Karp's patching algorithm on dense digraphs

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Abstract

We consider the following question. We are given a dense digraph D with n vertices and minimum in- and out-degree at least αn, where $\alpha > 1/2$ is a constant. The edges $E(D)$ of D are given independent edge costs $C(e), e \in E(D)$, such that (i) C has a density f that satisfies $f(x) = a + bx + O(x^2)$, for constants $a > 0, b$ as $x \to 0$ and such that in general either (ii) $\mathbb{P}(C \geq x) \leq \alpha e^{-\beta x}$ for constants $\alpha, \beta > 0$, or $f(x) = 0$ for $x > \nu$ for some constant $\nu > 0$. Let $C(i,j), i,j \in [n]$ be the associated $n \times n$ cost matrix where $C(i,j) = \infty$ if $(i,j) \notin E$. We show that w.h.p. (a small modification to) the patching algorithm of Karp finds a tour for the asymmetric traveling salesperson problem that is asymptotically equal to that of the associated assignment problem. The algorithm runs in polynomial time.

1 Introduction

Let $\mathcal{D}(\alpha,n)$ be the set of digraphs with vertex set $[n] = \{1,2,\ldots,n\}$ and with minimum in- and out-degree at least αn. The edges $E(D)$ of D are given independent edge costs $C(e), e \in E(D)$, such that (i) C has a density $f(x)$ where $f(x) = a + bx + O(x^2)$ as $x \to 0$ and such that in general either (ii) $\mathbb{P}(C \geq x) \leq \alpha e^{-\beta x}$ for constants $\alpha, \beta > 0$, or $f(x) = 0$ for $x > \nu$ for some constant $\nu > 0$. We say that such distributions are *acceptable*. The prime examples will be the uniform $[0,1]$ distribution ($a = 1, b = 0$) and the exponential mean 1 distribution $EXP(1)$ ($a = 1, b = -1, \alpha = \beta = 1$). Let $C(i,j), i,j \in [n]$ be the associated $n \times n$ cost matrix where $C(i,j) = \infty$ if $(i,j) \notin E(D)$. Here we are interested in using the relationship between the Assignment Problem (AP) and the Asymmetric Traveling Salesperson Problem (ATSP) associated with the cost matrix $C(i,j), i,j \in [n]$ to find a tour whose cost $\widehat{v}(ATSP)$ satisfies $v(AP) \leq v(ATSP) \leq \widehat{v}(ATSP) \leq (1+o(1))v(AP)$ w.h.p., where $v(\bullet)$ denotes the optimal cost. We say that the output of the algorithm is *asymptotically optimal* i.e. it produces a tour whose cost is at most $(1+o(1))$ times optimal

The problem AP is that of computing the minimum cost perfect matching in the complete bipartite graph $K_{n,n}$ when edge (i,j) is given a cost $C(i,j)$. Equivalently, when translated to the complete digraph $\vec{K}_n$ it becomes the problem of finding the minimum cost collection of vertex disjoint directed cycles that cover all vertices. The problem ATSP is that of finding a single cycle of minimum cost that covers all vertices. As such it is always the case that $v(ATSP) \geq v(AP)$. Karp [9] considered the case where $D = \vec{K}_n$. He showed that if the cost matrix comprised independent copies of the uniform $[0,1]$ random variable U then w.h.p. $v(ATSP) = (1+o(1))v(AP)$. He proves this by the analysis of a *patching* algorithm. Karp's result has been refined in [4], [7] and [10].

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Karp's Patching Algorithm: First solve AP to obtain a minimum cost perfect matching M and let $\mathcal{A}_M = \{C_1, C_2, \dots, C_\ell\}$ be the associated collection of vertex disjoint cycles covering $[n]$. Then *patch* two of the cycles together, as explained in the next paragraph. Repeat until there is one cycle.

A pair $e = (x, y), f = (u, v)$ of edges in different cycles C_1, C_2 are said to be a *patching pair* if the edges $e' = (u, y), f' = (x, v)$ both exist. In which case we can replace C_1, C_2 by a single cycle $(C_1 \cup C_2 \cup \{e', f'\}) \setminus \{e, f\}$. The edges e, f are chosen to minimise the increase in cost of the set of cycles.

Theorem 1. *Suppose that $D \in \mathcal{D}(\alpha), n$, $\alpha = 1/2 + \varepsilon$ where ε is a positive constant. Suppose that each edge e of D is given an independent cost drawn from an acceptable distribution Then w.h.p. $v(ATSP) = (1 + o(1))v(AP)$ and (a small modification to) Karp's patching algorithm finds a tour of the claimed cost in polynomial time.*

In an earlier paper Frieze and Michaeli [6] proved a similar result where $D = D_0 + R$ and $D_0 \in \mathcal{D}(\alpha, n), \alpha > 0$ and R is a set of $o(n^2)$ random edges. This being an example of the *perturbed model* introduced in Bohman, Frieze and Martin [2]. Our proof strategy follows that of [6] in some places and is significantly different in other places.

For the moment assume that C is distributed as $EXP(1)$ i.e $\mathbb{P}(X \geq x) = e^{-x}$. We will discuss more general distributions in Section 5.

Notation Let G denote the bipartite graph with vertex partition $A = \{a_1, a_2, \dots, a_n\}$, $B = \{b_1, b_2, \dots, b_n\}$ and an edge $\{a_i, b_j\}$ for every directed edge $(i, j) \in E(D)$. A matching M of G induces a collection $\mathcal{A}_M$ of vertex disjoint paths and cycles in D and vice-versa. If the matching is perfect, then there are only cycles.

2 Proof of Theorem 1

Notation We begin by solving AP. We prove the following:

Lemma 2. *W.h.p., the solution to AP contains only edges of cost $C(i, j) \leq \gamma_n = \frac{\log^4 n}{n}$.*

Lemma 3. *W.h.p., after solving AP, the number ν_C of cycles is at most $\nu_0 = n^{5/6}$.*

Bounding the number of cycles has been the most difficult task. Karp proves $O(\log n)$ w.h.p. for the complete digraph $\vec{K}_n$ and we conjecture this to be true here. Karp's proof relies on the key insight that if $D = \vec{K}_n$ then the optimal assignment comes from a uniform random permutation. This is not true in general.

Let L denote the set of edges of cost greater than γ_n. Given Lemmas 2, 3, for the purposes of proof, we temporarily replace costs $C(e), e \in L$ by infinite costs in order to solve AP. Lemma 2 implies that w.h.p. we get the same optimal assignment as we would without the cost changes. Having solved AP, the memoryless property of the exponential distribution, implies that the unused edges in L really have a cost which is distributed as $\gamma_n + EXP(1)$.

Let $\mathcal{C} = C_1, C_2, \dots, C_\rho$ be a cycle cover and let $c_i = |C_i|$ where $c_1 \leq c_2 \leq \dots \leq c_\rho$, $\rho \leq \nu_0$. Different edges in C_i give rise to disjoint patching pairs. We ignore the saving associated with deleting e, f and only look at the extra cost $C(e') + C(f')$ incurred.

Case 1: $\rho > \nu_1 = 3\varepsilon^{-1}$.
We see that if $c_\sigma < \varepsilon n/2 \leq c_{\sigma+1}$ then the number of possible patching pairs $\phi(\mathcal{C})$ satisfies

$$\phi(\mathcal{C}) \geq \sum_{i=1}^{\sigma} 2\,(\varepsilon n - c_i)\, c_i \geq \varepsilon n \sum_{i=1}^{\sigma} c_i \geq \varepsilon n \sigma \geq \varepsilon n(\rho - 2\varepsilon^{-1}). \tag{1}$$

Explanation: Having chosen $e = (x, y)$ in a cycle C_i we let U denote the in-neighbors of y in D that are not in C_i. Then let U' denote the predecessors of U in the cycles $\mathcal{C}$. Let V denote the out-neighbors of x. We have

$|U'|,|V| \geq \alpha n - c_i$ and so $|U' \cap V| \geq 2(\varepsilon n - c_i)$. Given $v \in U' \cap V$ we let $u \in U$ be its successor in the cycles. This gives us a pair $e' = (y,u)$, $f' = (x,v)$ that can be used for patching.

Let $\beta_\rho = \left(\frac{12 \log n}{\varepsilon \rho n}\right)^{1/2}$. Suppose we consider a sequence of patches where we always implement the cheapest patch. Let $\mathcal{E}_\rho$ denote the event that the cost of the ρth patch is at least $2(\gamma_n + \beta_\rho)$. Each edge of $\vec{K}_n$ will independently have cost exceeding $\gamma_n + \beta_\rho$ with probability of at most $e^{-\beta_\rho} \leq 1 - \beta_\rho/2$. We have

$$\mathbb{P}(\mathcal{E}_\rho) \leq (1-(1-e^{-\beta_\rho})^2)^{\varepsilon n(\rho - 2\varepsilon^{-1})} \leq \exp\left\{-\frac{\varepsilon n(\rho - 2\varepsilon^{-1})\beta_\rho^2}{4}\right\} \leq \frac{1}{n}.$$

It follows that

$$\begin{aligned}
\mathbb{P}(\exists \nu_1 \leq \rho \leq \nu_0 : \mathcal{E}_\rho) &\leq \sum_{\rho=\nu_1}^{\nu_0} \mathbb{P}(\mathcal{E}_\rho \mid \neg\mathcal{E}_{\rho+1} \wedge \cdots \wedge \neg\mathcal{E}_{\nu_0}) \\
&\leq \sum_{\rho=\nu_1}^{\nu_0} \frac{\mathbb{P}(\mathcal{E}_\rho)}{1 - \sum_{k=\rho+1}^{r_1} \mathbb{P}(\mathcal{E}_k)} \\
&\leq \sum_{\rho=\nu_1}^{\nu_0} \frac{n^{-1}}{1 - \nu_0 n^{-1}} = o(1).
\end{aligned}$$

W.h.p. the patches involved in these cases add at most the following to the cost of the assignment:

$$2\sum_{\rho=\nu_1}^{\nu_0}\left(\gamma_n + \left(\frac{12 \log n}{\varepsilon \rho n}\right)^{1/2}\right) \leq 2\nu_0\gamma_n + \left(\frac{96\nu_0 \log n}{\varepsilon n}\right)^{1/2} = o(1). \tag{2}$$

Case 2: $2 \leq \rho \leq 3\varepsilon^{-1}$ and $|C_1| \leq \varepsilon n/2$.
We can see that $\phi(\mathcal{C}) \geq \varepsilon n$ follows from (1) where we can just use the term $i = 1$. Let $\mathcal{E}_{\rho,2}$ denote event $\mathcal{E}_\rho$ when Case 2 holds.

$$\mathbb{P}(\mathcal{E}_{\rho,2}) \leq (1-(1-e^{-\beta_\rho})^2)^{\varepsilon n} \leq \exp\left\{-\frac{\varepsilon n \beta_\rho^2}{4}\right\} \leq \frac{1}{n}. \tag{3}$$

and we can proceed as for $\rho > 3\varepsilon^{-1}$.

Case 3: $2 \leq \rho \leq 3/\varepsilon$ and $|C_1| > \varepsilon n/2$.
It is here that we need to deviate from Karp's strategy. Let $p = 1/n^{1/10}$ and let R be the set of edges of D of cost at most $\gamma_n + p$. Each edge of cost more than γ_n has cost at most $\gamma_n + p$ with probability at least $p/2$. We will create a tour using the edges $\mathcal{C}$ and a bounded number of edges in R.

We begin by deleting two edges from each cycle C_i so that each of the two paths P_{2i-1}, P_{2i} created are within one of each other in size. Suppose now that path $P_i, i = 1,2,\ldots,2\rho$ is directed from x_i to y_i. We add edges $f_i = (y_i, x_{i+1})$ creating a tour T. Of course not all of the edges $F = \{f_1, f_2, \ldots, f_\rho\}$ will occur in D or even if they are, they may not be of low cost. In this case, an edge (i,j) is of low cost if it belongs to R.

Suppose now that f_1 is not a low cost edge of D. Let u be a low cost out-neighbor of y_1 and let v be the predecessor of u on the tour T. The Chernoff bounds imply that y_1 has $\Omega(n^{9/10})$ low cost out-neighbors. Now there are at least $2\varepsilon n$ pairs of vertex w,z such that (i) w is an out-neighbor of v, (ii) z is an in-neighbor of x_2 and z is the immediate predecessor of w on T and w.h.p. at least $\varepsilon n p^2 = \Omega(n^{4/5})$ pairs are such that both of the vertices in the pair are low cost neighbors. We can examine these edges in some order and w.h.p. we only have to examine at most $n^{1/10}\log n$ edges before we find a low cost edge. Once we have one, we replace T by $T' = T + (y_1,u) + (v,w) + (z,x_2) - (y_1,x_2) - (v,u) - (z,w)$, see Figure 1. This gives a tour $(y_1,u) + T(u \to z) + (z,x_2) + T(x_2 \to v) + (v,w) + T(w \to y_1)$. Having removed f_1, we apply the same procedure and remove f_2 and so on. We need to avoid looking at the same edge twice, but this is not a problem as we only have to remove $O(1)$ edges and thus at any time we have only looked at $O(n^{1/10}\log n)$ edges and w.h.p. we always have $\Omega(n^{4/5})$ choices.

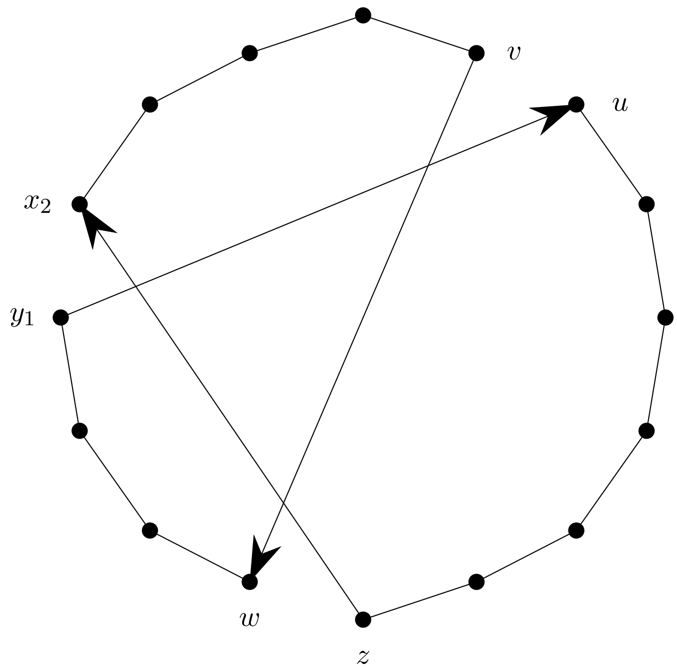

Figure 1: Removing (y_1, x_2)

So w.h.p. this case adds only $O(n^{-1/10})$ to the cost of the constructed tour. This completes the proof of Theorem 1, modulo proving Lemmas 2 and 3.

3 Proof of Lemma 2

We show that w.h.p. for any pair of vertices $a \in A, b \in B$ and any perfect matching between A and B that there is an M-alternating path from a to b that only uses non-M edges of cost at most $\frac{\log^3 n}{n}$. Furthermore this path uses $o(\log n)$ edges. This implies that alternately adding non-M edges and deleting M-edges will find a matching of A into B that doesn't use (a, b) and for which the cost of the added non-M edges is $o\left(\frac{\log^4 n}{n}\right)$. This implies that M^* does not contain edges of cost exceeding γ_n.

The idea of the proof is based on the fact that w.h.p. the sub-digraph induced by edges of low cost is a good expander. There is therefore a low cost path between every pair of vertices. Such a path can be used to replace an expensive edge.

Chernoff Bounds: We use the following inequalities associated with the Binomial random variable $Bin(N, p)$.

$$\mathbb{P}(Bin(N,p) \leq (1-\varepsilon)Np) \leq e^{-\varepsilon^2 Np/2}.$$
$$\mathbb{P}(Bin(N,p) \geq (1+\varepsilon)Np) \leq e^{-\varepsilon^2 Np/3} \qquad \text{for } 0 \leq \varepsilon \leq 1.$$
$$\mathbb{P}(Bin(N,p) \geq \gamma Np) \leq \left(\frac{e}{\gamma}\right)^{\gamma Np} \qquad \text{for } \gamma \geq 1.$$

Proofs of these inequalities are readily accessible, see for example [5]. We have the same bounds for the Hypergeometric distribution with mean Np. This follows from Theorem 4 of Hoeffding [8].

We will also use McDiamid's inequality, which can also be forund in [5]. Let $Z = Z(Y_1, Y_2, \ldots, Y_N)$ be a random variable that depends on independent random variables $Y_1, Y_2, \ldots, T_N$. Suppose that for every i and $\widehat{Y}_i \neq Y_i$,

$|Z(Y_1, Y_2, \ldots, Y_1, \ldots, Y_N)| - Z(Y_1, Y_2 \ldots, \widehat{Y}_1, \ldots, Y_N)| \leq c_i$. Then, for every $t \geq 0$,

$$\mathbb{P}(|\mathbb{E}(Z)| \geq t) \leq 2\exp\left\{-\frac{2t^2}{\sum_{i=1}^N c_i^2}\right\}. \tag{4}$$

Let $M^* = \{(a_i, \phi(a_i)) : i = 1, 2, \ldots, n\}$ denote the optimal solution to AP. Then let

$$\zeta = \alpha - \frac{\varepsilon}{2}; \qquad \delta = 3(\log\zeta^{-1} + 1); \qquad \mu = \frac{n}{\log^3 n}; \qquad \beta = \frac{1}{\mu}.$$

$$n_1 = \frac{\mu}{10}; \qquad n_2 = \frac{\alpha n}{1000}; \qquad n_3 = \zeta n; \qquad n_4 = n - \frac{n}{\delta\log^2 n}; \qquad n_5 = n - \log^2 n.$$

We let $E_\beta = \{(a_i, b_j) \in E(D) : \ C(i,j) \leq \beta\}$. Then for a set $S \subseteq A$ let

$$N_0(S) = \{b_j \in B : \exists a_i \in S \text{ such that } (a_i, b_j) \in E_\beta\}.$$

Lemma 4. *We have with probability* $1 - e^{-\Omega(\log^3 n)}$,

$$|N_0(S)| \geq \frac{\alpha n|S|}{3\mu} \qquad \textit{for all } S \subseteq A, 1 \leq |S| \leq n_1. \tag{5}$$

$$|N_0(S)| \geq \frac{\alpha n}{40} \qquad \textit{for all } S \subseteq A, n_1 < |S| \leq n_2. \tag{6}$$

$$|N_0(S)| \geq \zeta n + 1 \qquad \textit{for all } S \subseteq A, n_2 < |S| \leq n_3. \tag{7}$$

$$|N_0(S)| \geq n_4 \qquad \textit{for all } S \subseteq A, |S| > n_3. \tag{8}$$

$$n - |N_0(S)| \leq \frac{n - |S|}{\log n} \qquad \textit{for all } S \subseteq A, n_4 \leq |S| \leq n_5. \tag{9}$$

Proof. We first observe that for a fixed $S \subseteq A$, $s = |S| \geq 1$ we have that $|N_0(S)|$ dominates $Bin(\alpha n, q)$ in distribution, where $q = 1 - e^{-\beta s}$.

If $1 \leq s \leq n_1$ then $q \geq s/2\mu$. So,

$$\mathbb{P}(\neg(5)) \leq \sum_{s=1}^{n_1} \binom{n}{s} \mathbb{P}\left(Bin\left(\alpha n, \frac{s}{2\mu}\right) \leq \frac{\alpha n s}{3\mu}\right) \leq \sum_{s=1}^{n_1} \left(\frac{ne}{s}\right)^s e^{-\alpha s n/20\mu} = e^{-\Omega(\log^3 n)}.$$

If $n_1 < s \leq n_2$ then $q > 1/20$. So,

$$\mathbb{P}(\neg(6)) \leq \sum_{s=n_1}^{n_2} \binom{n}{s} \mathbb{P}\left(Bin\left(\alpha n, \frac{1}{20}\right) \leq \frac{\alpha n}{40}\right) \leq n\left(\frac{\alpha n e}{n_2}\right)^{n_2} e^{-\alpha n/80} = e^{-\Omega(n)}.$$

If $n_2 < s \leq n_3$ then $q \geq 1 - e^{-\alpha n/1000\mu}$. So,

$$\mathbb{P}(\neg(7)) \leq \sum_{s=n_2}^{n_3} \binom{n}{s} \mathbb{P}(Bin(\alpha n, q) \leq \zeta n) \leq 2^n \binom{\alpha n}{\zeta n} (1-q)^{(\alpha-\zeta)n} \leq 2^{2n} e^{-\alpha(\alpha-\zeta)n^2/1000\mu} = e^{-\Omega(n\log^3 n)}.$$

For (8) let $T, |T| = t = \frac{n}{\delta\log^2 n}$ denote a set of vertices with no N_0-neighbours in S. Each member of T has at least $3\varepsilon n/2$ G-neighbors in S. Thus,

$$\mathbb{P}(\neg(8)) \leq \sum_{s=n_3}^{n} \binom{n}{s}\binom{n}{t} e^{-3\beta t\varepsilon n/2} \leq 2^{n+o(n)} e^{-\Omega(n\log n)} = e^{-\Omega(n\log n)}.$$

For (9) let T play the same role as in (8), but with $t = |T| = \frac{n-s}{\log n}$. Each member of T has at least $n/2$ G-neighbors in S. Thus,

$$\mathbb{P}(\neg(9)) \leq \sum_{s=n_4}^{n_5} \binom{n}{s}\binom{n}{t} e^{-\beta t n/2} \leq \sum_{s=n_4}^{n_5} \left(\frac{ne}{n-s} \cdot \left(\frac{ne\log n}{n-s}\right)^{1/\log n} \cdot \exp\left\{-\frac{\log^2 n}{2}\right\}\right)^{n-s} = e^{-\Omega(\log^4 n)}.$$

$\square$

We can now proceed to the proof of Lemma 2.

We let $\vec{G}$ denote a digraph with vertex set $A \cup B$. The edges of $\vec{G}$ consist of $\vec{M} = \{(b_{\phi(a_i)}, a_i) : i \in [n]\}$ directed from B to A and edges $\vec{D} = \{(a_i, b_j) : (i,j) \in E, j \neq \phi(i)\}$ directed from A to B. The edge $(b_{\phi(a_i)}, a_i)$ of $\vec{M}$ is given cost $-C(i, \phi(i))$ for $i \in [n]$. The edge (a_i, b_j) of $\vec{D}$ is given cost $C(i,j)$. Paths in $\vec{G}$ are *alternating* in the sense that ignoring orientation, they alternate between being in M and not being in M.

Lemma 5. *Then q.s.*[1] *for all $a \in A, b \in B$, $\vec{G}$ contains an alternating path from a to b of total cost less than γ_n.*

Proof. Let an alternating path $P = (a = x_1, y_1, \dots, y_{k-1}, x_k, y_k = b)$ be *acceptable* if (i) $x_1, \dots, x_k \in A$, $y_1, \dots, y_k \in B$, (ii) $(x_{i+1}, y_i) \in M^*, i = 1, 2, \dots, k$ and (iii) $C(x_i, y_i) \leq \beta$, $i = 1, 2, \dots, k$. The existence of such a path with $k = o(\log n)$ implies the lemma.

Now consider the sequence of sets $S_0 = \{a_1\}, S_1, S_2, \dots, S_k \subseteq A, T_1, T_2, \dots, T_k \subseteq B$ defined as follows:

$T_i = N_0\left(\bigcup_{j<i} S_j\right)$ and $S_i = \phi^{-1}(T_i)$. Let $i_0 = \min\left\{i : (\alpha \log^3 n/3)^i \leq n_1\right\}$. It follows from (5) – (8) that q.s.

$$|T_i| > \left(\frac{\alpha \log^3 n}{3}\right)^i \text{ for } 1 \leq i \leq i_0, \quad |T_{i_0+1}| \geq \frac{\alpha n}{40}, \quad |T_{i_0+2}| \geq \zeta n + 1, \quad |T_{i_0+3}| \geq n - \frac{n}{\delta \log^2 n}.$$

If $b \in T_{i_0+3}$ then we have found an acceptable alternating path. If $b \notin T_{i_0+3}$ then using (9), $o(\log n)$ times we arrive at $k = o(\log n)$ such that $|T_{k-1}| \geq n_5 = n - \log^2 n$. If $b \notin T_{k-1}$ then we can use the fact that with probability $1 - e^{-\Omega(\log^3 n)}$, every vertex $b_j \in B$ has at least $(1 - o(1)) \log^3 n$ vertices $a_i \in A$ such that $C(i,j) \leq \beta$ to show that $b \in T_k$. This completes the proof of Lemma 5. □

We can now prove Lemma 2 as part of the following lemma.

Lemma 6. *The solution to AP contains only edges of cost $C(i,j) \leq \gamma_n$ q.s.*

Proof. Suppose that the solution M^* to AP contains an edge e of cost greater than $\gamma_n = \beta \log n$. Assume w.l.o.g. that $e = (a_1, b_1)$. It follows from Lemma 5 that there is an alternating path $P = (a_1, \dots, b_1)$ of cost at most γ_n. But then deleting e and the M^*-edges of P and adding the non-M^* edges of P to M^* creates a matching from A to B of lower cost than M^*, contradiction. □

4 Proof of Lemma 3

4.1 Linear programming formulation of AP

We consider the linear program $\mathcal{LP}$ for finding M^*. To be precise we let $\mathcal{LP}$ be the linear program

$$\text{Minimise} \sum_{i,j \in [n]} C(i,j) x_{i,j} \text{ subject to } \sum_{j \in [n]} x_{i,j} = 1,\ i \in [n],\ \sum_{i \in [n]} x_{i,j} = 1,\ j \in [n],\ x_{i,j} \geq 0.$$

The linear program $\mathcal{D}$ dual to $\mathcal{LP}$ is given by:

$$\text{Maximise} \sum_{i=1}^{n} u_i + \sum_{j=1}^{r} v_j \text{ subject to } u_i + v_j \leq C(i,j), i, j \in [n].$$

[1]A sequence of events $\mathcal{A}$ holds *quite surely* (q.s.) if $\mathbb{P}(\neg \mathcal{A}) = o(r^{-K})$ as $r \to \infty$ for any constant $K > 0$

4.2 Trees and bases

An optimal basis of $\mathcal{LP}$ can be represented by a spanning tree T^* of G that contains the perfect matching M^*, see for example Ahuja, Magnanti and Orlin [1], Chapter 11. We have that for every optimal basis T^*,

$$C(i,j) = u_i + v_j \text{ for } (a_i, b_j) \in E(T^*) \tag{10}$$

and

$$C(i,j) \geq u_i + v_j \text{ for } (a_i, b_j) \notin E(T^*). \tag{11}$$

Note that if λ is arbitrary then replacing u_i by $\widehat{u}_i = u_i - \lambda, i = 1, 2, \ldots, n$ and v_i by $\widehat{v}_i = v_i + \lambda, i = 1, 2, \ldots, n$ has no afffect on whether or not these constraints are satisfied. We say that $\mathbf{u}, \mathbf{v}$ and $\widehat{\mathbf{u}}, \widehat{\mathbf{v}}$ are equivalent. So, we can for example, assume when convenient, that $u_1 = 0$.

The next goal is to show that w.h.p. we can choose optimal dual variables of absolute value at most $2\gamma_n$. Let $\mathcal{E}$ be the event that given $u_s = 0$, we have $|u_i| \leq \gamma_n, |v_j| \leq 2\gamma_n$ for all i, j.

Lemma 7. *$\mathcal{E}$ occurs q.s.*

Proof. Let $\mathcal{E}_0$ be the event implied by Lemma 5. Fix a_i, b_j and let $P = (a_{i_1}, b_{j_1}, \ldots, a_{i_k}, b_{j_k})$ be the alternating path from a_i to b_j promised by $\mathcal{E}_0$. Then, using (10) and (11), we have

$$\gamma_n \geq C(P) = \sum_{l=1}^{k} C(i_l, j_l) - \sum_{l=1}^{k-1} C(i_{l+1}, j_l) \geq \sum_{l=1}^{k} (u_{i_l} + v_{j_l}) - \sum_{l=1}^{k-1} (u_{i_{l+1}}, v_{i_l}) = u_i + v_j. \tag{12}$$

Fix $u_s = 0$ for some s. For each $i \in [n]$ there is some $j \in [n]$ such that $u_i + v_j = C(i,j)$. This is because of the fact that a_i meets at least one edge of T and we assume that (10) holds. We also know that because (12) occurs that $u_{i'} + v_j \leq \gamma_n$ for all $i' \neq i$. It follows that $u_i - u_{i'} \geq C(i,j) - \gamma_n \geq -\gamma_n$ for all $i' \neq i$. Since i is arbitrary, we deduce that $|u_i - u_{i'}| \leq \gamma_n$ for all $i, i' \in [n]$. Since $u_s = 0$, this implies that $|u_i| \leq \gamma_n$ for $i \in [n]$. We deduce by a similar argument that $|v_j - v_{j'}| \leq \gamma_n$ for all $j, j' \in [n]$. Now because for the optimal matching edges $(i, \phi(i)), i \in [n]$ we have $u_i + v_{\phi(i)} = C(i, \phi(i))$, we see that $|v_j| \leq 2\gamma_n$ for $j \in [n]$. □

Lemma 6 bounds the cost of the edges in M^*. The next lemma proves the same bound for the costs of the other edges in T^*.

Lemma 8. *$C(e) \leq \gamma_n$ q.s. for all $e \in T^*$.*

Proof. We condition on the values $\mathbf{u}$,$\mathbf{v}$. Suppose there is an edge $e = (a_i, b_j) \in T^* \setminus M^*$ such that $C(i,j) > \gamma_n$. Suppose we replace $C(e)$ by γ_n and resolve AP. Lemma 6 implies that the matching will be unchanged. The optimal tree might change to $\widehat{T}$. Now the values $\mathbf{u}$,$\mathbf{v}$ are obtained from u_1 by taking the paths in $T^*, \widehat{T}$ and alternately adding and subtracting the edges in paths. Because the costs (and maybe the tree) have changed there will be a non-trivial sum of positive and negative costs that sum to zero or a one that sums to γ_n. Both of these possibilities have probability 0 and so $C(i,j) \leq \gamma_n$ with conditional probability 1. □

Fix $\mathbf{u}, \mathbf{v}$ and let $G^+ = G^+(\mathbf{u}, \mathbf{v})$ be the subgraph of G induced by the edges (a_i, b_j) for which $u_i + v_j \geq 0$. Let $f(\mathbf{u}, \mathbf{v})$ be the joint density of $\mathbf{u}$,$\mathbf{v}$.

Lemma 9. *Given $\mathbf{u}$,$\mathbf{v}$, M^* is a uniform random perfect matching of G^+.*

Proof. M is an optimal matching iff

$$u_i + v_j = C(i,j), \ \forall (i,j) \in M. \tag{13}$$

$$u_i + v_j \leq C(i,j),\ \forall (i,j) \notin M. \tag{14}$$

Equation (13) is complimentary slackness and equation (14) is dual feasibility. Also,

$$\mathbb{P}((13),(14) \mid \mathbf{u},\mathbf{v}) f(\mathbf{u},\mathbf{v}) = \prod_{i,j} e^{-(u_1+v_j)^+}, \tag{15}$$

which is independent of M. Now for costs C,

$$\mathbb{P}(\exists M_1 \neq M_2 \text{ satisfying } (13)) = 0.$$

This is because otherwise there will be a cycle where the total cost of the even edges equals the total cost of the odd edges. □

We need the following simple graph theoretic lemma:

Lemma 10. *Let v be a vertex of degree d in a graph G. Let T be a spanning tree of G maximum degree Δ and let $\rho \ll d$ be a postive integer. Then T contains at least $\lfloor d/\Delta^{\rho+1} \rfloor$ edge disjoint paths $P = (v_1, \dots, v_\rho)$ such that (i) $\{v, v_\rho\} \in E(G)$ and (ii) P constitutes the last ρ edges in the path from v to v_ρ. We refer to these paths as* useful *paths.*

Proof. We prove this by induction on d, with $d = \Delta^{\rho+1}$ as the base case. If $d > \Delta^{\rho+1}$ then we iteratively remove leaves of T that are not adjacent to v in G. We then choose a leaf w at maximal distance from v. Let the path from v to w be $v = x_0, x_1, \dots, x_k = w$. Deleting the edges of the tree rooted at $x_{k-\rho-1}$ after removing the edge $\{x_{k-\rho-1}, x_{k-\rho}\}$ yields a tree with at least $d - \Delta^{\rho+1}$ vertices and at least $d/\Delta^{\rho+1} - 1$ paths of length ρ. □

We need to know that w.h.p. the minimum in- and out-degree in G^+ is high. We fix a tree T and condition on $T^* = T$. For $i = 1, 2, \dots, n$ let $L_{i,+} = \{j : (i,j) \in E(G^+)\}$ and let $L_{j,-} = \{i : (i,j) \in E(G^+)\}$. Then for $i = 1, 2, \dots, n$ let $\mathcal{A}_{i,+}$ be the event that $|L_{i,+}| \leq n/\log^{25} n$ and let $\mathcal{A}_{j,-}$ be the event that $|L_{j,-}| \leq n/\log^{25} n$.

Lemma 11. *Fix a spanning tree T of G. Let $\mathcal{E}$ be the event of of Lemma 7.*

$$\mathbb{P}((\mathcal{A}_{i,+} \vee \mathcal{A}_{i,-}) \wedge \mathcal{E} \mid T^* = T) = O(n^{-anyconstant}) \text{ for } i = 1, 2, \dots, r.$$

Proof. We assume that $C(i,j) \leq \gamma_n$ for $(a_i, b_j) \in T$. The justification for this is Lemma 8. The number of edges in G of cost at most γ_n incident with a fixed vertex is dominated by $Bin(n, \gamma_n)$ and so q.s. the maximum degree in G can be bounded $2\log^4 n$. This degree bound applies to the trees we consider.

Let $Y = \{C(i,j) : (a_i, b_j) \in E(T)\}$ and let $\delta_1(Y)$ be the indicator for $\mathcal{A}_{s,+} \wedge \mathcal{E}$. Let $\mathcal{B}$ be the event that (11) holds. Now Y determines $\mathbf{u}$,$\mathbf{v}$ and $\mathcal{B}$ determines that T is an optimal basis tree. We fix $u_s = 0$ and write,

$$\mathbb{P}(\mathcal{A}_{s,+} \wedge \mathcal{E} \mid \mathcal{B}) = \frac{\int \delta_1(Y) \mathbb{P}(\mathcal{B} \mid Y) dY}{\int \mathbb{P}(\mathcal{B} \mid Y) dY} \tag{16}$$

Then we note that since $(a_i, b_j) \notin E(T)$ satisfies the condition (11),

$$\begin{aligned} \mathbb{P}(\mathcal{B} \mid Y) &= \prod_{(a_i,b_j)} (\mathbb{P}(C(i,j) \geq (u_i(Y) + v_j(Y))^+)) \\ &= e^{-W}, \end{aligned} \tag{17}$$

where $W = W(Y) = \sum_{(a_i,b_j)} (u_i(Y) + v_j(Y))^+$.

We first observe that McDiarmid's inequality 4 implies that

$$\mathbb{P}(|W - \mathbb{E}(W)| \geq t) \leq 2 \exp\left\{ -\frac{t^2}{4n^3\gamma_n^2} \right\}. \tag{18}$$

To see this, we view the random variable W as a function of $2n-1$ random variables, each independently distributed as $EXP(1)$ conditioned on being at most γ_n. (The variables are the costs of the tree edges.) If we change the value of one variable then we change W by at most $2n\gamma_n$. To see this, suppose that in this change the cost of edge $e=\{a_{i_1},b_{j_1}\}$ goes from $C(e)$ to $C(e)+\xi, |\xi|\leq\gamma_n$. The effect on $\mathbf{u}$,$\mathbf{v}$, under the assumption that u_{i_1} does not change is as follows: (i) $v_j\leftarrow v_j+\xi$ for all $j\in[n]$ and (ii) $u_i\leftarrow u_i-\xi$ for all $i\in[n]\setminus\{i_1\}$. So, u_i+v_j changes only for $i=i_1$.

We have, using Holder's inequality with $p=n^{3/4}$, that

$$\begin{aligned}\int_Y \delta_1(Y)\mathbb{P}(\mathcal{B}\mid Y)\,dY &= \int_Y e^{-W}\delta_1(Y)\,dY\\ &\leq \left(\int_Y e^{-pW/(p-1)}\,dY\right)^{(p-1)/p}\left(\int_Y \delta_1(Y)^p\,dY\right)^{1/p}\\ &= e^{-\mathbb{E}(W)}\left(\int_Y e^{-p(W-\mathbb{E}(W))/(p-1)}\,dY\right)^{(p-1)/p}\left(\int_Y \delta_1(Y)^p\,dY\right)^{1/p}. \end{aligned}\tag{19}$$

We also have

$$\int_Y \mathbb{P}(\mathcal{B}\mid Y)dY = e^{-\mathbb{E}(W)}\int_Y e^{-(W-\mathbb{E}(W))}dY \tag{20}$$

Putting $t=n^{2/3}$ in (18), we see that if $\Omega_1=\left\{Y:|W-\mathbb{E}(W)|\leq n^{2/3}\right\}$ then $Y\in\Omega_1$ q.s. Conditioning on $Y\in\Omega_1$ we have that since $p=n^{3/4}$,

$$\int_{Y\in\Omega_1} e^{-p(W-\mathbb{E}(W))/(p-1)}\,dY \sim \int_{Y\in\Omega_1} e^{-(W-\mathbb{E}(W))}\,dY.$$

Combining this with (16), (19) and (20) we see that

$$\mathbb{P}(\mathcal{A}_{s,+}\wedge\mathcal{E}\mid\mathcal{B}) \lesssim \mathbb{P}(\mathcal{A}_{s,+}\wedge\mathcal{E})^{1/p}\sim\mathbb{P}(\mathcal{A}_{s,+}\mid\mathcal{E})^{1/p}. \tag{21}$$

For the remainder of the lemma we assume that the $C(i,j)$ for $(a_i,b_j)\in T$ satisfy $C\leq\gamma_n$ and that $\mathcal{E}$ holds. Denote this conditioning by $\mathcal{F}$. Note that if $\mathcal{B}$ occurs and (10) holds then $T^*=T$. Let b_j be a neighbor of a_s in G^+ and let $P_j=(i_1=s,j_1,i_2,j_2,\ldots,i_k,j_k=j)$ define the path from a_s to b_j in T. Then it follows from (10) that $v_{j_l}=v_{j_{l-1}}-C(i_l,j_{l-1})+C(i_l,j_l))$. Thus v_j is the final value S_k of a random walk $S_t=X_0+X_1+\cdots+X_t, t=0,1,\ldots,k$, where $X_0\geq 0$ and each $X_t,t\geq 1$ is the difference between two independent copies of $EXP(1)$ that are conditionally bounded above by γ_n. Assume for the moment that $k\geq 5$ and let $x=u_{i_{k-4}}\in[-\gamma_n,\gamma_n]$. Given x we see that there is some positive probability $p_0=p_0(x)=\mathbb{P}(x+X_{k-1}+C(i_k,j_k)>0\mid\mathcal{F})$. Let $\eta=\min\{x\geq-2\gamma_n:p_0(x)\}>0$.

The number of edges in G^+ of cost at most γ_n incident with a fixed vertex is dominated by $Bin(n,\gamma_n)$ and so w.h.p. the maximum degree of the trees we consider can be bounded by $2\log^4 n$. So the number of vertices in T at distance at most 5 from a_s in T is $O(\log^{20} n)$. This will justify assuming $k\geq 5$ above. Lemma 10 with $\rho=5$ and $d=\alpha n$ implies that there are $n/(64\log^{24} n)$ choices of b_j giving rise to edge disjoint useful paths. We know that each such j belongs to $L_{i,+}$ with probability at least η, conditional on $\mathcal{E}$, but conditionally independent of the other useful j's. We see from the Chernoff bounds that

$$\mathbb{P}(\mathcal{A}_{s,+}\wedge\mathcal{E}\mid\mathcal{B})\lesssim\mathbb{P}(Bin(n/(64\log^{24} n),\eta)\leq n/\log^{25} n)^{1/p}\leq e^{-\Omega(n^{1-o(1)}/p)}.$$

Taking the union bound over choices of $s\leq r$ and $+,-$ proves the lemma. □

We will need a another similar lemma. Let $M(i,j)=\{k:(i,k),(k,j)\in E(G^+)\}$ and let $\mathcal{M}(i,j)$ be the event that $|M(i,j)|\leq\frac{n}{\log^{30} n}$.

Lemma 12.

$$\mathbb{P}\left(\mathcal{M}(i,j)\wedge\mathcal{E}\ \mid\ T^*=T\right)=O(n^{-anyconstant})\ \textit{for } i=1,2,\ldots,r.$$

Proof. We first note that for all i,j there are at least $2\varepsilon n$ indices k such that G contains (a_i,b_k) and (a_k,b_j) as edges. It follows from the proof of Lemma 10 with $v=a_i,\rho=5,d=2\varepsilon n$ that we can find $m=2\varepsilon n/(64\log^{24}n)$ useful disjoint paths $P_1,P_2,\ldots,P_m$ of length 5. Suppose that P_k begins at a_{p_k} and ends at b_{q_k} for $k\in[m]$. Next let $Q_1,Q_2,\ldots,Q_m$ be the paths in T from b_j to a_{q_k} and let $R_k,k\in[m]$ denote the last 5 edges of Q_k. Suppose that R_k begins at b_{r_k} and ends at a_{q_k} for $k\in[m]$. We must have $P_k\cap R_k\neq\emptyset$ for at most one value of k, else T has a cycle. A vertex can be on at most $10\log^4 n$ distinct R_k. So we can find at least $m_1=(m-1)/(10\log^4 n)$ indices $k_1,k_2,\ldots,k_{m_1}$ corresponding to disjoint R_k's. We apply the argument of the previous lemma to finish the proof. We get to the equivalent of (21) by replacing $\mathcal{A}_{i,+}$ with $\mathcal{M}(i,j)$. We fix u_i and v_j. Then for each $\ell\in[m_1]$ in turn we condition on the value $x_\ell=u_{p_\ell}$ and $y_\ell=v_{r_\ell}$ and then argue that there is a positive probability that $u_i+v_{k_\ell}\geq 0$ and $u_{k_\ell}+v_j\geq 0$. Concentration of the number of ℓ where this happens follows from the Chernoff bounds as before. This completes the proof of the lemma. $\square$

4.3 Analysis of a Markov chain

In order to understand the number of cycles associated with a perfect matching of G^+, we study a Markov chain introduced by Broder [3]. The state space Ω of this chain is the set of perfect and near-perfect matchings of G^+. (Here a matching is *near-perfect* if it contains $n-1$ edges.) If M is only near-perfect, we let $e_M=\{a_M,b_M\}$ where $a\in A,b\in B$ are the unique pair of vertices notcovered by an edge of M. If M is perfect then we let $M=\{(a,\phi_M(a))\}$ for bijection ϕ_M. If M is near perfect, ϕ_M will only be defined for vertices of A covered by M.

Broder Chain

begin
 Choose $e=(x,y)$ uniformly from $E(G^+)$;
 Choose M_0 uniformly from Ω; $t\leftarrow 0$;
 repeat $t\leftarrow t+1$ **forever**
 begin
  If M_{t-1} is near-perfect **then**
  begin
   If $x=a_{M_{t-1}}$ and $y=b_{M_{t-1}}$ then $M_t\leftarrow M_{t-1}\cup\{e\}$;
   If $x=a_{M_{t-1}}$ and $y\neq b_{M_{t-1}}$ then $M_t\leftarrow (M_{t-1}\cup\{e\})\setminus\big\{(x,\phi_{M_{t-1}}(x))\big\}$;
   If $x\neq a_{M_{t-1}}$ and $y=b_{M_{t-1}}$ then $M_t\leftarrow (M_{t-1}\cup\{e\})\setminus\Big\{(\phi^{-1}_{M_{t-1}}(y),y)\Big\}$;
   Otherwise $M_t\leftarrow M_{t-1}$;
  end
  If M_{t-1} is perfect **then**
  begin
   If $e\in M_{t-1}$ then $M_t\leftarrow M_{t-1}\setminus\{e\}$;
   Otherwise $M_t\leftarrow M_{t-1}$;
  end
 end

It is not difficult to see that for $t\geq 0$, M_t is uniformly chosen from Ω. For most of the time, M_t is near-perfect and is distributed as a random perfect matching less a randomly chosen edge. In which case the number of cycles $\nu_C(M_t)$ in D associated with M_t is distributed as one less than the number of cycles associated with a random perfect matching of G^+.

Let a cycle C be small if $|C|\leq\ell_1=n^{4/5}$ and let σ_t denote the number of vertices on small cycles. Let δ_t be the increase in the number of small cycles when going from M_{t-1} to M_t. We must of course have $\mathbb{E}(\delta_t)=0$. Suppose now that M_t is near-perfect. It follows from Lemma 11 that $\mathbb{E}(\delta_t)\leq\ell_1\log^{25}n/n$. This is because M_{t-1} induces a path P, from i to j say, plus a set of vertex disjoint cycles covering $[n]$. In iteration t, there is a random choice of at least

$n/\log^{25} n$ neighbors b_k of a_i such that $u_i + v_k \geq 0$. Of these choices, at most ℓ_1 lead to the creation of a new small cycle. On the other hand, Lemma 12 implies that

$$0 = \mathbb{E}(\delta_t + \delta_{t+1}) \leq \frac{2\ell_1 \log^{25} n}{n} - \mathbb{E}(\sigma_t) \cdot \frac{n}{\log^{30} n} \cdot \frac{1}{n^2}.$$

It follows that $\mathbb{E}(\sigma_t) \leq 2n^{4/5} \log^{55} n$. The Markov inequality implies that $\sigma_t \leq \frac{1}{2}n^{5/6}$ w.h.p.

We can now complete the proof of Lemma 3. W.h.p. there are at most $n^{1/6}$ large and at most $\frac{1}{2}n^{5/6}$ small cycles. □

5 More general distributions

Replacing $C(i,j)$ by $aC(i,j)$ yields an acceptable distribution where $a = 1$. We assume then that $a = 1$.

Suppose first that the cost density function can be re-expressed as $f(x) = e^{-bx+O(x^2)}$ as $x \to 0$, where $b \neq 0$. We let $F(x) = \mathbb{P}(C \geq x) = b^{-1}e^{-bx+O(x^2)}$ as $x \to 0$. In this case we run Karp's algorithm with the given costs. Let $E^+ = \{(i,j) : u_i + v_j \geq 0\}$. Equation (15) becomes

$$\mathbb{P}((13),(14) \mid \mathbf{u},\mathbf{v}) f(\mathbf{u},\mathbf{v}0 = \prod_{(a_i,b_j)\in E^+\setminus M} F(u_i+v_j) \prod_{(a_i,b_j)\in M} f(u_i+v_j).$$

So, for matchings M_1, M_2 we have where ,

$$\begin{aligned}
\frac{\mathbb{P}(M^* = M_1 \mid \mathbf{u},\mathbf{v})}{\mathbb{P}(M^* = M_2 \mid \mathbf{u},\mathbf{v})} &= \prod_{(a_i,b_j)\in M_1\setminus M_2} \frac{f(u_i+v_j)}{F(u_i+v_j)} \prod_{(a_i,b_j)\in M_2\setminus M_1} \frac{F(u_i+v_j)}{f(u_i+v_j)} \\
&= \prod_{(a_i,b_j)\in M_1\setminus M_2} be^{O(\gamma_n^2)} \prod_{(a_i,b_j)\in M_2\setminus M_1} b^{-1}e^{O(\gamma_n^2)} \\
&= e^{O(n\gamma_n^2)} = o(1).
\end{aligned}$$

So, we replace uniformity by asymptotic uniformity and this is enough for the proof to go through.

When $b = 0$, such as when C is uniform $[0,1]$ then we proceed as follows: In the analysis above we have assumed that $f(x) = e^{-x}$ i.e. that the costs are distributed as exponential mean 1, $EXP(1)$. We extend the analysis to costs C with density function $f(x) = 1 + O(x^2)$ as $x \to 0$ as follows: Given $C(e) = x$, we define $\widehat{C}(e) = y = y(x)$ where $\widehat{C}(e)$ is exponential mean 1 and $\mathbb{P}(\widehat{C}(e) \leq y) = \mathbb{P}(C(e) \leq x)$ i.e. $1 - e^{-y} = \mathbb{P}(C(e) \leq x) = x + O(x^2)$, for small x. We then find that $y + O(y^2) = x + O(x^2)$ and so $x = y + O(y^2)$.

We run the algorithm with C replaced by $y(C)$. We have shown that w.h.p. the tour found by the heuristic has cost $\sum_{i=1}^n \widehat{U}_i$ where $\widehat{U}_i \leq \gamma_n$ for all i and so the corresponding C costs U_i satisfy $U_i = \widehat{U}_i + O(\widehat{U}_i^2)$. Consequently, the increase in cost of using C over $\widehat{C}$ is $O(n\gamma_n^2) = o(1)$. Of course it would be more satisfying to apply the algorithm directly to C and we conjecture that the proof can be modified to verify this.

6 Final Remarks

We have extended the proof of the validity of Karp's patching algorithm to dense graphs with minimum in- and out-degree at least αn, $\alpha > 1/2$ and independent edge weights C. The case $\alpha = 1/2$ looks very challenging.

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