List chromatic number of the square of a sparse random graph

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Abstract

We show that w.h.p the list chromatic number χ_ℓ of the square of $G_{n,p}$ for $p = c/n$ is asymptotically equal to the maximum degree $\Delta(G_{n,p})$. Since $\chi(G^2_{n,p}) \leq \chi_\ell(G^2_{n,p})$, this also improves an earlier result of Garapaty et al [6] who proved that $\chi(G^2_{n,p}) \leq 6 \cdot \Delta(G_{n,p})$ w.h.p.

1 Introduction

The Erdős-Rényi random graph $G_{n,p}$ for a positive integer n and a real number $p \in [0, 1]$ is defined as an n-vertex graph in which each pair of vertices $\{u, v\}$ is connected by an edge uv with probability p, independently of all other pairs. Let $p = c/n$ where $c > 0$ is a constant. The chromatic number of $G_{n,p}$ is well-understood, at least for sufficiently large c. Łuczak [7] proved that if $G = G_{n,p}$ then $\chi(G) \sim \frac{c}{2\log c}$. This was refined by Achlioptas and Naor [1], and further improved later by Coja-Oghlan and Vilenchik [3]. The list chromatic number χ_ℓ is defined as the smallest number k such that if each vertex is assigned a list of k colors, there is always a valid proper coloring from those lists. Alon, Krivelevich and Sudakov [2] showed that $\chi_\ell(G_{n,p}), p = c/n$ is $\Theta\left(\frac{c}{2\log c}\right)$.

The square of a graph G is obtained from G by adding edges for all pairs of vertices at distance two from each other. Atkinson and Frieze [4] showed that w.h.p. the independence number of $G_2 = G^2_{n,p}$ is asymptotically equal to $\frac{4n\log c}{c^2}$, for large c. Garapaty, Lokshtanov, Maji and Pothen [6] studied the chromatic number of powers of $G_{n,p}$. Let $\Delta = \Delta(G_{n,p}) \sim \frac{\log n}{\log\log n}$ be the maximum degree in $G = G_{n,p}$ (for a proof of this known claim about the maximum degree, see for example [5], Theorem 3.4). Garapaty et al proved, in the case of the square G_2 of $G_{n,p}$ with $p = c/n$, that $\chi(G_2) \leq 6 \cdot \frac{\log n}{\log\log n}$ w.h.p. In this work, we strengthen this bound and prove a more general result about the list chromatic number.

Theorem 1. *Let $p = c/n$ where $c > 0$ is a constant. Let G_2 denote the square of $G_{n,p}$. Then w.h.p., $\chi(G_2) \sim \chi_\ell(G_2) \sim \Delta(G_{n,p}) \sim \frac{\log n}{\log\log n}$.*

We show that w.h.p. G_2 is q-list-colorable with $q = (1 + 3\theta^{1/3})\Delta$ where $\theta = o(1)$ is given in (1), establishing the upper bound. Note that for every graph G, $\chi(G) \leq \chi_\ell(G)$. Since the neighbors of a vertex in $G_{n,p}$ form a clique in the square graph G_2, the lower bound of Δ in the theorem is trivial.

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Remark 1. *The value of c does not contribute to the main term in the claim of Theorem 1. Thus we would expect that we could replace $p = c/n$ by $p \leq \omega/n$ for some slowly growing function $\omega = \omega(n) \to \infty$. Indeed, a careful examination of the proof below verifies this so long as $c = o(\log \log n)$.*

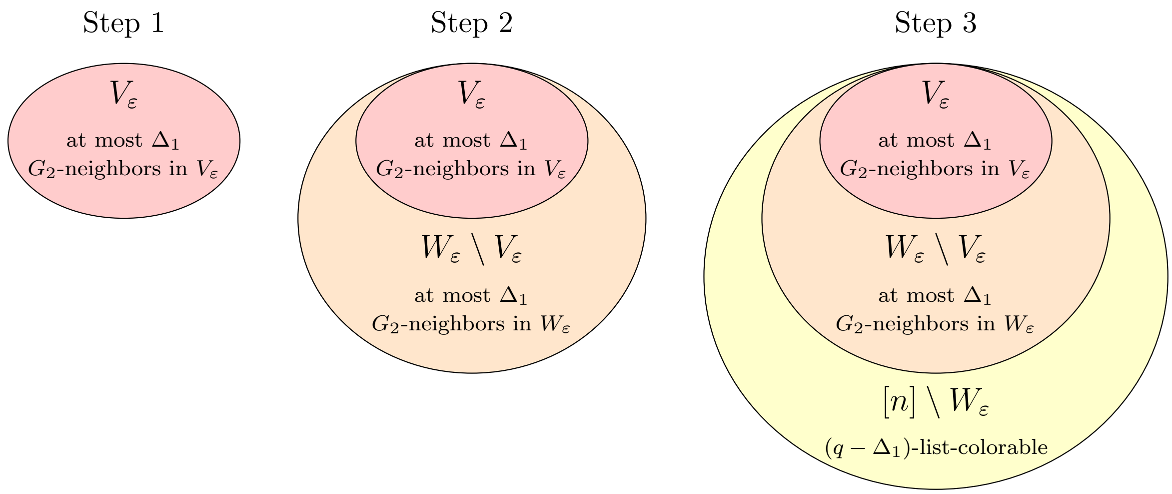


Figure 1: The set of colored vertices in G_2 at the end of each step

1.1 Overview of the proof

The main idea of the proof is to color the vertices $[n]$ of the square graph G_2 by dividing them into parts and assigning colors in a carefully chosen order greedily, as shown in Figure 1, where $\Delta_1 \sim \Delta$. When assigning color to a vertex v, we ensure that the number of already colored neighbors of v in G_2 is less than q. We can then use a greedy coloring strategy to obtain the claimed upper bound q on the list chromatic number.

From here onward, we use 'neighbors' and 'degree' specifically for the neighbors and degree of a vertex in $G_{n,p}$. Similarly, we use 'G_1-neighbors' and 'G_1-degree' when $G_1 = G_{n,p}$. We specify 'G_2-neighbors' or 'G_2-degree' when referring to the neighbors or the degree of a vertex in the square graph. We define V_ε as the set of vertices of 'high' degree (at least ε fraction of maximum degree) and W_ε as their closed neighborhood, for a carefully chosen ε. In particular, the aforementioned order of coloring vertices is:

Step 1. All the vertices with a high degree (V_ε)

Step 2. Neighbors of all high-degree vertices ($W_\varepsilon \setminus V_\varepsilon$)

Step 3. Remaining vertices ($[n] \setminus W_\varepsilon$)

We bound the number of G_2-neighbors of V_ε within V_ε by $\Delta_1 = \left(1 + 2\theta^{1/3}\right)\Delta$ in Corollary 4, ensuring step 1 of the coloring. The number of G_2-neighbors of $W_\varepsilon \setminus V_\varepsilon$ within W_ε is bounded by Δ_1 in Corollary 5, which ensures the completion of step 2. For step 3, we prove that G_2 restricted to $[n] \setminus W_\varepsilon$ can be list-colored with a small number of colors in Corollary 9. This number of extra colors along with the bound on the number of G_2-neighbors of $[n] \setminus W_\varepsilon$ in W_ε given by Corollary 5 is smaller than q, completing step 3.

Corollaries 4 and 5 are proved using the structural results of Lemmas 2 and 3. The list-coloring claim for $[n] \setminus W_\varepsilon$ in Corollary 9 is proved by establishing that any subset $S \subseteq [n] \setminus W_\varepsilon$ contains at most $(6 + 2c)\varepsilon\Delta|S|$ edges in G_2. For 'large' sets S, this is proved in Lemma 6 and 'small' sets are handled by Lemmas 7 and 8.

1.2 Organization of the paper

In Section 2, we provide the proofs of Corollaries 4, 5 and 9 that are required for the three steps of the coloring, and mention statements of all the supporting lemmas used in these proofs. Section 3 contains the explicit details of the greedy coloring strategy and proofs of all the lemmas mentioned in Section 2. We conclude with Section 4, which mentions some directions for future work and our remarks on the result.

2 Proof of Theorem 1

Let $d(v)$ and $N(v)$ denote the degree and neighborhood of v in $G_{n,p}$ respectively, and let Δ be the maximum degree of a vertex.

We can use the following high probability bounds for Δ taken from [5], Theorem 3.4:

$$\frac{\log n}{\log\log n}\left(1-\frac{3\log\log\log n}{\log\log n}\right)\leq\Delta\leq\frac{\log n}{\log\log n}\left(1+\frac{3\log\log\log n}{\log\log n}\right)$$

This implies that w.h.p.

$$n^{1-\theta}\leq\Delta^{\Delta}\leq n^{1+\theta}\text{ where }\theta=\frac{4\log\log\log n}{\log\log n}. \tag{1}$$

For the above value of $\theta=o(1)$, we fix

$$\varepsilon=\theta^{1/2}.$$

For each $0<\alpha\leq 1$, we define $V_\alpha=\{v:d(v)\geq\alpha\Delta\}$ as the set of vertices with degree at least an α fraction of the maximum degree. Let W_α denote the closed neighborhood of V_α, i.e., the neighbors of V_α in $G_{n,p}$ along with the vertices V_α. A subset of vertices in V_ε with sum of degrees comparable to our bound q will be of interest. Thus, define a set of 'good' m-tuples of degrees as

$$L_m=\left\{(\ell_1,\ell_2,\ldots,\ell_m)\in\{\varepsilon\Delta,\varepsilon\Delta+1,\ldots,\Delta\}^m\ :\sum_{i=1}^{m}\ell_i\geq(1+\theta^{1/3})\Delta\right\}.$$

2.1 Bounding the number of G_2-neighbors in W_ε

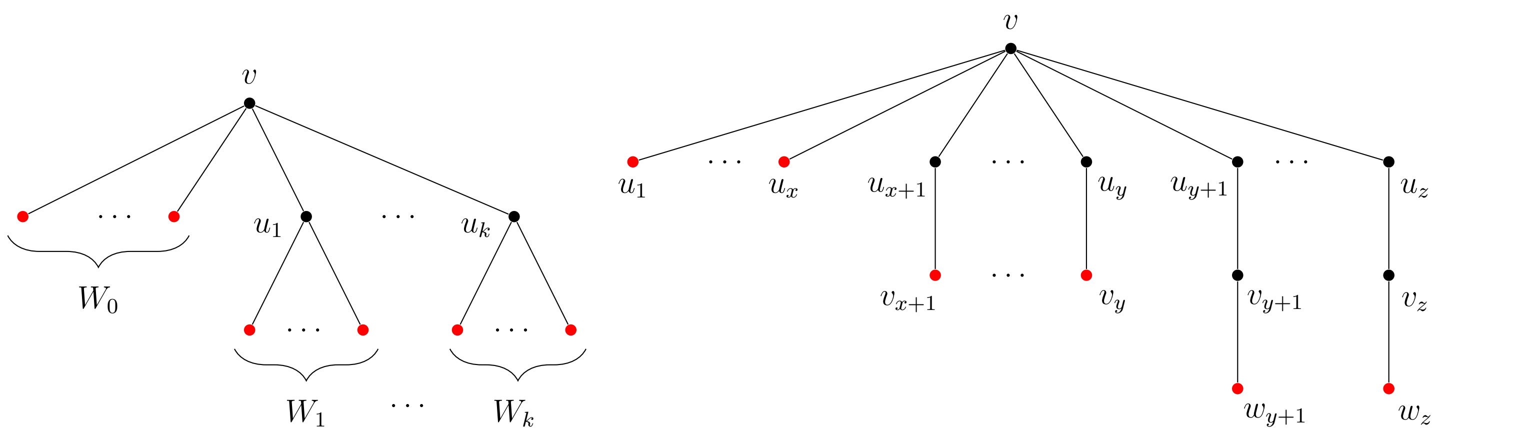
The following two lemmas are needed to analyze the coloring of vertices in W_ε. We prove them in Section 3.

Lemma 2. *W.h.p., $v,w\in V_{2/3}$ implies that $dist(v,w)\geq 10$. (Here $dist(.,.)$ is graph distance in $G_{n,p}$.)*

Lemma 3. *Suppose that $m\leq 2/\varepsilon$. Then w.h.p. there does not exist a connected subset $S\subseteq[n]$ of $G_{n,p}$ with at most $3m$ vertices, which contains an m-sized subset of vertices w_i with a 'good' m-tuple of degrees, i.e., $(d(w_i),i=1,2,\ldots,m)\in L_m$.*

Corollary 4. *A vertex $v\in[n]$ has at most $\Delta_1=\left(1+2\theta^{1/3}\right)\Delta$ G_2-neighbors in V_ε, w.h.p.*

Proof. Suppose v has more than Δ_1 G_2-neighbors in V_ε. Let T be the tree obtained by Breadth-First Search to depth two from v in $G_1=G_{n,p}$. We may assume this is a tree by ignoring the edges revisiting an explored vertex at depth two, if it has multiple parents at depth one. Remove all leaves from T that are not in V_ε and repeat, as shown in Figure 2a. We are left with a set of G_1-neighbors W_0 of v in V_ε and set of G_1-neighbors $u_1, u_2, \ldots, u_k$ of v that are not in V_ε. In addition we have sets $W_1, W_2, \ldots, W_k \subseteq V_\varepsilon$ such that u_i is a G_1-neighbor of all vertices in $W_i, i = 1, 2, \ldots, k$. The G_2-degree of v is given by $D = \sum_{w \in W_0} d(w) + k + \sum_{i=1}^{k} |W_i| > \Delta_1$. The number of G_2-neighbors of v in V_ε is $m_1 = \sum_{i=0}^{k} |W_i|$ and the tree T contains $1 + k + m_1 \leq 2m_1 + 1$ vertices. Let $W = \bigcup_{i=0}^{k} W_i$ and add v to W if $v \in V_\varepsilon$. Then let $W = \{w_1, w_2, \ldots, w_m\}$ where $m = m_1 + \mathbf{1}_{v \in V_\varepsilon}$. If $m \leq 2/\varepsilon$ then by letting T take the place of S in Lemma 3, we reach a contradiction. Otherwise, $M = \sum_{i=1}^{m} d(w_i) \geq m d_{\min}$ where $d_{min} = \min\{d(w) : w \in W\}$. But $d_{min} \geq \varepsilon\Delta$ and so $M > 2\Delta$. It follows from Lemma 2 that $d_{min} < 2\Delta/3$ and so we can reduce m by one, keeping $M > 4\Delta/3$. If $m > 2/\varepsilon$ after this update, we rewrite the expression for M and repeat the argument. We eventually reduce m to below $2/\varepsilon$ while keeping $M > (1 + \theta^{1/3})\Delta$. But now we contradict Lemma 3 as discussed before. $\square$



(a) T from Corollary 4

(b) T_1 from Corollary 5

Figure 2: Breadth-First-Search trees after deletions (red nodes are in V_ε).

A similar argument bounds the number of G_2-neighbors in W_ε for vertices outside V_ε.

Corollary 5. *A vertex $v \notin V_\varepsilon$ has at most $\Delta_1 = \left(1 + 2\theta^{1/3}\right)\Delta$ G_2-neighbors in W_ε, w.h.p.*

Proof. Suppose v has more than Δ_1 G_2-neighbors in W_ε. Let T be the tree obtained by Breadth-First Search to depth three from v in $G_1 = G_{n,p}$. Again, assuming this to be a tree involves ignoring the edges revisiting an already explored vertex, if it has multiple parents at smaller depths. Let this tree have levels $L_0 = \{v\}, L_1, L_2, L_3$. Let the G_1-neighbors of v be $\{u_1, u_2, \ldots, u_k\}$. Let $F_{i,t}$ for $t = 2, 3$ denote the vertices in L_t separated from v in T by u_i.

We now define a subtree T_1 of T that will take the place of S in Lemma 3. To obtain T_1 we do the following: suppose that $u_1, u_2, \ldots, u_x$ are the neighbors of v in V_ε. Delete $F_{i,2} \cup F_{i,3}$ from T for $i \in [1, x]$. Now suppose that $X_i = F_{i,2} \cap V_\varepsilon \neq \emptyset$ for $i \in [x+1, y]$ and that $F_{i,2} \cap V_\varepsilon = \emptyset$ for $i \in [y+1, k]$. Choose one vertex $v_i \in X_i$ for each $i \in [x+1, y]$ and delete $X_i \setminus \{v_i\}$ and their children from T. Suppose also that $Y_i = F_{i,3} \cap V_\varepsilon \neq \emptyset$ for $i \in [y+1, z]$. Choose one vertex $v_i \in X_i$ along with one neighbor w_i in Y_i for each $i \in [y+1, z]$ and delete $X_i \setminus \{v_i\}$, $Y_i \setminus \{w_i\}$, and their children from T. For $i \in [z+1, k]$, we delete u_i and the vertices $F_{i,2} \cup F_{i,3}$ from T. As shown in Figure 2b, T_1 is the tree that survives these deletions. All the leaves of T_1 are in V_ε.

The number of G_2-neighbors of vertex v in W_ε is at most

$$D = \sum_{i=1}^{z} d(u_i) + (k - z) \leq \sum_{i=1}^{x} d(u_i) + (z - x)\varepsilon\Delta + (k - z) \leq \sum_{i=1}^{x} d(u_i) + (z - x + 1)\varepsilon\Delta.$$

Our assumption is that $D > \Delta_1$. Now let $M = \sum_{w \in V(T_1) \cap V_\varepsilon} d(w)$. Then,

$$M \geq \sum_{i=1}^{x} d(u_i) + (z-x)\varepsilon\Delta \geq D - \varepsilon\Delta > (1 + 2\theta^{1/3} - \theta^{1/2})\Delta > (1 + \theta^{1/3})\Delta.$$

This is due to the fact that we get a contribution of at least $\varepsilon\Delta$ from each surviving member of $F_{i,2}$ and $F_{j,3}$ for $i \in [x+1, y]$ and $j \in [y+1, z]$. The number of vertices N in the tree T_1 satisfies

$$N \leq 1 + x + 2(y-x) + 3(z-y) \leq 3|V(T_1) \cap V_\varepsilon|.$$

Put $m = |V(T_1) \cap V_\varepsilon|$. If $m \leq 2/\varepsilon$, the tree T_1 contradicts Lemma 3. Assume then that $m > 2/\varepsilon$, hence $M \geq m\varepsilon\Delta > 2\Delta$. If the smallest degree in $V(T_1) \cap V_\varepsilon$ is at least $2\Delta/3$, then it follows from Lemma 2 that T consists only of $u_1 \in V_{2/3}$ and the neighbors of u_1. In this case, the corollary holds trivially. Otherwise, we can delete a vertex of degree less than $2\Delta/3$ and reduce m by one, keeping $M > (1 + \theta^{1/3})\Delta$. Similar to the proof of Corollary 4, this leads to a contradiction. $\square$

2.2 Extra colors required for $[n] \setminus W_\varepsilon$

The next part of our strategy is to bound the number of G_2-edges contained in any set S that is disjoint from W_ε. We prove a high probability bound of $(6 + 2c)\varepsilon\Delta|S|$, which will imply Corollary 9 as desired.

For 'large' sets S, the following lemma can be invoked for the desired bound.

Lemma 6. *The total number of edges in G_2 is less than $c(c+1)n$ w.h.p.*

The proof of Lemma 6 can be found in Section 3.

For $2 \leq s \leq n$ let ν_s be the maximum number of G_2-edges in a set of size s.

$$\text{If } k_0 = 2/\varepsilon^2 \text{ then } \nu_s \leq 10k_0c^3s \text{ for } s \geq n/(10ck_0). \tag{2}$$

In particular, the above statement derived using Lemma 6 makes the notion of 'large' precise for sets S. For further analysis, we restrict to the case of $|S| \leq n/(10ck_0) = n\varepsilon^2/(20c)$.

Let $e(S)$ denote the number of edges inside S in $G_{n,p}$. We first show that the expected number of sets S with more than $\frac{\varepsilon\Delta}{2}|S|$ edges is $o(1)$. Let X_S denote the number of such sets of size at most $s_0 = n\varepsilon^2/(20c)$.

$$\begin{aligned}
\mathbb{E}\left(X_S\right) &\leq \sum_{s=4}^{s_0} \binom{n}{s} \mathbb{P}\left([s] \text{ contains } \frac{\varepsilon\Delta s}{2} \text{ edges}\right) \\
&\leq \sum_{s=4}^{s_0} \left(\frac{ne}{s}\right)^s \mathbb{E}\left(\text{number of sets of } \frac{\varepsilon\Delta s}{2} \text{ or more edges in [s]}\right) \\
&\leq \sum_{s=4}^{s_0} \left(\frac{ne}{s}\right)^s \binom{\binom{s}{2}}{\varepsilon\Delta s/2} \left(\frac{c}{n}\right)^{\varepsilon\Delta s/2} \qquad (3) \\
&\leq \sum_{s=4}^{s_0} \left(\frac{ne}{s}\right)^s \left(\frac{es^2c}{\varepsilon\Delta sn}\right)^{\varepsilon\Delta s/2} \\
&\leq \sum_{s=4}^{s_0} \left(\frac{ne}{s}\right)^s \left(\frac{e\varepsilon}{20\Delta}\right)^{\varepsilon\Delta s/2}.
\end{aligned}$$

Let $u_s = \left(\frac{ne}{s}\right)^s \left(\frac{e\varepsilon}{20\Delta}\right)^{\varepsilon\Delta s/2}$. If $s \leq \log^2 n$, then $u_s \leq n^{-1/2}$ implying $\sum_{s=4}^{\log^2 n} u_s \leq \frac{\log^2 n}{n^{1/2}} = o(1)$. If $s \geq \log^2 n$, then $u_s \leq (e/20)^{\log^2 n}$ so $\sum_{s=\log^2 n}^{s_0} u_s \leq n(e/20)^{\log^2 n} = o(1)$. Thus, $e(S) \leq \frac{\varepsilon\Delta}{2}|S|$ for all S w.h.p.

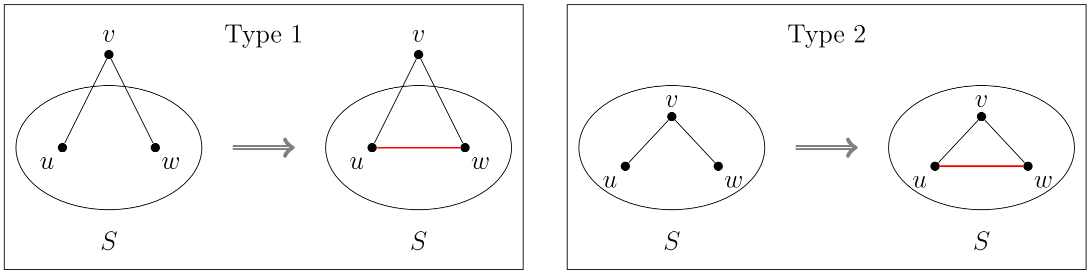


Figure 3: The two types of edges introduced in a set S when squaring graph G

We now focus on the 'new' edges in G_2, which were not present in $G_{n,p}$. For $u, w \in S$, a new edge uw appears in S when squaring $G_{n,p}$ only due to the existence of a common neighbor v. From containment of such a neighbor in S, the new edges in G_2 can be of two types seen in Figure 3.

Type 1. $v \notin S$, bounded by Lemma 7

Type 2. $v \in S$, bounded by Lemma 8

If $S \cap W_\varepsilon = \emptyset$ then a vertex outside S has at most $\varepsilon\Delta$ neighbors in S. For a fixed set S let $a_{k,S}$ denote the number of (vertex, set) pairs (v, T) where $v \notin S$ and $T = N(v) \cap S$ with $|T| = k$.

Lemma 7. *The following holds w.h.p. Let $A_1(S) = \sum_{k \leq \varepsilon\Delta} a_{k,S} k^2$ bound the number of G_2-edges of type 1 in S. Then $A_1(S) \leq 5\varepsilon\Delta|S|$ for all S with $|S| \leq n/(10ck_0)$.*

We now deal with the number of edges of type 2, i.e., uw coming from paths uvw of length two where $\{u, v, w\} \subseteq S$. Denote the number of such paths by $A_2(S)$.

Lemma 8. *W.h.p., $A_2(S) \leq (2c + 1/2)\varepsilon\Delta|S|$ for all S with $|S| \leq n/(10ck_0)$.*

We provide proofs of the above two lemmas in Section 3.

It follows from lemmas 7 and 8 and the upper bound on $e(S)$ obtained using 3, we have w.h.p.,

$$e(S) + A_1(S) + A_2(S) \leq (6 + 2c)\varepsilon\Delta|S|. \tag{4}$$

From 2 and 4, we have established that any $S \subseteq [n] \setminus W_\varepsilon$ contains at most $(6 + 2c)\varepsilon\Delta|S|$ edges w.h.p.

Corollary 9. *The vertices of $[n] \setminus W_\varepsilon$ can be list-colored with lists of at most $(6 + 2c)\varepsilon\Delta + 1$ colors.*

Proof. We derive the required result by proving a more general statement with the principle of strong mathematical induction: for any vertex set V, if every subset $S \subseteq V$ contains at most $r|S|$ edges then V is $(r+1)$-list-colorable. Clearly if $|V| \leq r + 1$ then the result holds as every vertex has at most r neighbors. Assume the result holds for all vertex sets of size $1, 2, .., t$ for some $t \in \mathbb{N}$. For a set V of size $|V| = t + 1$, we have at most $r(t + 1)$ edges inside. Then, there exists a vertex v of degree $\leq r$. The induced subgraph on $V \setminus \{v\}$ has vertex set of size t with the required property on edges, and hence is $(r + 1)$-list-colorable by the induction hypothesis. We assign colors to all the vertices in $V \setminus \{v\}$ first. Now since v has degree $\leq r$, there is at least one color available for v from its list of size $r + 1$, which is not used by any neighbors in $V \setminus \{v\}$. $\square$

3 List coloring of G_2 and proofs of lemmas

Given the above results on the number of neighbors, we can list-color G_2 as follows:

(1) We list-color V_ε with lists of size $q = \Delta(1 + 3\theta^{1/3})$ colors. We do this greedily by arbitrarily ordering the vertices in V_ε and coloring a vertex with the lowest index color available. Corollary 4 implies that any vertex v has at most $\Delta\left(1 + 2\theta^{1/3}\right)$ G_2-neighbors in V_ε and so there will be an unused color.

(2) We list-color $W_\varepsilon \setminus V_\varepsilon$ with lists of size $q = \Delta(1 + 3\theta^{1/3})$ greedily, i.e., we arbitrarily order the vertices in $W_\varepsilon \setminus V_\varepsilon$ and color a vertex with the lowest index color available. Corollary 5 implies that any vertex $v \notin V_\varepsilon$ has at most $\Delta\left(1 + 2\theta^{1/3}\right)$ G_2-neighbors in W_ε and so there will be an unused color.

(3) We then list-color $[n] \setminus W_\varepsilon$ with at most $\Delta\left(1 + 2\theta^{1/3} + (6 + 2c)\theta^{1/2}\right) + 1$ colors greedily. This follows similarly from Corollaries 5 and 9. Since $\theta = o(1)$, the size of lists is bounded above by $q = \Delta(1 + 3\theta^{1/3})$.

3.1 Proof of Lemma 2

Proof. To bound the probability of having $dist(v, w) < 10$ for two vertices, we can count all paths of length k between them in $\binom{n}{k}k!$ ways, for $k = 1, 2..., 9$. We also need $v, w \in V_{2/3}$, so we can multiply by square of the probability of a vertex having a high degree. Let $\ell_0 = 2\Delta/3 - 10$. We have

$$\begin{aligned}
\mathbb{P}(\exists v, w \in V_{2/3} : dist(v, w) < 10) &\leq \sum_{k=1}^{9} \binom{n}{k} k! p^{k-1} \left(\sum_{\ell=\ell_0}^{n-1} \binom{n}{\ell} p^\ell (1-p)^{n-10-\ell} \right)^2 \\
&\leq \sum_{k=1}^{9} n c^{k-1} n^{-4/3+o(1)} = o(1).
\end{aligned}$$

□

3.2 Proof of Lemma 3

Proof. There are $\binom{n}{s}$ choices for S, where $|S| = s \in [m, 3m]$. There are at most s^{s-2} choices for a spanning tree of S. We can choose the vertices of large degree in $\binom{s}{m}$ ways. Let D be the sum of degrees of the chosen m vertices. The probability that these vertices have large degrees can be bounded by a product, so

$$\begin{aligned}
\mathbb{P}(\exists S) &\leq \sum_{m=2}^{2/\varepsilon} \sum_{s=m}^{3m} \binom{n}{s} s^{s-2} p^{s-1} \binom{s}{m} \sum_{D \geq (1+\theta^{1/3})\Delta} \sum_{\ell_1 + \cdots + \cdots \ell_m = D} \prod_{i=1}^{m} \left(\sum_{k=\ell_i}^{n-s} \binom{n-s}{k} p^k (1-p)^{n-s-k} \right) \\
&\leq \sum_{m=2}^{2/\varepsilon} \sum_{s=m}^{3m} \binom{n}{s} s^{s-2} p^{s-1} 2^s \sum_{D \geq (1+\theta^{1/3})\Delta} \sum_{\ell_1 + \cdots + \cdots \ell_m = D} \prod_{i=1}^{m} n^{-\ell_i/\Delta + O(\theta)} \\
&\leq \frac{2n}{c} \sum_{m=2}^{2/\varepsilon} \left(\frac{2ec}{3m} \right)^{3m} \sum_{D \geq (1+\theta^{1/3})\Delta} \binom{D-1}{m-1} n^{-D+O(\theta m)} \\
&\leq \frac{2n}{c} \sum_{m=2}^{2/\varepsilon} \left(\frac{2ec}{3m} \right)^{3m} \left(\frac{(1+\theta^{1/2})\Delta}{m} \right)^m n^{-(1+\theta^{1/3} - O(\theta^{1/2}))}
\end{aligned}$$

$$\leq n^{1+o(\theta^{1/2})-(1+\theta^{1/3}-O(\theta^{1/2}))} = o(1).$$

□

3.3 Proof of Lemma 6

Proof. Let $d(i)$ denote the degree of vertex i in $G_{n,p}$. The expected number of edges in G_2 is

$$\mathbb{E}\left(\sum_{i=1}^{n} \frac{d(i)(d(i)+1)}{2}\right) = \frac{n}{2}\sum_{j=1}^{n-1} j(j+1)\binom{n-1}{j} p^j (1-p)^{n-1-j} = \frac{c^2(n-1)(n-2)+cn^2}{2n}.$$

To show concentration around the mean, we use the following theorem from Warnke [8]:

Theorem 10. *Let $X = (X_1, X_2, \ldots, X_N)$ be a family of independent random variables with X_k taking values in a set Λ_k. Let $\Omega = \prod_{k\in[N]} \Lambda_k$ and suppose that $\Gamma \subseteq \Omega$ and $f : \Omega \to \mathbf{R}$ are given. Suppose also that whenever $\mathbf{x}, \mathbf{x}' \in \Omega$ differ only in the k-th coordinate*

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq \begin{cases} c_k & \textit{if } \mathbf{x} \in \Gamma. \\ d_k & \textit{otherwise.} \end{cases}$$

If $W = f(X)$, then for all reals $\gamma_k > 0$,

$$\mathbb{P}(W \geq \mathbb{E}(W) + t) \leq \exp\left\{-\frac{t^2}{2\sum_{k\in[N]}(c_k + \gamma_k(d_k - c_k))^2}\right\} + \mathbb{P}(X \notin \Gamma)\sum_{k\in[N]} \gamma_k^{-1}.$$

We use Theorem 10 with $N = n$, defining $X_i = \{j < i : \{j, i\}$ is an edge of $G_{n,p}\}$ for $i = 1, 2, \ldots, n$, letting $f(X) = W = \sum_{i=1}^{n} \frac{d(i)(d(i)+1)}{2}$ be the total number of edges in $G^2_{n,p}$, and $\Gamma = \{\Delta(G_{n,p}) \leq \log n\}$. The condition of $\mathbf{x}, \mathbf{x}'$ differing only in the k-th coordinate can be seen as missing an edge. In this case, we can assign $c_k = \log^2 n$, $d_k = n^2$ and $\mathbb{P}(X \notin \Gamma) \leq (\log n)^{-\frac{1}{2}\log n}$. Then we can take $\gamma_k = n^{-4}$ for $k \in [n]$ and $t = n^{2/3}$ to complete the proof of Lemma 6. □

3.4 Proof of Lemma 7

Proof. Let $|S| \leq n/(10ck_0)$. We will prove that w.h.p.:

(a) For $k_0 < k_1 < k_2 \leq \varepsilon\Delta$, for all sets $S \subseteq [n] \setminus V_\varepsilon$,

$$\sum_{k=k_1}^{k_2} a_{k,S} \leq \frac{(1+\varepsilon)|S|}{k_1}.$$

(b) $a_{k,S} \leq (10c)^{k_0}|S|$ for $k \leq k_0$.

We have, where $M_{u,k_1,k_2} = \left\{(x_2, \dots, x_{\varepsilon\Delta}) : \sum_{k=k_1}^{k_2} x_k = u\right\}$.

$$\begin{aligned}
&\mathbb{P}\left(\exists S, |S| = s \leq n/(10ck_0) : \sum_{k=k_1}^{k_2} a_{k,S} \geq t\right) \\
&\leq \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \binom{n}{s} \sum_{u \geq t} \sum_{\mathbf{x} \in M_{u,k_1,k_2}} \binom{n}{x_{k_1}, \dots, x_{k_2}, n-u} \prod_{k=k_1}^{k_2} \left(\binom{s}{k}\left(\frac{c}{n}\right)^k\right)^{x_k} \\
&\leq \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \sum_{u \geq t} \sum_{\mathbf{x} \in M_{u,k_1,k_2}} \binom{n}{x_{k_1}, \dots, x_{k_2}, n-u} \prod_{k=k_1}^{k_2} \left(\frac{sec}{k_1 n}\right)^{k_1 x_k} \\
&= \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \sum_{u \geq t} \left(\frac{sec}{k_1 n}\right)^{k_1 u} \sum_{\mathbf{x} \in M_{u,k_1,k_2}} \binom{n}{x_{k_1}, \dots, x_{k_2}, n-u} \\
&\leq \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \sum_{u \geq t} \left(\frac{sec}{k_1 n}\right)^{k_1 u} \binom{n}{u}.
\end{aligned}$$

Putting $t = (1+\varepsilon)s/k_1$, we have, for large k_1 i.e. for $k_1 > 2/\varepsilon$,

$$\begin{aligned}
&\mathbb{P}\left(\exists S, |S| = s \leq n/(10ck_0) : \sum_{k=k_1}^{k_2} a_{k,S} \geq \frac{(1+\varepsilon)s}{k_1}\right) \\
&\leq \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \sum_{u \geq t} \left(\frac{sec}{k_1 n}\right)^{k_1 u} \binom{n}{u} \\
&\leq 2 \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \left(\frac{sec}{k_1 n}\right)^{(1+\varepsilon)s} \binom{n}{(1+\varepsilon)s/k_1} \\
&\leq 2 \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \left(\frac{sec}{k_1 n}\right)^{(1+\varepsilon)s} \left(\frac{nek_1}{(1+\varepsilon)s}\right)^{(1+\varepsilon)s/k_1} \\
&= 2 \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\left(\frac{s}{n}\right)^{\varepsilon-(1+\varepsilon)/k_1} \frac{e^{2+\varepsilon+(1+\varepsilon)/k_1} c^{1+\varepsilon}}{k_1^{(1+\varepsilon)(k_1-1)/k_1}}\right)^s = o(n^{-2}),
\end{aligned}$$

verifying (a).

We also have

$$\begin{aligned}
\mathbb{P}(\exists S, |S| = s \leq n/(10ck_0) : a_{k,S} \geq t) &\leq \binom{n}{s}\binom{n}{t}\left(\binom{s}{k}\left(\frac{c}{n}\right)^k\right)^t \\
&\leq \left(\frac{ne}{s}\right)^s \left(\frac{ne}{t}\right)^t \cdot \left(\frac{se}{k} \cdot \frac{c}{n}\right)^{kt}. \qquad (5)
\end{aligned}$$

We put $t = (10c)^{k_0}s$ for $2 \leq k < k_0$. Then we have, where $L = (10c)^{k_0}$,

$$\mathbb{P}(\exists S, |S| = s \leq n/(10ck_0) : 2 \leq k \leq k_0, a_{k,S} \geq Ls) \leq \left(\frac{ne}{s}\right)^s \cdot \left(\frac{ne}{Ls}\right)^{Ls} \cdot \left(\frac{sec}{kn}\right)^{Lks}$$

$$
\begin{aligned}
&= \left(\left(\frac{s}{n}\right)^{L(k-1)-1} \cdot e \cdot \left(\frac{e}{L}\right)^{L} \cdot \left(\frac{ec}{k}\right)^{Lk} \right)^{s} \\
&\leq \left(\left(\frac{s}{n}\right)^{L(k-1)-1} \cdot e^{cL+2} \right)^{s}. \qquad (6)
\end{aligned}
$$

If $k \geq 3$ then the bracketed term σ_s in (6) is at most $\left(\frac{se^c}{n}\right)^L = o(1)$ and so $\sum_{s\geq 1} \sigma_s^s = o(1)$. If $k = 2$ we write $\sigma_s = \left(\frac{se^c}{n}\right)^{L-1} \cdot e^{c+2} = o(1)$, as well. This verifies (b).

So, by dividing $[1, \varepsilon\Delta]$ into intervals of size $\varepsilon\Delta/2^i$ for $i \geq 1$, we get that for all $|S| \leq n/(10ck_0)$,

$$
\begin{aligned}
A_1(S) &\leq \sum_{k=2}^{k_0} (10c)^{k_0} k^2 s + \sum_{i\geq 1} \frac{(1+\varepsilon)s}{\varepsilon\Delta/2^i} \cdot \frac{(\varepsilon\Delta)^2}{2^{2i-2}} \\
&\leq (2/\varepsilon)^3 (10c)^{2/\varepsilon} s + 4(1+\varepsilon)\varepsilon\Delta s \qquad (7) \\
&\leq 5\varepsilon\Delta s. \qquad (8)
\end{aligned}
$$

□

3.5 Proof of Lemma 8

Proof. The number of such paths is equal to $\sum_{v\in S} \frac{d_S(v)(d_S(v)+1)}{2}$ where $d_S(v)$ is the degree of v in S.

$$
A_2(S) = \sum_{v\in S, d(v)\leq \varepsilon\Delta} \frac{d_S(v)(d_S(v)+1)}{2} \leq \varepsilon\Delta \sum_{v\in S} \frac{d_S(v)+1}{2} = \varepsilon\Delta(e(S) + |S|/2), \qquad (9)
$$

where $e(S)$ is the number of $G_{n,p}$ edges entirely contained in S. Now

$$
\begin{aligned}
\mathbb{P}(\exists S, s = |S| \leq n/(10ck_0) : e(S) \geq 2cs) &\leq \sum_{s=2}^{n/(10ck_0)} \binom{n}{s} \binom{\binom{s}{2}}{2cs} \left(\frac{c}{n}\right)^{2s} \leq \sum_{s=2}^{n/(10ck_0)} \left(\frac{ne}{s} \cdot \left(\frac{se}{4n}\right)^2 \right)^s \\
&= \sum_{s=2}^{n/(10ck_0)} \left(\frac{s}{n} \cdot \frac{e^3}{16} \right)^s = o(n^{-1}).
\end{aligned}
$$

So,

$$
A_2(S) \leq \varepsilon\Delta(2c + 1/2)|S| \text{ for all } |S| \leq n/(10ck_0),\ w.h.p.
$$

□

4 Conclusions

While we have shown that $\chi_\ell(G_2) \sim \Delta$ w.h.p., it is possible that $\chi(G_2) = \Delta + 1$ w.h.p. This would be quite pleasing, but we are not confident enough to make this a conjecture. It is of course interesting to further consider $\chi(G_2)$ or $\chi_\ell(G_2)$ when $np \to \infty$. Note that when $np \gg n^{1/2}$, the diameter of $G_{n,p}$ is equal to 2 w.h.p., in which case $G_2 = K_n$. One can also consider higher powers of $G_{n,p}$ as was done in [4] and [6]. Such considerations are more technically challenging, especially with list coloring.

References

[1] D. Achlioptas and A. Naor, The two possible values of the chromatic number of a random graph, *Annals of Mathematics* 162 (2005) 1335-1351.

[2] N. Alon, M. krivelevich and B. Sudakov, List coloring of random and pseudo-random graphs, *Combinatorica* 19 (1999) 453-472.

[3] A. Coja–Oghlan and D. Vilenchik, Chasing the k-colorability threshold, Proceedings of the 54th IEEE Annual Symposium on Foundations of Computer Science (2013).

[4] A.M. Frieze and G. Atkinson, On the b-independence number of sparse random graphs, *Combinatorics, Probability and Computing* 13 (2003), 295-310.

[5] A.M. Frieze and M. Karoński, Introduction to Random Graphs, Cambridge University Press (2015).

[6] K. Garapaty, D. Lokshtanov, H. Maji and A. Pothen, The Chromatic Number of Squares Of Random Graphs, *Journal of Combinatorics* 14 (2023) 507-537.

[7] T. Łuczak, The chromatic number of random graphs, *Combinatorica* 11 (1991) 45-54.

[8] L. Warnke, On the Method of Typical Bounded Differences, *Combinatorics, Probability and Computing* 25 (2016) 269–299.