ &\leq 2m \sum_{k\in K_i} \frac{D(k)}{k} \left(\frac{d-1}{d}\right)^{(1+\epsilon/2)k}. \end{aligned}$$

The main term occurs at $i = 3$. Using (14), (17), the fact that $(nep(d-1))/(kd))^k$ is maximized at $k = np(d-1)/d$, and $m = dn \log n(1 + o(1))$ **whp**, we see that

$$\begin{aligned} S_3 &\leq \frac{8m}{n^{d-1}} \sum_{k=c_0np}^{\Delta_0} \left(\frac{nep}{k}\right)^k \left(\frac{d-1}{d}\right)^{(1+\epsilon/2)k} \\ &\leq 8m\ \Delta_0\ e^{-\epsilon c_0 np/2d} \\ &= o(t_0). \end{aligned} \tag{99}$$

Note that $K_0 = 0$. We next consider the cases $i = 1, 2$. For $i = 1$, we refer first to Lemma 4(i-a). If $d - 1 \geq (\log n)^{-1/3}$ then $K_1 = \emptyset$. If $d - 1 < (\log n)^{-1/3}$, then $D(k) \leq (\log \log n)^2$, from (15). In this case $t_0 = O((1/(d-1)))dn \log n$. Thus

$$\begin{aligned} S_1 &\leq m \sum_{k\in K_1} \frac{D(k)}{k} \left(\frac{d-1}{d}\right)^{(1+\epsilon/2)k} \\ &\leq m \sum_{k=1}^{15} \frac{(\log\log n)^2}{k} \left(\frac{d-1}{d}\right)^{(1+\epsilon/2)k} \\ &= O(t_0)(\log\log n)^2 (d-1)^{-\epsilon/2} \\ &= o(t_0) \end{aligned} \tag{100}$$

For $i = 2$, by Lemma 4 if $d - 1 < (\log n)^{-1/3}$ and $k \geq 16$, and using (16) we have $D(k) \leq$

$(\log n)^4$. Thus

$$\begin{aligned} S_2 &\leq m \sum_{k \in K_2} \frac{D(k)}{k} \left(\frac{d-1}{d}\right)^{(1+\epsilon/2)k} \\ &\leq O(t_0) \sum_{k \in K_2} \frac{\log^4 n}{k}(d-1) \left(\frac{d-1}{d}\right)^{(1+\epsilon/2)k} \\ &= O(t_0) \log^4 n (\log n)^{-(19/3+\epsilon/8)} \\ &= o(t_0). \end{aligned} \tag{101}$$

If $d-1 \geq (\log n)^{-1/3}$ then by Lemma 4(i-a) $\min\{k \in K_2\} \geq (\log n)^{1/2}$, and $|K_2| = O(\log \log n)$. Thus, as d is bounded

$$\begin{aligned} S_2 &= O(t_0) \sum_{k \geq (\log n)^{1/2}} \frac{\log \log n}{k}(d-1) \left(\frac{d-1}{d}\right)^{(1+\epsilon/2)k} \\ &= o(t_0) \end{aligned} \tag{102}$$

The upper bound on cover time of $C_u \leq t_0 + o(t_0)$ now follows from (98)–(102).

8.2 Lower Bound on the Cover Time

For $np = d \log n$, let $t_1 = (1-\epsilon) d \log\left(\frac{d}{d-1}\right) n \log n$. Here $\epsilon \to 0$ sufficiently slowly so that all inequalities claimed below are valid.

Case 1: $np \leq n^\delta$ where $0 < \delta \ll \eta$ is a positive constant.
Let $k^* = (d-1)\log n$, and let $V^* = \left\{v : \deg^-(v) = k^* \textit{ and } \deg^+(v) = d \log n\right\}$. **Whp** the size $|V^*| \geq n^* = \frac{n^{\gamma_d}}{4\pi \log n (d(d-1))^{1/2}}$ (see Lemma 4(ii)). Let us first work assuming $d \geq 1.05$. In this case $\gamma_d = (d-1)\ln(d/(d-1)) \geq .15$ and we write $n^* = n^{\gamma_d - o(1)}$. The maximum degree in D is at most $\Delta_0 = O(np)$ and so V^* contains a sub-set V_1^* of size $n^{\gamma_d/2}$ such that $v, w \in V_1^*$ and $x \in V$ implies

$$dist(x,v) + dist(x,w) \;>\; \Lambda/100. \tag{103}$$

$$dist(y,x) + dist(x,y) \;>\; \Lambda/50, \;\; for \; y = v, w. \tag{104}$$

Here "dist" refers to directed distance in $D_{n,p}$ and recall that $\Lambda = \log_{np} n$.

Each $v \in V_1^*$ has $\pi_v \sim \frac{d-1}{dn}$ and so we can choose a subset V^{**} of size $\geq n^{\gamma_d/3}$ such that if $v_1, v_2 \in V^{**}$ then

$$|\pi_{v_1} - \pi_{v_2}| \leq \frac{1}{n \log^{10} n}. \tag{105}$$

Indeed, suppose that $\pi_v \in \left[\frac{d-1}{2dn}, \frac{2(d-1)}{dn}\right]$ for $v \in V_1^*$. Divide this interval into $\log^{10} n$ equal sized sub-intervals and then use the pigeon-hole principle.

Now choose $u \notin V^{**}$ and let $V^\dagger$ denote the set of vertices in V^{**} that have not been visited by $\mathcal{W}_u$ by time t_1. Then $\mathbf{E}(|V^\dagger|) \to \infty$, as the following calculation shows;

$$\mathbf{E}(|V^\dagger|) \geq n^{\gamma_d/3}\left(\exp\left\{-\frac{(1+o(1))k^* t_1}{m}\right\} - o(e^{-\Omega(t_1/T)})\right) - T,$$

where the last term accounts for possible visits before time T.

Now assume that $1 + o(1) \leq d \leq 1.05$. In these circumstances we have $n^* = \log^\omega n$ where $\omega \to \infty$, see (21). Equations (103), (104) now hold for all $v, w \in V^*$. This follows from Lemma 14 because the vertices of V^* are small. The size of V^{**} is at least $n^*/(\log n)^{10}$ and we can again write

$$\begin{aligned}\mathbf{E}(|V^\dagger|) &\geq \frac{n^*}{(\log n)^{10}}\left(\exp\left\{-\frac{(1+o(1))k^* t_1}{m}\right\} - o(e^{-\Omega(t_1/T)})\right) - T \\ &\to \infty.\end{aligned}$$

As in previous papers, see for example [5], we will finish our proof by using, the Chebyshev inequality to show that $V^\dagger \neq \emptyset$ **whp**, thus completing the proof of Theorem 1. This will follow if we can prove that

$$\mathbf{Var}(|V^\dagger|) = o(\mathbf{E}(|V^\dagger|^2) + O(|V^{**}|^2 n^{-2}) = o(\mathbf{E}(|V^\dagger|)^2).$$

To establish this inequality, we will show that if $v, w \in V^{**}$ then

$$\mathbf{Pr}(\boldsymbol{A}_v(t_1) \cap \boldsymbol{A}_w(t_1)) \leq (1 + o(1))\mathbf{Pr}(\boldsymbol{A}_v(t_1))\mathbf{Pr}(\boldsymbol{A}_w(t_1)). \tag{106}$$

To prove this, we identify vertices v, w into a "supernode" σ to obtain a digraph D_σ with $n-1$ vertices. In this digraph σ has in-degree $deg^-(v) + deg^-(w) = 2k^*$ and out-degree $2d\log n$.

The stationary distribution of D_σ.
Let π^* denote the vector of steady states in D_σ. The arguments we used in Sections 4 and 5 remain valid in D_σ, and thus

$$\pi^*_\sigma \sim (1 - o(1))\frac{2k^*}{m}.$$

However, we need to be more precise. For a vertex x of D_σ let

$$\hat{\pi}_x = \begin{cases} \pi_x & x \neq \sigma \\ \pi_v + \pi_w & x = \sigma \end{cases}.$$

We will prove for all $x \in V(D_\sigma)$, that

$$|\pi^*_x - \hat{\pi}_x| = O\left(\frac{1}{n(\log n)^8}\right). \tag{107}$$

Proof of (107)**.**
Let $\xi = \hat{\pi} - \pi^*$ be the difference between $\hat{\pi}$ and π^*. Let P^* be the transition matrix of the walk on D_σ, then

$$P^*(x,y) = \begin{cases} P(x,y) & x,y \neq \sigma \\ (P(v,y)+P(w,y))/2 & x=\sigma \\ P(x,v)+P(x,w) & y=\sigma \end{cases}.$$

Let ξ' be the transpose of ξ. It follows from the steady state equations that

$$(\xi' P^*)_x = \begin{cases} \hat{\pi}_x - \pi^*_x & x \notin N^+(\{v,w\}) \\ \hat{\pi}_x - \pi^*_x + \frac{\pi_w - \pi_v}{2} P(v,x) & x \in N^+(v) \\ \hat{\pi}_x - \pi^*_x + \frac{\pi_v - \pi_w}{2} P(w,x) & x \in N^+(w) \end{cases}.$$

We rewrite this as

$$\xi'(I - P^*) = \eta' \tag{108}$$

where $\eta_x = 0$ for $x \notin N^+(\{v,w\})$ and $|\eta_x| \leq |\pi_v - \pi_w|/2$ otherwise.

Multiplying (108) on the right by $M = \sum_{t=0}^{T-1} (P^*)^t$ we have

$$\xi'(I - P^*)M = \xi'(I - (P^*)^T) = \eta' M. \tag{109}$$

Let

$$(P^*)^T = \Pi + E \tag{110}$$

where Π is the $(n-1) \times (n-1)$ matrix with each row equal to $(\pi^*)'$. The definition of T implies that each entry of E has absolute value bounded by n^{-3}.

Now write $\xi = \alpha\pi^* + \zeta$ where $\zeta \perp \pi^*$. It follows from $(\pi^*)'P^* = (\pi^*)'$ and (109) that

$$(\alpha\pi^* + \zeta)'(I - (P^*)^T) = \zeta'(I - (P^*)^T) = \zeta'(I - \Pi - E) = \eta' M.$$

Now

$$\zeta'(I - E) = \zeta'(I - (P^*)^T + \Pi) = \eta' M + \zeta' \Pi.$$

As $\zeta \perp \pi^*$ this implies that

$$\zeta'(I - E)\zeta = \eta' M \zeta. \tag{111}$$

Note that

$$|\eta' M \zeta| \leq \sum_{t=0}^{T-1} |\eta'(P^*)^t \zeta| \leq T|\eta||\zeta|, \tag{112}$$

where $|z|$ denotes the ℓ_2 norm of z.

Now

$$|\zeta'(I - E)\zeta| \geq |\zeta|^2 - |\zeta' E \zeta| \geq |\zeta|^2 - n^{-3} \left(\sum_{i=1}^{n-1} |\zeta_i| \right)^2 \geq |\zeta|^2 (1 - n^{-2}). \tag{113}$$

It follows from (111), (112) and (113) that

$$|\zeta|^2(1-n^{-2}) \leq T|\eta||\zeta|$$

and so using (105) we find that

$$|\zeta| = O\left(\frac{1}{n(\log n)^8}\right). \tag{114}$$

Now let $\mathbf{1}$ denote the $(n-1)$-vector of 1's. Then

$$0 = 1 - 1 = (\hat{\pi} - \pi^*)'\mathbf{1} = \xi'\mathbf{1} = \alpha + \zeta'\mathbf{1}.$$

Using (114) this gives

$$|\alpha| \leq |\mathbf{1}|\,|\zeta| = O\left(\frac{1}{n^{1/2}(\log n)^8}\right).$$

Now $\xi_x = \alpha\pi_x^* + \zeta_x$ for all x and so

$$\xi_x^2 \leq 2\alpha^2(\pi_x^*)^2 + 2\zeta_x^2 = O\left(\frac{1}{n(\log n)^{16}} \cdot \frac{1}{n^2} + \frac{1}{n^2(\log n)^{16}}\right) = O\left(\frac{1}{n^2(\log n)^{16}}\right).$$

This completes the proof of (107). □

Proof of (106)**.**
For $v \in V^{**}$, we first tighten (89) to

$$R_v = 1 + o(1/(\log n)^2). \tag{115}$$

Assume first that $np \leq \log^{10} n$. Then (103) and (104) imply that for $1 \leq t \leq (\log n)^{2/3}$, vertex v will be at distance $\geq 2\log^{2/3} n - t$ from $\mathcal{W}_v(t)$. Then once the walk is at a vertex w within distance $\log^{2/3} n$ of v its chance of getting closer is only $O(1/\log n)$. This being true with at most one exception for a vertex of low out-degree. The probability that there is a time t such that $\mathcal{W}_v$ is within $\log^{2/3} n$ of v and it makes 10 steps closer to v in the next 100 steps is $O(T/\log^9 n) = O(1/\log^7 n$. This implies (115). If $np \geq \log^{10} n$ then we use $R_v \leq 1 + (1+o(1))T/np$.

Similarly,

$$R_\sigma = 1 + o(1/(\log n)^2). \tag{116}$$

The mixing time T in what follows is the maximum of the mixing times for D and the maximum over v, w for D_σ. Using the suffix $\mathbf{Pr}_\sigma$ to denote probabilities related to random walks in D_σ and using (107), it follows that

$$\begin{aligned}
\mathbf{Pr}_\sigma(\boldsymbol{A}_\sigma(t_1)) &\leq \exp\left\{-\frac{(1+O(T\pi_\sigma^*))\pi_\sigma^* t_1}{m}\right\} - o(e^{-\Omega(t_1/T)}) \\
&\leq \exp\left\{-\frac{(1+o(1/\log n))(\pi_v+\pi_w)t_1}{m}\right\} - o(e^{-\Omega(t_1/T)}) \\
&= (1+o(1))\mathbf{Pr}(\boldsymbol{A}_v(t_1))\mathbf{Pr}(\boldsymbol{A}_w(t_1)).
\end{aligned} \tag{117}$$

But, using rapid mixing in D_σ,

$$\begin{aligned}\mathbf{Pr}_\sigma(\boldsymbol{A}_\sigma(t_1)) &= \sum_{x\neq\sigma} P_{\sigma,u}^T(x)\mathbf{Pr}_\sigma(\mathcal{W}_x(t)\neq\sigma, 1\le t\le t_1-T)\\ &= \sum_{x\neq\sigma}(\pi_x^*+O(n^{-3}))\mathbf{Pr}_\sigma(\mathcal{W}_x(t)\neq\sigma, 1\le t\le t_1-T)\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbf{Pr}(\boldsymbol{A}_v(t_1)\cap\boldsymbol{A}_w(t_1)) &= \sum_{x\neq v,w} P_u^T(x)\mathbf{Pr}(\mathcal{W}_x(t)\neq v,w, T\le t\le t_1)\\ &= \sum_{x\neq v,w}(\pi_x+O(n^{-3}))\mathbf{Pr}(\mathcal{W}_x(t)\neq v,w, 1\le t\le t_1-T)\end{aligned}$$

But,

$$\mathbf{Pr}_\sigma(\mathcal{W}_x(t)\neq\sigma, T\le 1\le t_1-T)=\mathbf{Pr}(\mathcal{W}_x(t)\neq v,w,1\le t\le t_1-T)$$

because random walks from x that do not meet v,w or σ have the same measure in both digraphs.

It follows that

$$\begin{aligned}&\mathbf{Pr}(\boldsymbol{A}_v(t_1)\cap\boldsymbol{A}_w(t_1))-\mathbf{Pr}_\sigma(\boldsymbol{A}_\sigma(t_1))\\ &=\sum_{x\neq v,w}(\pi_x-\pi_x^*+O(n^{-3}))\mathbf{Pr}(\mathcal{W}_x(t)\neq v,w,1\le t\le t_1-T)\\ &\le O\left(\frac{1}{n\log^8 n}\right)\sum_{x\neq v,w}\mathbf{Pr}(\mathcal{W}_x(t)\neq v,w,1\le t\le t_1-T)\\ &\le O\left(\frac{1}{n\log^8 n}\right)\sum_{x\neq v,w}\frac{P_u^T(x)}{P_u^T(x)}\mathbf{Pr}(\mathcal{W}_x(t)\neq v,w,1\le t\le t_1-T)\\ &\le O\left(\frac{1}{n\log^8 n}\right)O(n\log n)\sum_{x\neq v,w}P_u^T(x)\mathbf{Pr}(\mathcal{W}_x(t)\neq v,w,1\le t\le t_1-T)\end{aligned}$$

since $P_u^T(x)=\Omega(1/n\log n)$

$$\le O\left(\frac{1}{\log^7 n}\right)\mathbf{Pr}(\boldsymbol{A}_v(t_1)\cap\boldsymbol{A}_w(t_1)). \tag{118}$$

Equations (117) and (118) together imply (106). □

Case 2: $np\ge n^\delta$**.**
In this range we take $t_1=(1-\epsilon)n\log n$ and let V^* be the set of vertices of degree $\lfloor np\rfloor$. A simple second moment calculation shows that **whp** we have $|V^*|=\Omega((np)^{1/2-o(1)})$. We then

choose ϵ so that $\mathbf{E}(|V^{\dagger}|) \geq (np)^{1/4}$. It is then only a matter of verifying (106). The details are as in the previous case.

This completes the proof of Theorem 1. □

Acknowledgement: We thank several referees whose insight and hard work has helped to make this paper (hopefully) correct and more readable.

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