Karp’s patching algorithm on dense digraphs

Alan Frieze

Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh PA 15213

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Abstract

We consider the following question. We are given a dense digraph \( D \) with minimum in- and out-degree at least \( \alpha n \), where \( \alpha > 1/2 \) is a constant. The edges of \( D \) are given edge costs \( C(e), e \in E(D) \), where \( C(e) \) is an independent copy of the uniform \([0, 1]\) random variable \( U \). Let \( C(i, j), i, j \in [n] \) be the associated \( n \times n \) cost matrix where \( C(i, j) = \infty \) if \((i, j) \not\in E(D)\). We show that w.h.p. the patching algorithm of Karp finds a tour for the asymmetric traveling salesperson problem that is asymptotically equal to that of the associated assignment problem. Karp’s algorithm runs in polynomial time.

1 Introduction

Let \( D(\alpha) \) be the set of digraphs with vertex set \([n]\) and with minimum in- and out-degree at least \( \alpha n \). We are given a digraph \( D \in D(\alpha) \). The edges of \( D \) are given independent edge costs \( C(e), e \in E(D) \), where \( C(e) \) is a copy of the uniform \([0, 1]\) random variable \( U \). Let \( C(i, j), i, j \in [n] \) be the associated \( n \times n \) cost matrix where \( C(i, j) = \infty \) if \((i, j) \not\in E(D)\). One is interested in the relationship between the optimal cost of the Assignment Problem (AP) and the Asymmetric Traveling Salesperson Problem (ATSP) associated with a cost matrix \( A(i, j), i, j \in [n] \).

The problem AP is that of computing the minimum cost perfect matching in the complete bipartite graph \( K_{n,n} \) when edge \((i, j)\) is given a cost \( C(i, j) \). Equivalently, when translated to the complete digraph \( \vec{K}_n \) it becomes the problem of finding the minimum cost collection of vertex disjoint directed cycles that cover all vertices. The problem ATSP is that of finding a single cycle of minimum cost that covers all vertices. As such it is always the case that \( v(\text{ATSP}) \geq v(\text{AP}) \) where \( v(.) \) denote the optimal cost. Karp considered the case where \( D = \vec{K}_n \). He showed that if the cost matrix comprised independent copies of \( U \) then w.h.p. \( v(\text{ATSP}) = (1 + o(1))v(\text{AP}) \). He proves this by the analysis of a patching algorithm. Karp’s result has been refined in \([2], [5] \) and \([8] \).

Karp’s Patching Algorithm: First solve AP to create a collection \( C_1, C_2, \ldots, C_\ell \) of vertex disjoint cycles covering \([n]\). Then patch two of the cycles together, as explained in the next paragraph. Repeat until there is one cycle.

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Let \( E(C) \) denote the set of edges in these cycles. A pair \( e = (x, y), f = (u, v) \) in different cycles \( C_1, C_2 \) are said to be a patching pair if the edges \( e' = (u, y), f' = (x, v) \) both exist. In which case we can replace \( C_1, C_2 \) by a single cycle \( (C_1 \cup C_2 \cup \{(u, y), (x, v)\}) \setminus \{e, f\} \). The edges \( e, f \) are chosen to minimise the increase in cost of the set of cycles.

**Theorem 1.** Suppose that \( D \in D(\alpha) \) where \( \alpha > 1/2 \) is a positive constant. Suppose that each edge of \( D \) is given a independent uniform \([0, 1]\) cost. Then w.h.p. \( v(\text{ATSP}) = (1 + o(1))v(\text{AP}) \) and Karp’s patching algorithm finds a tour of the claimed cost in polynomial time.

## 2 Proof of Theorem 1

We begin by solving AP. We prove the following:

**Lemma 2.** W.h.p., after solving AP, the number \( \nu_C \) of cycles \( \ell \leq \ell_0 = n^{3/4} \log^2 n \).

**Lemma 3.** W.h.p., the solution to AP contains only edges of cost \( A(i, j) \leq \alpha_0 = \frac{\log^4 n}{n} \).

Given these lemmas, the proof is straightforward. We can begin by replacing costs \( X(e) \leq \alpha_0 \) by infinite costs in order to solve AP. Lemma 3 implies that w.h.p. we get the same optimal assignment as we would without cost changes Having solved AP, the unused edges in \( E(D) \) of cost greater than \( \alpha_0 \) have a cost which is uniform in \([\alpha_0, 1]\) and this is dominated by \( \alpha_0 + U \).

Let \( C = C_1, C_2, \ldots, C_n \) be a cycle cover and let \( k_i = |C_i| \) where \( k_1 \leq k_2 \leq \cdots \leq k_m, m \leq \ell_0 \). If \( e \in C_i \) then there at least \( \varepsilon n - k_i \) edges \( f \) such that \( e, f \) make a patching pair. Different edges in \( C_i \) give rise to disjoint patching pairs. We ignore the saving associated with deleting \( e, f \) and only look at the extra cost \( X(e') + X(f') \) incurred. We see that if \( k_i < \varepsilon n/2 \leq k_{i+1} \) then the number of possible patching pairs is at least

\[
\phi(C) = \frac{1}{2} \sum_{i=1}^{r} k_i (\varepsilon n - k_i) \geq \frac{\varepsilon n}{2} \sum_{i=1}^{r} k_i \geq \frac{\varepsilon n}{2} \geq \frac{\varepsilon n (m - 2\varepsilon^{-1})}{2}.
\]

Let \( \alpha_1 = \frac{1}{mn^{1/4}} \). Suppose we consider a sequence of patches where we always implement the cheapest patch. Let \( \mathcal{E}_i \) denote the event that the cost of the \( i \)th patch is at most \( 2(\alpha_0 + \alpha_1) \). We claim that as long as \( m > 2\varepsilon^{-1} \) we have

\[
\mathbb{P}(\mathcal{E}_i) \geq 1 - \ell_0 e^{-\varepsilon n^{1/16}/(16 \log^2 n)} = o(n^{-1}). \tag{1}
\]

Indeed,

\[
\mathbb{P}(\neg \mathcal{E}_i \mid \mathcal{E}_{i-1}) = \mathbb{P} \left( \text{cheapest patch has cost greater than } 2(\alpha_0 + \alpha_1) \mid \mathcal{E}_{i-1} \right) \leq \frac{(1 - \alpha_0^2)^{\varepsilon n(m - 2\varepsilon^{-1})}}{\mathbb{P}(\mathcal{E}_{i-1})} \leq \mathbb{P}(\mathcal{E}_{i-1})^{-1} e^{-\varepsilon n^{1/16}/(8 \log^2 n)} \tag{2}
\]

This implies that

\[
\mathbb{P}(\neg \mathcal{E}_i) \leq e^{-\varepsilon n^{1/16}/(8 \log^2 n)} + \mathbb{P}(\neg \mathcal{E}_{i-1}).
\]

Given that \( \mathbb{P}(\mathcal{E}_0) = 1 \), this gives an inductive proof of (1).

When \( m \leq 2\varepsilon^{-1} \), we have

\[
\phi(C) \geq \varepsilon n \quad \text{and} \quad \mathbb{P}(\neg \mathcal{E}_i \mid \mathcal{E}_{i-1}) \leq \mathbb{P}(\mathcal{E}_{i-1})^{-1} e^{-\varepsilon (1-\varepsilon) n^2 \alpha_1^2} = \mathbb{P}(\mathcal{E}_{i-1})^{-1} e^{-\varepsilon n^{3/8}/\log^4 n} \tag{3}
\]
and we can proceed as for \( m > 2\varepsilon^{-1} \).

It follows from (2) and (3) and the union bound that w.h.p.

\[
v(\text{ATSP}) \leq v(\text{AP}) + 2 \sum_{m=1}^{\ell_{\alpha}} \left( \frac{\log^4 n}{n} + \frac{1}{mn^{1/6}} \right) = (1 + o(1))v(\text{AP}).
\]

The last equality follows from the fact that w.h.p. \( v(\text{AP}) > (1 - o(1))\zeta(2) > 1 \) where the lower bound of \((1 - o(1))\zeta(2)\) comes from [1].

## 3 Proof of Lemma 2

Let \( G \) denote the bipartite graph with vertex partition \( A = \{a_1, a_2, \ldots, a_n\} \), \( B = \{b_1, b_2, \ldots, b_m\} \) and an edge \( \{a_i, b_j\} \) for every directed edge \((i, j) \in D\). We let \( A_r = \{a_1, a_2, \ldots, a_\sigma\} \) and we let \( M_r \) be the minimum cost matching of \( A_r \) into \( B \) and let \( B_r \) be the \( B \)-endpoints of the edges in \( M_r \). We obtain \( M_r \) from \( M_{r-1} \) by finding an augmenting path \( P = (a_r, \ldots, a_\sigma, b_{\phi(r)}) \) from \( a_r \) to \( B \setminus B_{r-1} \) of minimum additional weight. So, in this notation, \( M_r \) matches \( A_r \) with \( \{b_{\phi(i)}, i = 1, 2, \ldots, r\} \).

The matching \( M_{r-1} \) induces a collection \( C_{r-1} \) of vertex disjoint paths and cycles in \( D \). An augmenting path \( P \) with respect to \( M_{r-1} \) changes this collection in the following way. We first add an edge \( \{a_r, b_k\} \) for some index \( k \). If \( b_k \) is isolated in \( M_r \) then \( C_r = C_{r-1} \) plus the edge \((r, k)\). Otherwise, suppose that \((a_\sigma, b_k) \in M_r\). This means that adding the edge \((r, k)\) to \( C_r \) increases the in-degree of \( k \) to two. So, we delete the edge \((s, k)\) and then continue along \( P \) to examine the other edge \((a_\sigma, b_k)\) incident with \( a_\sigma \) in the path. This continues until we reach \( a_\sigma \). We then add the edge \((a_\sigma, b_{\phi(r)})\). In the digraph \( D \) this either means that the added edge closes a cycle or connects two paths into one.

Suppose now that \( a_\sigma \) has \( \delta_r \) neighbors in \( B \setminus B_{r-1} \). Conditional on the history of the algorithm, each of these \( \delta_r \) vertices is equally likely to be \( b_{\phi(r)} \). It follows that the probability the edge \((\sigma, \phi(r))\) closes a cycle of length at most \( \ell_1 \) (to be defined below) is at most \( \ell_1/\delta_r \). (We close a cycle if in \( D \), vertex \( \sigma \) is the tail of a path \( P \) in the digraph induced by the matching and \( \phi(r) \) is within \( \ell_1 \) of \( \sigma \) on \( P \).) It follows that

\[
E(\nu_C) \leq \frac{n}{\ell_1} + E \left( \sum_{r=1}^{n} \frac{\ell_1}{\delta_r} \right).
\]  

Here \( n/\ell_1 \) bounds the number of large cycles induced by \( M_n \) and the sum bounds the expected number of small cycles ever created. We will prove below in Section 4 that

\[
P(\delta_r \leq \delta) \leq \frac{\delta \nu_1}{\theta_r},
\]

where

\[
\nu_1 = 2\log^4 n \quad \text{and} \quad \theta_r = \min \{\alpha n, n - r\}.
\]

It follows that for any choice of \( \gamma_r \) we have

\[
E \left( \frac{1}{\delta_r} \right) \leq \sum_{\delta=1}^{\gamma_r} \frac{P(\delta_r \leq \delta)}{\delta} + \frac{1}{\gamma_r} \nu_1 \leq \gamma_r \nu_1 \frac{1}{\theta_r} + \frac{1}{\gamma_r} = 2 \left( \frac{\nu_1}{\theta_r} \right)^{1/2},
\]
if we take the best choice for $\gamma_r$ of $\left(\theta_r/\nu_1\right)^{1/2}$. From which we deduce that

$$\mathbb{E}(\nu_C) \leq \frac{n}{\ell_1} + 2\ell_1 \left( \sum_{r=1}^{(1-\alpha)n} \nu_1^{1/2} + \sum_{r=(1-\alpha)n}^{n-1} \frac{\nu_1^{1/2}}{(n-r)^{1/2}} \right) \leq \frac{n}{\ell_1} + 8\ell_1 \left( \frac{\nu_1 n}{\alpha} \right)^{1/2}. \quad (6)$$

The optimal value for $\ell_1$ is $\frac{n^{1/4} \nu^{1/4}}{8^{1/2} \nu^{1/4}}$. Plugging this into (6) we get that $\mathbb{E}(\nu_C) = O\left(n^{3/4} \log n\right)$. Lemma 2 follows from the Markov inequality.

4 Proof of Lemma 3

Chernoff Bounds: We use the following inequalities associated with the Binomial random variable $Bin(n,p)$.

$$\mathbb{P}(Bin(n,p) \leq (1 - \varepsilon)np) \leq e^{-\varepsilon^2 np/2}.$$

$$\mathbb{P}(Bin(n,p) \geq (1 + \varepsilon)np) \leq e^{-\varepsilon^2 np/3} \quad \text{for } 0 \leq \varepsilon \leq 1.$$

$$\mathbb{P}(Bin(n,p) \geq \gamma np) \leq \left( \frac{e}{\gamma} \right)^{\gamma np} \quad \text{for } \gamma \geq 1.$$

Proofs of these inequalities are readily accessible, see for example [4].

We use the notation

$$\mu = \frac{n}{\log^3 n}; \quad w_0 = \frac{1}{\mu}; \quad w_1 = w_0 \log n = \frac{\log^4 n}{n}.$$

The aim of this section is to show that w.h.p. no edges of weight more than $w_1$ are used in the construction of $M_n$. For a set $S \subseteq A$ we let

$$N_0(S) = \{b \in B : (a,b) \in E(G) \text{ and } w(a,b) \leq w_0 \text{ for some } a \in S\}.$$

Let

$$\beta = \frac{\alpha}{2} + \frac{1}{4} \text{ and } \gamma = 3(\log \beta^{-1} + 1).$$

and let

$$r_1 = \frac{\mu}{10}; \quad r_2 = \frac{\alpha n}{1000}; \quad r_3 = \beta n; \quad r_4 = n - \frac{n}{\gamma \log^2 n}; \quad r_5 = n - \log^2 n.$$

**Lemma 4.** W.h.p. we have

$$|N_0(S)| \geq \frac{\alpha n |S|}{3\mu} \quad \text{for all } S \subseteq A, 1 \leq |S| \leq r_1. \quad (7)$$

$$|N_0(S)| \geq \frac{n}{40} \quad \text{for all } S \subseteq A, r_1 < |S| \leq r_2. \quad (8)$$

$$|N_0(S)| \geq \beta n + 1 \quad \text{for all } S \subseteq A, r_2 < |S| \leq r_3. \quad (9)$$

$$|N_0(S)| \geq r_4 \quad \text{for all } S \subseteq A, |S| > r_3. \quad (10)$$

$$n - |N_0(S)| \leq \frac{n - |S|}{\log n} \quad \text{for all } S \subseteq A, |S| \geq r_4. \quad (11)$$
Proof. We first observe that for a fixed \( S \subseteq A, 1 \leq |S| \leq \beta n \) we have \( |N_0(S)| \) stochastically dominates \( \text{Bin}(\alpha n, q) \) where \( q = 1 - (1 - w_0)^s \). If \( 1 \leq |S| \leq r_1 \) then \( q \geq s/2\mu \). So,
\[
\mathbb{P}(\neg(7)) \leq \sum_{s=1}^{r_1} \binom{n}{s} \mathbb{P}
\left( \text{Bin}(\alpha n, q) \leq \frac{\alpha ns}{3\mu} \right) \leq \sum_{s=r_1}^{r_2} \left( \binom{ne}{s} \right)^s e^{-sn/20\mu} = o(1).
\]
If \( r_1 < |S| \leq r_2 \) then \( q > 1/20 \). So,
\[
\mathbb{P}(\neg(8)) \leq \sum_{s=r_1}^{r_2} \binom{n}{s} \mathbb{P}
\left( \text{Bin}(\alpha n, 1/20) \leq \frac{\alpha n}{40} \right) \leq \left( \frac{\alpha ne}{r_2} \right)^{r_2} e^{-\alpha n/80} = o(1).
\]
If \( r_2 < |S| \leq r_3 \) then \( q \geq 1 - e^{-\alpha n/2000\mu} \). So,
\[
\mathbb{P}(\neg(9)) \leq \mathbb{P}(\text{Bin}(\alpha n, q) \leq \beta n) \leq \left( \frac{\beta n}{\alpha n} \right) (1 - q)^{\beta n} \leq 2^n e^{-\alpha (\alpha - \beta) n^2/2000\mu} = o(1).
\]
For (10) let \( T, |T| = t = \frac{n}{\gamma \log^2 n} \) denote a set of vertices with no neighbours in \( S \). Each member of \( B \) has at least \( 3\varepsilon n/2 \) \( G \)-neighbors in \( S \). Thus,
\[
\mathbb{P}(\neg(10)) \leq \sum_{s=r_3}^{n} \binom{n}{s} \frac{n}{t} \left( 1 - w_0 \right)^{3\varepsilon n/2} \leq 2^{n-o(n)} e^{-\Omega(n \log n)} = o(1).
\]
For (11) let \( T \) play the same role but with \( t = |T| = \frac{n-s}{\log^2 n} \). Then,
\[
\mathbb{P}(\neg(11)) \leq \sum_{s=r_4}^{r_5} \binom{n}{s} \frac{n}{t} \left( 1 - w_0 \right)^{3\varepsilon n/2} \leq \sum_{s=r_4}^{r_5} \left( \frac{ne}{n - s} \cdot \left( \frac{n \log n}{n - s} \right)^{1/\log n} \right) \exp \left\{ -\frac{3\varepsilon \log^2 n}{2} \right\} = o(1).
\]

\[\square\]

Lemma 5. W.h.p., no edge of length at least \( w_1 \) appears in any \( M_r, r \leq n \)

Proof. We first consider \( 1 \leq r \leq r_1 \). Choose \( a \in A_r \) and let \( S_0 = \{ a \} \) and let an alternating path \( P = (a = u_1, v_1, \ldots, u_{k-1}, v_k, \ldots) \) be acceptable if (i) \( u_1, \ldots, u_k, \ldots \in A, v_1, \ldots, v_{k-1}, \ldots \in B \), (ii) \( (u_{i-1}, v_i) \in M_r, i = 1, 2, \ldots \) and (iii) \( w(u_i, v_i) \leq w_0, i = 1, 2, \ldots \). Now consider the sequence of sets \( S_0 = \{ a_0 \}, S_1, S_2, \ldots, S_t, \ldots \) defined as follows:

Case (a): \( N_0(S_i) \subseteq \phi(A_r) \). In this case we define \( S_{i+1} = \phi^{-1}_r(T_i) \), where \( T_i = N_0(S_i) \). By construction then, every vertex in \( S_j, j \leq i + 1 \) is the endpoint of some acceptable alternating path.

Case (b): \( T_i \setminus \phi(A_r) \neq \emptyset \). In this case there exists \( b \in T_i \) which is the endpoint of some acceptable augmenting path.

It follows from (7) applied to \( S \) that w.h.p. there exists \( k = o(\log n) \) such that \( |N_0(S_k)| > r \) and so Case (b) holds. This implies that if \( 1 \leq r \leq r_1 \) then \( w(a, \phi_r(a)) \leq kw_0 \) for all \( a \in A_r \). For if \( w(a, \phi_r(a)) > kw_0 \) then there are at least \( \Omega(rn/\mu) \) choices of \( b \in B \setminus \phi(A_r) \) such that we can reduce the matching cost by deleting \( (a, \phi_r(a)) \) and changing \( M_r \) via an acceptable augmenting path from \( a \) to \( b \). The extra cost of the edges added in this path is \( o(w_0 \log n) \).

Now consider \( r_1 < r \leq r_2 \). We know that w.h.p. there is \( k = o(\log n) \) such that \( |S_k| > r_1 \) and that by (8) we have that w.h.p. \( |N_0(S_{k+1})| > n/40 > r \) and we are in Case (b) and there is a low cost augmenting path.
for every $a$, as in the previous case. When $r_2 < |S_k| \leq r_3$ we use the same argument and find by (9) we have w.h.p. $N_0(S_{k+1}) > r_3 \geq r$ and there is a low cost augmenting path. Similarly for $r_3 < r \leq r_4$, using (10) and for $r_4 < r \leq r_5$ using (11), $o(\log n)$ times. When $r > r_5$ we can use the fact that w.h.p. for every vertex $b \in B$ has at least $(1 - o(1)) \log^3 n$ vertices $a \in A$ such that $w(a, b) \leq w_0$. Finally note that the number of edges in the augmenting paths we find is always $o(\log n)$. \hfill \Box

Now,
\[ \mathbb{P}(\exists a \in A : |\{e : v \in e, X_e \leq w_1\}| \geq \nu_1) \leq \mathbb{P}\left(\text{Bin}\left(n, \frac{\log^4 n}{n}\right) \geq \nu_1\right) = O(n^{-2}). \] (12)

Let $\zeta_a$ denote the number of times that vertex $a$ takes the role of $a_\sigma$. It follows from (12) that w.h.p.
\[ \zeta_a \leq \nu_1, \text{ for all } a \in A. \] (13)

We now use (13) to prove (5). Consider now how a vertex $a \in A$ loses neighbors in $B \setminus B_r$. It can lose up to $\nu_1$ for the times when $a = a_\sigma$. Otherwise, it loses a neighbor when $a_\sigma \neq a$ chooses a common neighbor with $a$. The important point here is that this choice depends on the structure of $G$, but not on the weights of edges incident with $a$. It follows that the cheapest neighbors at any time are randomly distributed among the current set of available neighbors. To get to the point where $a_\sigma = a$ and $\delta_r \leq \delta$, we must have at least one of the $\nu_1$ original cheapest neighbors occurring in a random $\delta$ subset of a set of size at least $\theta_r = \min\{\alpha n, n - r\}$. The probability of this is $\nu_1\delta/\theta_r$.

5 Final Remarks

We have extended the proof of the validity of Karp’s patching algorithm to dense graphs with minimum in- and out-degree at least $\alpha n$, $\alpha > 1/2$ and uniform $[0, 1]$ edge weights. It is a routine exercise to extend the analysis to costs with a distribution function $F(x)$ that satisfies $F(x)/x \searrow 1$ as $x \to 0$. Janson [6] describes a nice simple coupling in the case of shortest paths. Our argument fails if $\alpha \leq 1/2$. In this case the assignment problem may not be feasible. One can consider adding randomly weighted random edges as in Frieze [3], but there seems to be a technical difficulty at present in extending the results in this direction.

References

