Colorful Hamilton cycles in random graphs

Debsoumya Chakraborti*[1], Alan Frieze†[2], and Mihir Hasabnis[2]

[1]Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, South Korea
[2]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, USA
Email: debsoumya@ibs.re.kr, alan@random.math.cmu.edu, mhasabni@andrew.cmu.edu

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Abstract

Given an n vertex graph whose edges have colored from one of r colors $C = \{c_1, c_2, \ldots, c_r\}$, we define the Hamilton cycle color profile $hcp(G)$ to be the set of vectors $(m_1, m_2, \ldots, m_r) \in [0, n]^r$ such that there exists a Hamilton cycle that is the concatenation of r paths $P_1, P_2, \ldots, P_r$, where P_i contains m_i edges. We study $hcp(G_{n,p})$ when the edges are randomly colored. We discuss the profile close to the threshold for the existence of a Hamilton cycle and the threshold for when $hcp(G_{n,p}) = \{(m_1, m_2, \ldots, m_r) \in [0, n]^r : m_1 + m_2 + \cdots + m_r = n\}$.

1 Introduction

We are given an n-vertex graph where each edge is colored from a set $C = \{c_1, c_2, \ldots, c_r\}$. The Hamilton cycle color profile $hcp(G)$ is defined to be the set of vectors $\mathbf{m} \in \mathbf{M} = \{\mathbf{m} \in [0, n]^r, m_1 + \cdots + m_r = n\}$ such that there exists a Hamilton cycle H such that H is the concatenation of r paths $P_1, P_2, \ldots, P_r$, where P_i contains m_i edges.

Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be positive constants that sum to one and α denote $(\alpha_1, \alpha_2, \ldots, \alpha_r)$. Let $G_{n,p}^{\boldsymbol{\alpha}}$ denote the random graph $G_{n,p}$ where each edge e is independently given a random color $c(e) \in C = \{c_1, c_2, \ldots, c_r\}$ where the color $c(e)$ of edge e satisfies $\mathbb{P}(c(e) = c_i) = \alpha_i$.

Randomly colored random graphs have been studied recently in the context of (i) rainbow matchings and Hamilton cycles, see for example [2], [5], [10], [13] [16]; (ii) rainbow connection see for example [8], [14], [15], [19], [17]; (iii) pattern colored Hamilton cycles, see for example [1], [9]. This paper is closely related to Frieze [11] and Chakraborti and Hasabanis [4] where edge colored matchings are the topic of interest. This paper can be considered to be a contribution in the same genre. Our first theorem considers $G_{n,p}$ where p is close to the Hamiltonicity threshold.

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Theorem 1. *Fix $r \geq 2$ and positive real numbers $\beta, \alpha_1, \alpha_2, \ldots, \alpha_r$ where $\sum_{i=1}^r \alpha_i = 1$. If $p \geq \frac{\log n + r \log\log n + \omega}{n}$ where $\omega = \omega(n) \to \infty$ as $n \to \infty$, then w.h.p. $hcp(G_{n,p}^{\boldsymbol{\alpha}}) \supseteq \mathbf{M}_\beta = \{\mathbf{m} \in \mathbf{M} : m_i \geq \beta n, i \in [r]\}$.*

We will for convenience assume that $\omega = o(\log\log n)$ and note that this also implies the theorem for larger ω. We next discuss why the factor r in the definition of p cannot be replaced by anything smaller in Theorem 1. The importance of the factor r lies in the fact that it implies that the minimum degree is at least $r+1$ w.h.p. and if we replace $\omega = o(\log\log n)$ by $-\omega$ then w.h.p. there will be at least $e^{\omega}/2$ vertices of degree r. In which case there will w.h.p. be $e^{\omega - r}/2$ vertices of degree r, all of whose incident edges have a distinct color. Thus, it is not possible to have a Hamilton cycle made from the concatenation of r monochromatic paths.

Our next theorem considers when to expect $G_{n,p}$ to have a full Hamilton cycle color profile. For brevity, let $\alpha_{\min} = \min\{\alpha_1, \ldots, \alpha_r\}$.

Theorem 2. *Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_r$ are as in Theorem 1 and that $p \geq \frac{\log n + \log\log n + \omega}{\alpha_{min} n}$, where $\omega = \omega(n) \to \infty$ as $n \to \infty$. Then, w.h.p. $hcp(G_{n,p}^{\boldsymbol{\alpha}}) = \mathbf{M}$.*

If $p \leq \frac{\log n + \log\log n - \omega}{\alpha_{min} n}$, then w.h.p. the subgraph of $G_{n,p}$ induced by the edges of color 1 has a vertex of degree one, assuming that $\alpha_{\min} = \alpha_1$.

We finally consider directed versions of the above two theorems. Let $D_{n,p}^{\boldsymbol{\alpha}}$ denote the random digraph in which each edge of the complete digraph $\vec{K}_{n,p}$ occurs with probability p and is randomly colored as above. We use the coupling argument of McDiarmid [18] to prove the following couple of theorems.

Theorem 3. *Suppose that $r, \beta, \alpha_1, \alpha_2, \ldots, \alpha_r$ are as in Theorem 1. If $p \geq \frac{\log n + r \log\log n + \omega}{n}$ where $\omega = \omega(n) \to \infty$ as $n \to \infty$, then w.h.p. $hcp(D_{n,p}^{\boldsymbol{\alpha}}) \supseteq \mathbf{M}_\beta = \{\mathbf{m} \in \mathbf{M} : m_i \geq \beta n, i \in [r]\}$.*

Theorem 4. *Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_r$ are as in Theorem 1 and that $p \geq \frac{\log n + \log\log n + \omega}{\alpha_{min} n}$, where $\omega = \omega(n) \to \infty$ as $n \to \infty$. Then, w.h.p. $hcp(D_{n,p}^{\boldsymbol{\alpha}}) = \mathbf{M}$.*

Note that Theorems 3 and 4 probably carry an extra $\frac{\log\log n}{n}$ in the values of p. This is inherent in the use of McDiarmid's argument.

2 Preliminaries

Throughout the paper, for the sake of clarity of presentation, we systematically omit the floor and ceiling signs when they are not crucial. This paper is organized in the following way. We start with a few standard properties of random graphs in the current section, which will be useful to prove our main results. We prove Theorems 1 and 2 in the next two sections, and prove Theorems 3 and 4 in Section 5. We defer the proofs of some structural lemmas for random graphs to Section 6.

In the following we distinguish between events of two kinds. Those that do not depend on $\mathbf{m}$ and we show that they occur with probability $1 - o(1)$, i.e., w.h.p. Those events that do depend on $\mathbf{m}$ where we need to prove that they occur with probability $1 - o(n^{-r})$ in order to use the union bound on the 'bad' events over all choices of $\mathbf{m} \in \mathbf{M}$ (note that $|\mathbf{M}| = \Theta(n^r)$). We say that such events occur *w.v.h.p.*

The following lemma will be used in the proof of both Theorems 1 and 2.

Lemma 5. *Suppose that $p = \frac{(c+o(1))\log n}{n}$ where c is constant. Then the following properties hold in $G_{n,p}$:*

B1 *Suppose that $S \subseteq [n]$ and $|S| = \Omega(n)$. For a vertex $v \in [n]$, we let $d_S(v)$ denote the number of neighbors of v in S. Then, $|B(p,S)| \le n^{1-c|S|/4n}$ w.v.h.p., where $B(p,S) = \left\{v \in [n] : d_S(v) \le \frac{c|S|\log n}{20n}\right\}$.*

B2 *Let* **SMALL** $= B(p,[n])$*. Then w.h.p., $v, w \in$* **SMALL** *implies that $dist(v,w) \ge 3$ in $G_{n,p}$. (Here dist refers to graph distance.)*

B3 *Fix S as in B1. Then w.v.h.p., every $v \in [n]$ is within distance 10 of at most $\frac{10rn}{c|S|}$ vertices in $B(p,S)$.*

B4 *If $p = \frac{\log n + r\log\log n + \omega}{n}$ with $\omega = \omega(n) \to \infty$ as $n \to \infty$, then $G_{n,p}$ has minimum degree at least $r+1$ w.h.p.*

B5 *W.v.h.p., there exists an edge between S_1 and S_2 for every $S_1, S_2 \subseteq [n]$ such that $|S_1|, |S_2| \ge \frac{n(\log\log n)^2}{\log n}$ and $S_1 \cap S_2 = \emptyset$.*

This lemma is proved in Section 6.1.

3 Proof of Theorem 1

Fix a vector $\mathbf{m} \in \mathbf{M}_\beta$ and let $\mu_i = m_i/n$ for $i = 1, 2, \ldots, r$ and let $\mu_{\min} = \min\{\mu_i\}$. Partition the vertex set $[n]$ into $V_1, V_2, \ldots, V_r$, where V_1 contains the first m_1 elements (i.e., $V_1 = [m_1]$), V_2 contains the next m_2 elements, and so on.

We let $p_1 = \frac{\log n + r\log\log n + \omega/2}{n}$ and have $1 - p = (1-p_1)(1-p_2)$ so that $p_2 \approx \omega/2n$. Let $d(v)$ denote the degree of v in G_{n,p_1} and let $d_i(v) = |\{u \in V_i : uv \text{ has color } i\}|$, for $i = 1, 2, \ldots, r$. Define the following sets:

$$A_{\mathbf{m}} = \left\{v : \exists i \in [r], d_i(v) \le \frac{\mu_i \alpha_i \log n}{25}\right\}. \tag{1}$$

$$B = \left\{v : d(v) \le \frac{50r}{\beta\alpha_{\min}}\right\}. \tag{2}$$

Lemma 6.
(a) W.h.p. simultaneously, for all $\mathbf{m} \in \mathbf{M}_\beta$,

$$|A_{\mathbf{m}}| \le rn^{1-\alpha_{\min}\mu_{\min}}. \tag{3}$$

(b) W.h.p. simultaneously, for all $\mathbf{m} \in \mathbf{M}_\beta$, every $v \in [n]$ is within distance 10 of at most $\frac{10r}{\alpha_{\min}\mu_{\min}}$ vertices of $A_{\mathbf{m}}$.

(c) The following is w.h.p. true simultaneously for all choices of $\mathbf{m} \in \mathbf{M}_\beta$: every pair of vertices $u \in A_{\mathbf{m}}$ and $w \in B$ are at distance at least three in G_{n,p_1}.

Parts (a) and (b) of this lemma are straightforward corollaries of Properties **B1** and **B3** respectively. Proving Part (c) is more subtle and it is done in Section 6.2.

In some sense, the vertices v in the set A_m are dangerous (and we need to be careful how we place them in the Hamilton cycle). We do this by first finding vertex disjoint paths of length two with the vertices in $A_{\mathbf{m}}$ as middle vertex and then later we make sure to include those paths in the Hamilton cycle.

We now give an outline of the way we will construct a Hamilton cycle in several steps. Later we will elaborate on why these steps are valid, assuming the high probability events stated in Lemmas 5 and 6.

Step 1 For each $v \in A_{\mathbf{m}}$, choose a path $Q_v = (w_1, v, w_2)$ where $w_1, w_2 \notin A_{\mathbf{m}}$ and both edges of Q_v have the same color, c_j, say. Move v, w_1, w_2 to V_j and move three vertices in $V_j \setminus (A_{\mathbf{m}} \cup N(v))$ to the sets originally containing v, w_1, w_2, in order to keep the sizes of the V_i's unchanged. Let $\mathcal{Q} = \{Q_v\}$ and let $\mathcal{Q}_i \subseteq \mathcal{Q}$ be the set of paths contained in $V_i, i = 1, 2, \ldots, r$. The paths $Q_v, v \in A_{\mathbf{m}}$ can be chosen to be vertex disjoint.

Following this step, for $i = 1, 2, \ldots, r$ let G_i denote the subgraph of G_{n,p_1} with vertex set V_i and edges of color i.

Step 2 For each $1 \leq i \leq r$, execute a restricted *rotation-extension* algorithm where at all times we ensure that for all $Q \in \mathcal{Q}_i$, the current path either contains Q or is vertex disjoint from Q. In this way, create a Hamilton path H_i through V_i for $i = 1, 2, \ldots, r$.

Step 3 Connect the Hamilton paths constructed in Step 2 into a Hamilton cycle.

3.1 Validation of Step 1

Property **B4** and the pigeonhole principle imply that for each $v \in A_{\mathbf{m}}$, we can choose two neighbors w_1, w_2 such that the edges vw_1, vw_2 have the same color, c_j, say. We must now consider disjointness of the paths w_1vw_2. If $v_1, v_2 \in B$ then Property **B2** ensures that Q_{v_1}, Q_{v_2} are vertex disjoint. If $v_1 \in A_{\mathbf{m}}$ and $v_2 \in B$ then we can use Lemma 6(c) to argue that Q_{v_1}, Q_{v_2} are vertex disjoint. If $v \in A_{\mathbf{m}} \setminus B$, then Lemma 6(b) implies that we have at least $\frac{50r}{\beta\alpha_{\min}} - \frac{20r}{\alpha_{\min}\mu_{\min}} \geq \frac{30r}{\beta\alpha_{\min}}$ choices of neighbors w which are not used in any other paths in $\mathcal{Q}$. Once again by pigeonhole principle, we can choose two neighbors w_1, w_2 such that vw_1, vw_2 have the same color and will give us a required path Q_v.

3.2 Validation of Step 2

Call a neighbor w of a vertex v *bad* if $(\{w\} \cup N(w)) \cap A_{\mathbf{m}} \neq \emptyset$. In Step 1, only the bad neighbors of $v \notin A_{\mathbf{m}}$ can reduce the V_i-neighborhood of v. Lemma 6(b) implies that for each $v \notin A_{\mathbf{m}}$, the number of neighbors of v in G_i can drop by at most $\frac{20r}{\alpha_{\min}\mu_{\min}}$. Thus, the vertices of G_i, not in $A_{\mathbf{m}}$, have degree at least $\frac{\mu_{\min}\alpha_{\min}\log n}{25} - \frac{20r}{\alpha_{\min}\mu_{\min}} \geq \frac{\mu_{\min}\alpha_{\min}\log n}{26}$.

3.2.1 Expansion properties

We need to show that each G_i has certain expansion properties. We have the following properties of $G_i \subseteq G_{n,p_1}$, which will be verified in Section 6. For a set $S \subseteq V_i$, let $N_i(S) = \{w \in V_i \setminus S : \exists v \in S \text{ } s.t. \text{ } vw \in E(G_i)\}$.

Lemma 7. *The following properties hold for all $1 \leq i \leq r$ w.v.h.p.*

(a) For every set $S \subseteq V_i \setminus A_{\mathbf{m}}$ with $|S| \leq \alpha_{\min}\mu_{\min}n/200\log n$, we have that $|N_i(S)| \geq |S|\mu_{\min}\alpha_{\min}\log n/100$.

(b) For every set $S \subseteq V_i \setminus A_{\mathbf{m}}$ with $|S| \leq \mu_{\min}^2\alpha_{\min}^2 n/10^5$, we have that $|N_i(S)| \geq 3|S|$.

(c) G_i is connected.

This lemma is proved in Section 6.3.

3.2.2 Step 2: Constructing Hamilton paths in G_i

We now validate Step 2 in a stronger sense. More precisely, we prove that there are many Hamilton paths in each G_i. This will later be of use in gluing them together to obtain a Hamilton cycle of G. Let

$$n_0 = \frac{\mu_{\min}^2 \alpha_{\min}^2 n}{10^5}.$$

Lemma 8. *W.h.p. simultaneously, for all $\mathbf{m} \in \mathbf{M}_\beta$, the following two events occur in $G_{n,p}$: Each G_i has at least n_0 vertices v for which there are at least n_0 Hamilton paths with one end point v such that the other end points are pairwise distinct.*

Proof. Although by now extension-rotation is a standard procedure for attacking Hamilton cycle problems, we briefly describe it here. Given a path $P = (x_1, x_2, \dots, x_k)$ an *extension* is simply the creation of a new path $P + (x_k, y)$ where $x_k y$ is an edge and $y \notin V(P)$. If $1 < i \le k-2$ and $x_k x_i$ is an edge then we create a new path $(x_1, x_2, \dots, x_i, x_k, x_{k-1} \dots, x_{i+1})$ of the same length as P by a *rotation* with *fixed endpoint* x_1. We let $END = END(P, x_1)$ denote the set of vertices that can be the endpoint of a path created by a sequence of rotations.

We modify the above constructions on G_i by adding the restriction that for each $Q \in \mathcal{Q}_i$, the paths generated either contain Q or are vertex disjoint from Q. We can do this by always adding or deleting both edges of such a path in any change. Any rotation that would result in deleting one edge of such a path is neglected. Under the assumption that P is a longest path, so that there are no extensions, Pósa [20] proved that $|N(END)| < 2|END|$ and then accounting crudely for the interiors of the paths of $\mathcal{Q}$ we see that the endpoint sets satisfy

$$|N(END)| \le 2|END| + \min\left\{2|\mathcal{Q}_i|, \frac{20r}{\mu_{\min}\alpha_{\min}}|END|\right\}.$$

Since $|\mathcal{Q}_i| \le |A_{\mathbf{m}}| \ll n/\log^2 n$ for each $\mathbf{m} \in \mathbf{M}_\beta$ (by (3)), we can deduce from Lemma 7 that w.h.p. for each $\mathbf{m} \in \mathbf{M}_\beta$, the endpoint sets are of size at least n_0. We show next that with the use of G_{n,p_2}, we can prove that each G_i has a Hamilton cycle w.h.p. More precisely, suppose that $E(G_{n,p_2}) = F = \{f_1, f_2, \dots, f_\sigma\}$ where w.h.p. $\sigma \ge \omega n/3$. Partition F into $r+1$ sets $F_0, F_1, \dots, F_r$ of almost equal size.

Condition on the high probability events in Lemmas 5, 6, and 7. Now given a path P of length $\ell < m_i - 1$ in G_i, we make a series of rotations with one endpoint fixed until either the endpoint set END reaches n_0 in size, or we generate a path that can be extended. Assume the former. Then for each $v \in END$, there is a path P_v of length ℓ and one endpoint being v. We then try to find a longer path by doing rotations and extensions with v as the fixed endpoint. We do this for all $v \in END$. If we never extend a path then we terminate with n_0 vertices END and for each $v \in END$, a set of n_0 paths with distinct endpoints END_v. Observe next that adding an edge $f = vw$ where $w \in END_v$ will enable us to create a path of length $\ell + 1$. This is because adding f creates a cycle C of length $\ell + 1$. Because G_i is connected we can find a path of length $\ell + 1$ by adding an edge $g_1 = ww_1$ and deleting an edge $g_2 = ww_2$ where $g_1 \in E(C)$ and $w_2 \notin V(C)$. The edge f is referred to as a *booster*.

If we go through the edges of F_i one by one, we see that each edge has probability at least $\gamma = \alpha_{\min} n_0^2 / 3n^2$ of being a booster. This bound holds given the previous edges examined. Thus the probability we fail to obtain a Hamilton path in each G_i is bounded by the probability that the binomial random variable $B(\sigma/(r+1), \gamma) < n$, which is bounded by $e^{-\Omega(n)}$. After a simple application of union bound, this shows that w.h.p. for each $\mathbf{m} \in \mathbf{M}_\beta$, we can find Hamilton paths in each G_i.

Finally, note that once we have found one Hamilton path we can find n_0 vertices, each of which are the endpoints of n_0 Hamilton paths. Note that w.h.p. we will only need to examine $O(n)$ edges of F in this construction. The remaining edges can be used to glue Hamilton paths into a Hamilton cycle of $G = G_{n,p}$. $\square$

3.3 Step 3: Connecting the Hamilton paths together

In the final step, our goal is to show that w.h.p. we can choose Hamilton paths P_i of G_i with endpoints x_i and y_i for $i = 1, 2, \ldots, r$, such that for each i, the edge $y_i x_{i+1}$ exists and is colored with c_{i+1}. We begin by choosing n_0 hamilton paths in G_1 all with vertex x'_1, say as one endpoint.

Assume inductively, that we have chosen $P_1, P_2, \ldots, P_{i-1}$ plus n_0 Hamilton paths $Q_1, Q_2, \ldots, Q_{n_0}$ of G_i, all with endpoint x_i (or x'_1 if $i = 1$). Now choose a set END_{i+1} of size n_0 such that each $v \in END_{i+1}$ is the endpoint of n_0 Hamilton paths of G_{i+1} with distinct endpoints. We now use the edges of G_{n,p_2} to find a vertex $x_{i+1} \in END_{i+1}$ such that there is an edge yx_{i+1} of color c_{i+1}, where $y \neq x_i$ is an endpoint of one of the paths $Q_1, Q_2, \ldots, Q_{n_0}$. As we go through the edges of F_0 we see that we find such an edge with probability at least γ. It follows that w.h.p. for each $\mathbf{m} \in \mathbf{M}_\beta$, we find the required edge after at most $\log^2 n$ steps. Repeating this argument r times we see that w.h.p. for each $\mathbf{m} \in \mathbf{M}_\beta$, there are n_0 Hamilton paths of G made up of correctly colored paths of length $m_1 - 1, m_2, \ldots, m_{r-1}$ plus one of n_0 Hamilton paths $H_1, H_2, \ldots, H_{n_0}$ of G_r, all with x_r as an endpoint.

We now do rotations in G_1, starting with P_1 and keeping the endpoint y_1 fixed and generate n_0 paths $J_1, J_2, \ldots, J_{n_0}$. We then search for an edge $y_r x_1$ of color c_1 such that y_r is an endpoint of an H_k and x_1 is an endpoint of a J_l. We can find one w.h.p. for each $\mathbf{m} \in \mathbf{M}_\beta$ by examining $\log^2 n$ edges of F_0 and we are done with the proof of Theorem 1.

4 Proof of Theorem 2

To prove Theorem 2, we will deal with small and large m_i separately. Fix a vector $\mathbf{m} \in \mathbf{M}$. Let $J = \{j : m_j \leq n/4r\}$ and let $\sigma = |J| < r$. Assume without loss of generality that $J = [\sigma]$. We focus initially on the small m_j and construct paths P_j for each $j \in J$ in such a way that we can construct the remaining long paths using the previous strategy to glue the long and short paths together.

We partition $[n]$ into $V^*, V_{\sigma+1}, \ldots, V_r$, where V^* contains the first $\frac{n}{2}$ elements, $V_{\sigma+1}$ contains the next $\frac{n}{2} \cdot \frac{m_{\sigma+1}}{\sum_{i>\sigma} m_i}$ elements, $V_{\sigma+2}$ contains the next $\frac{n}{2} \cdot \frac{m_{\sigma+2}}{\sum_{i>\sigma} m_i}$ elements, and so on. We construct internally vertex-disjoint short paths with the colors in J, only using the vertices in V^*. We further partition the set V^* to accommodate different colors in J. We partition V^* into $\sigma + 1$ almost equal parts $V_0, V_1, \ldots, V_\sigma$, where V_0 contains the first $\frac{n}{2(\sigma+1)}$ elements, V_1 contains the next $\frac{n}{2(\sigma+1)}$ elements, and so on. (Observe that if $j \leq \sigma$ then $m_j \leq \frac{n}{4r} < \frac{n}{2(\sigma+1)}$.) Let μ_i be such that $|V_i| = \mu_i n$ for $i = 0, 1, \ldots, r$.

We let $p_1 = \frac{\log n + \log\log n + \omega/2}{\alpha_{min} n}$ and then let p_2, p_3 satisfy $1 - p = (1 - p_1)(1 - p_2)(1 - p_3)$ so that $p_2 = p_3 \approx \omega/4\alpha_{min} n$. For each $v \in [n]$, let $d_0(v) = |\{u \in V_0 : uv \text{ has color } c_1\}|$ and for $i = 1, 2, \ldots, r$, let $d_i(v) = |\{u \in V_i : u, v \text{ has color } c_i\}|$.

We now give an outline of the way we will construct a Hamilton cycle in several steps. The strategy is similar to that used for the proof of Theorem 1, except for the way we deal with short paths. Let now

$$A_{\mathbf{m}} = \left\{ v : \exists i > \sigma : d_i(v) \le \frac{\mu_i \alpha_i \log n}{25} \right\}.$$
$$B = \left\{ v : d_r(v) \le \frac{400r}{\beta \alpha_{\min}} \right\}.$$

Lemma 9. *W.h.p. simultaneously, for all choices of* $\mathbf{m}$*, every pair of vertices* $u \in A_{\mathbf{m}}$ *and* $v \in B$ *are at distance at least three.*

This lemma is proved in Section 6.4.

Step 1 For each $v \in A_{\mathbf{m}}$, choose two neighbors $w_1, w_2 \notin A_{\mathbf{m}}$ of v such that vw_1 and vw_2 have the color c_r and let Q_v be the path w_1vw_2. Then, move v, w_1, and w_2 to V_r. Lemma 9 and Properties **B3** and **B4** and the fact that $|V_r| = \Omega(n)$ implies that we can choose the pairs w_1, w_2 such that the paths in $\mathcal{Q} = \{Q_v\}$ are vertex disjoint. After this step for $i = 1, \ldots, \sigma$, denote the new V_i's by V_i'.

Step 2 Construct a path P of the form $P_1P_2 \ldots P_\sigma$, where P_j is a path using only the vertices of V_j' as internal vertices, and with edges of color c_j. P_1 has length $m_1 - 1$ and P_j has length m_j for $j = 2, \ldots s$. For all vertices in V^* that are not used in P, place them arbitrarily into V_i's and denote them by V_i'''s and ensure that each V_i' has size m_i, for $i \notin J$.

Let G_i denote the subgraph induced by the edges of color c_i in V_i', for $i = 1, 2, \ldots, r$. (Note that it is possible to have $i \notin J$ and $j \in J$ such that V_i' and V_j' are not disjoint, however this does not pose any problem in our arguments.) For $i = \sigma + 1, \ldots, r$, the graphs G_i have minimum degree at least $\frac{\mu_i \alpha_i \log n}{25} - \frac{20r}{\mu_{\min}} \ge \frac{\mu_{\min} \alpha_{\min} \log n}{26}$ by construction.

Step 3 For each $\sigma + 1 \le i \le r$, execute the restricted rotation-extension algorithm to ensure that for all $Q \in \mathcal{Q}$, the current path either contains Q or is vertex disjoint from Q. In this way, create a Hamilton path H_i of color i.

Step 4 Connect the Hamilton paths into a Hamilton cycle.

We already validated Step 1 in its description. We now elaborate on and validate Step 2. To obtain $P_1, P_2, \ldots, P_\sigma$ we use the following lemma. (See Ben-Eliezer, Krivelevich, and Sudakov [3].)

Lemma 10. *Let* G *be a connected graph with* N *vertices such that for every pair of disjoint sets* S *and* T *with* $|S| = |T| = M$ *there is an edge joining* S *and* T*. Then for every* $v \in V(G)$*, there is a path of length* $N - 2M$ *with one endpoint* v*.*

We need the following lemma which enables us to apply Lemma 10 on the graphs G_i for $i = 1, 2, \ldots, \sigma$.

Lemma 11. *W.h.p. simultaneously, for all choices of* $\mathbf{m}$*, for each* $i = 1, 2, \ldots, \sigma$*, we have the following:*

(a) G_i *is connected and*

(b) There is an edge in G_i *between every pair of disjoint sets* S *and* T *with* $|S| = |T| = n_1 = \frac{n(\log \log n)^2}{\log n}$.

This will be proved in Section 6.5.

4.1 Construction of paths $P_1, P_2, \ldots, P_\sigma$

We condition on the high probability events in the above lemmas. We assume here that $m_i \geq 1$ for all i, because otherwise we are just dealing with fewer colors. Fix a starting vertex $v_1 \in V_1'$. It follows from Lemmas 10 and 11 that there is a path P_1 of length $m_1 - 1$ starting at v_1 and using only the vertices in V_1', all of whose edges have color c_1 (we use Lemma 10 with $N = \frac{n}{2(\sigma+1)}$ and $M = n_1$). Suppose then that we have constructed paths $P_1, P_2, \ldots, P_k, k < \sigma$ where P_{j-1}, P_j share an endpoint and the edges of P_j are colored c_j for $1 \leq j \leq k$. (If $m_1 = 1$ then we can take P_0 to be an endpoint of P_1.) Let u_k denote the endpoint of P_k that is not in P_{k-1} and v_{k+1} be a c_{k+1}-neighbor of u_k in V_{k+1}' (such a neighbor exists because of the fact that V_k' contains only vertices outside of $A_{\mathbf{m}}$). Then it follows from Lemmas 10 and 11 that there is a path P_{k+1} of length m_{k+1} starting at v_{k+1} and using only the vertices in V_{k+1}', all of whose edges have color c_{k+1}. We end the path P_σ with a vertex $v_{\sigma+1} \in V_{\sigma+1}$.

The path P_1 has length $m_1 - 1$. There are at least $\ell_0 = \frac{1}{2}\alpha_{k+1} \log n$ choices of vertex $v_0 \in V_0$ such that $v_0 v_1$ is an edge of color c_1. This finishes the construction of ℓ_0 choices for P_1 and a single choice of each of $P_2, \ldots, P_\sigma$ giving us ℓ_0 choices for $P = P_1 P_2 \ldots P_\sigma$ where the starting vertices v_0 are distinct. Denote this collection of paths by $\mathcal{P}$. The paths $P = P_1 P_2 \ldots P_\sigma$ use vertices from $V^* \setminus A_{\mathbf{m}}$ except one of the end points $v_{\sigma+1} \in V_{\sigma+1}$.

4.2 Construction of $P_{\sigma+1}, \ldots, P_r$ and the Hamilton cycle

Steps 3 and 4 can be validated in the exact same way as was done in Sections 3.2.2 and 3.3 and Lemma 8 continues to hold. We can therefore w.h.p. construct a Hamilton path P in $G_{n,p_1} \cup G_{n,p_2}$ with the correct color scheme. We can in fact find n_0 choices for one endpoint v of P (corresponding to P_r) and we have ℓ_0 choices for the other endpoint w (corresponding to P_1). The probability that none of the $n_0 \ell_0$ possible edges vw occur in $E(G_{n,p_2})$ is at most $(1-p_2)^{n_0 \ell_0} \leq e^{-c\omega \log n} = o(n^{-r})$ for some absolute constant $c > 0$. This completes the proof of Theorem 2.

5 Proof of Theorems 3 and 4

We can consider both theorems simultaneously. Let $q = p(1-p)$ and note that $G_{n,q}^{\boldsymbol{\alpha}}$ satisfies the conditoins of Theorems 1 and 2. Suppose that the claim in Theorem 1 (or 2) holds with probability $1 - \varepsilon_n$ where $\varepsilon_n \to 0$ as $n \to \infty$. Let $\Omega(\alpha)$ denote the set of colorings of the edges of K_n with the following property: if $c \in \Omega(\alpha)$ is fixed and we construct $G_{n,q}$ then the claimed Hamilton cycles exist with probability at least $1 - \varepsilon_n^{1/2}$. Let $\mathcal{F}$ denote the failure of the property described in Theorem 1 (or 2). Then,

$$\begin{aligned}\varepsilon_n = \mathbb{P}(\mathcal{F}) &= \mathbb{P}(\mathcal{F} \mid c \in \Omega(\alpha))\mathbb{P}(c \in \Omega(\alpha)) + \mathbb{P}(\mathcal{F} \mid c \notin \Omega(\alpha))\mathbb{P}(c \notin \Omega(\alpha)) \\ &\geq \varepsilon_n^{1/2} \mathbb{P}(c \notin \Omega(\alpha)).\end{aligned}$$

It follows that $\mathbb{P}(c \notin \Omega(\alpha)) \leq \varepsilon_n^{1/2}$. So, let c be a fixed coloring in $\Omega(\alpha)$ that we will use to color edges. Now let $e_i = \{u_i, v_i\}, i = 1, 2, \ldots, N = \binom{n}{2}$ be an arbitrary ordering of the edges of K_n. We couple the construction of $G_{n,q}^{\boldsymbol{\alpha}}, q = p(1-p)$ with $D_{n,p}^{\boldsymbol{\alpha},*}$, a subgraph of $D_{n,p}^{\boldsymbol{\alpha}}$. For each i, we generate two independent Bernouilli random variables, B_{u_i,v_i} and B_{v_i,u_i}, each with probability of success p. If exactly one of these variables has value one then we include the corresponding directed edge in $D_{n,p}^{\boldsymbol{\alpha},*}$ and give it the color $c(e_i)$.

Consider the following sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$ of random edge colored digraphs. In Γ_i, for $j \leq i$, we first tentatively include (u_j, v_j) and (v_j, u_j) independently with probability p and include the corersponding edge only if exactly one is chosen. In which case give it color $c(e_j)$. For $j > i$ we include both $(u_j, v_j), (v_j, u_j)$ with probability q and neither of $(u_j, v_j), (v_j, u_j)$ with probability $1 - q$.

Now Γ_0 is distributed as $G^{\boldsymbol{\alpha}}_{n,q}$ and Γ_N is distributed as a subgraph of $D^{\boldsymbol{\alpha}}_{n,p}$. We argue that

$$\mathbb{P}(\Gamma_i \in \mathcal{F}) \geq \mathbb{P}(\Gamma_{i+1} \in \mathcal{F}) \text{ for } 0 \leq i < N. \tag{4}$$

Given (4) we see that we have Theorems 3 and 4. So let us verify (4). Following [18], we condition on the existence or non-existence of (u_j, v_j) or (v_j, u_j) for $j \neq i + 1$, in both models, $\Gamma_i, \Gamma_{i+1|}$. Let $\mathcal{C}$ denote this conditioning. Then, one of (a), (b), (c) below occurs:

(a) There is a desiredly colored Hamilton cycle (in both Γ_i, Γ_{i+1}) that does not use either of (u_{i+1}, v_{i+1}) or (v_{i+1}, u_{i+1}).

(b) Not (a) and there exists a desiredly colored Hamilton cycle if at least one of (u_{i+1}, v_{i+1}) or (v_{i+1}, u_{i+1}) is present, or

(c) There does not exist a desiredly colored Hamilton cycle even if both of (u_{i+1}, v_{i+1}) and (v_{i+1}, u_{i+1}) are present.

(a) and (c) give the same conditional probability of Hamiltonicity in Γ_i, Γ_{i+1}, 1 and 0 respectively. In Γ_i (b) happens with probability q. In Γ_{i+1} we consider two cases (i) exactly one of $(u_{i+1}, v_{i+1}), (v_{i+1}, u_{i+1})$ yields Hamiltonicity and in this case the conditional probability is again q and (ii) either of $(u_{i+1}, v_{i+1}), (v_{i+1}, u_{i+1})$ yields Hamiltonicity and in this case the conditional probability is $1 - (1 - p)^2 - p^2 = 2q$. Note that we will never require that **both** $(u_{i+1}, v_{i+1}), (v_{i+1}, u_{i+1})$ occur. In summary, we have proved that

$$\mathbb{P}(D^{\boldsymbol{\alpha},*}_{n,p} \in \mathcal{F}) \leq \varepsilon_n^{1/2}. \tag{5}$$

6 Structural lemmas

In this section, we prove the various structural properties of random graphs that have been used throughout this paper. We begin with the following: let $0 < \gamma < 1$ and $g = \lfloor 1/\gamma \rfloor$ and let $W_1, W_2, \ldots, W_g$ be consecutive intervals in $[n]$ where $|W_i| = \lfloor \gamma n \rfloor$ for $1 \leq i < g$. Let $d_{i,j}(v)$ denote the number of neighbors w of vertex v in W_j such that $c(vw) = c_i$. Here $G = G_{n,p_1}$ with $p_1 \approx \frac{c \log n}{n}, c \geq 1$. Let

$$A^*_{\mathbf{m}} = \left\{ v : \exists i \in [r], j \in [g-1] : d_{i,j}(v) \leq \frac{\gamma c \alpha_i \log n}{20} \right\}.$$

$$B_1 = \left\{ v : d(v) \leq \frac{5r}{\gamma \alpha_{\min}} \right\}.$$

$$B_2 = \left\{ v : d_r(v) \leq \frac{5r}{\gamma \alpha_{\min}} \right\}.$$

Lemma 12.

(a) In Section 3 with $c = 1$, $\gamma = \beta/10$, we have that $A_{\mathbf{m}} \subseteq A^*_{\mathbf{m}}$ and $B_1 = B$.

(b) In Section 4 with $c = 1/\alpha_{\min}$, $\gamma = 1/80r$, we have that $A_{\mathbf{m}} \subseteq A^*_{\mathbf{m}}$ and $B_2 = B$.

Proof. It is clear that $B_1 = B$ in (a) and $B_2 = B$ in (b).

(a) If $v \in A_{\mathbf{m}}$, then there is some $i \in [r]$ such that $d_i(v) \leq \frac{\mu_i \alpha_i \log n}{25}$, i.e., there are at most $\frac{\mu_i \alpha_i \log n}{25}$ many c_i-colored edges between v and V_i. Recall that V_i was defined so that it consists of $m_i \geq \beta n$ consecutive elements from $[n]$. Hence, there are j, k such that $P_j \cup P_{j+1} \cup \cdots \cup P_{j+k-1} \subseteq V_i$ and $V_i \setminus (P_j \cup P_{j+1} \cup \cdots \cup P_{j+k-1})$ has at most $2\beta/10$ elements. Thus,

$$k\gamma n = |P_j \cup P_{j+1} \cup \cdots \cup P_{j+k-1}| \geq m_i - \beta/5 \geq 4m_i/5.$$

Suppose for the sake of contradiction that $v \notin A^*_{\mathbf{m}}$. Then, $d_{i,j+l}(v) > \frac{\gamma \alpha_i \log n}{25}$ for all $l = 0, 1, \ldots, k-1$. Thus, we have the following (recall that $m_i = \mu_i n$):

$$d_i(v) \geq \sum_{l=0}^{k-1} d_{i,j+l}(v) > \frac{k\gamma\alpha_i \log n}{20} \geq \frac{\mu_i \alpha_i \log n}{25},$$

giving us a contradiction.

(b) The proof is essentially the same as for part (a). □

Lemma 13.

(a) If $c = 1$, $\gamma = \beta/10$, then w.h.p. every pair of vertices $u \in A^*_{\mathbf{m}}$ and $v \in B_1$ are at distance at least three.

(b) If $c = \alpha_{\min}$, $\gamma = 1/80r$, then w.h.p. every pair of vertices $u \in A^*_{\mathbf{m}}$ and $v \in B_2$ are at distance at least three.

Proof.

(a) The probability that there are vertices $u \in A^*_{\mathbf{m}}$ and $v \in B_1$ at distance at most two can be bounded by

$$\sum_{i=1}^{2} n^i \left(\frac{c\log n}{n}\right)^i \sum_{k=1}^{5r/\gamma\alpha_{\min}} \binom{n-2-i}{k} p^k (1-p)^{n-2-i-k} \sum_{i=1}^{r}\sum_{j=1}^{l}\sum_{k=1}^{\gamma c \alpha_i \log n/20} \binom{\gamma n}{k} (p\alpha_i)^k (1-p\alpha_i)^{\gamma n - 2 - i - k}$$
$$= o(1).$$

(b) This is similar. □

6.1 Proof of Lemma 5

B1 Let $Z = |B(p,S)|$ and $L = \log n$ and $A = \frac{c|S|\log n}{20n}$. Then,

$$\mathbb{E}\left(\binom{Z}{L}\right) \le \binom{n}{L}\left(\sum_{i=0}^{A}\binom{|S|-L}{i}p^i(1-p)^{|S|-L-i}\right)^L \tag{6}$$
$$\le \binom{n}{L}\left(2\binom{|S|}{A}p^A(1-p)^{|S|-A}\right)^L$$
$$\le \binom{n}{L}\left(2\left(\frac{|S|e}{A}\cdot\frac{c\log n}{n}\cdot e^{o(1)}\right)^A e^{-c|S|\log n/n}\right)^L$$
$$\le \binom{n}{L}((21e)^{\log n/20}n^{-1+o(1)})^{Lc|S|/n}$$
$$\le \frac{n^{L-2c|S|L/3n}}{L!}.$$

Explanation for (6)**:** Having chosen a set X of L vertices, we bound the probability that the set is contained in $B(p,S)$ by the probability that the vertices in X each have at most A neighbors in $S \setminus X$.

So, from the Markov inequality

$$\mathbb{P}(Z \ge n^{1-c|S|/4n}) \le \frac{\mathbb{E}\left(\binom{Z}{L}\right)}{\binom{n^{L-c|S|/4n}}{L}} \le \frac{n^{L-o(1)-2c|S|L/3n}}{n^{L-c|S|L/4n}} \le n^{-c|S|L/3n} = o(n^{-r}).$$

B2 Let $\ell = c\log n/20$.

$$\mathbb{P}(\exists v,w \in \textbf{SMALL} : \mathrm{dist}(v,w) \le 3) \le \sum_{j=2}^{3} n^j p^{j-1}\left(\sum_{i=0}^{\ell}\binom{n-j}{i}p^i(1-p)^{n-j-i}\right)^2$$
$$\le \left(cn\log n + c^2 n\log^2 n\right)\cdot(n^{-2/3})^2 = o(1).$$

B3 If this property fails then there is a connected set T of at most $t_0 = 1 + \frac{100rn}{c|S|}$ vertices that contains a set T_1 of size $t_1 = \frac{10rn}{c|S|}$ vertices, each of which has at most $s_0 = \frac{c|S|\log n}{20n}$ neighbors in $S \setminus T$. The probability of this can be bounded by

$$\binom{n}{t_0}t_0^{t_0-2}p^{t_0-1}\binom{t_0}{t_1}\left(\sum_{i=0}^{s_0}\binom{|S|}{i}p^i(1-p)^{|S|-i}\right)^{t_1} \le (c\log n)^{t_0}\left(2\left(\frac{|S|ep}{s_0}\right)^{s_0}e^{-c|S|\log n/n}\right)^{t_1}$$
$$= (c\log n)^{t_0}(2(20e)^{s_0}e^{-20s_0})^{t_1}$$
$$\le (c\log n)^{t_0}e^{-15s_0t_1} = o(n^{-r}).$$

B4 Proof of this can be found in Chapter 3 of [12].

B5 Let $s_1 = \frac{n(\log\log n)^2}{\log n}$. The probability of existence a pair of disjoint sets S_1, S_2 of size s_1 with no edge between them can be bounded by

$$\binom{n}{s_1}^2(1-p)^{s_1^2} \le \left(\frac{n^2e^2}{s_1^2e^{s_1p}}\right)^{s_1} = o(n^{-r}).$$

6.2 Proof of Lemma 6

(a) This follows from Property **B1**.

(b) This follows from Property **B3**.

(c) This follows from Part (a) of Lemmas 12 and 13.

6.3 Proof of Lemma 7

We first prove the following lemma bounding the edge density of small sets.

Lemma 14. *In $G_{n,p}$ with $p \approx \frac{c \log n}{n}$, w.v.h.p. for each $S \subseteq [n]$ satisfying $|S| \leq s_0 = \rho n / \log n$, we have that $e(S) \leq e\rho c |S| \log n$, where $e(S)$ denotes the number of edges contained in S.*

Proof. The probability that there exists a set S with more edges than claimed can be bounded by

$$\sum_{s=e\rho c \log n}^{s_0} \binom{n}{s} \binom{\binom{s}{2}}{e\rho cs \log n} p^{e\rho cs \log n} \leq \sum_{s=e\rho c \log n}^{s_0} \left(\frac{ne}{s} \cdot \left(\frac{s^2 ec \log n}{2e\rho csn \log n} \right)^{e\rho c \log n} \right)^s$$
$$\leq \sum_{s=e\rho c \log n}^{s_0} \left(\left(\frac{s}{n} \right)^{1-2/e\rho c \log n} \left(\frac{1}{2\rho} \right) \right)^{ce\rho s \log n}$$
$$\leq \sum_{s=e\rho c \log n}^{s_0} \left(\frac{e^{o(1)}}{2} \right)^{ce\rho s \log n} = o(n^{-r}).$$

□

(a) Suppose that there exists S with $|S| \leq \alpha_{\min}\mu_{\min} n/200 \log n$ that does not satisfy Part (a) of Lemma 7. Let $T = N_i(S)$. Then $|S \cup T| \leq |S|(1 + \mu_{\min}\alpha_{\min} \log n/100) \leq \mu_{\min}\alpha_{\min} n/199$ and $e(S \cup T) \geq |S|\mu_{\min}\alpha_{\min} \log n/52$, contradicting Lemma 14, with $c = 1$ and $\rho = \alpha_{\min}\mu_{\min}/199$.

(b) Suppose now that S is a set with $\alpha_{\min}\mu_{\min} n/200 \log n \leq |S| \leq \mu_{\min}^2 \alpha_{\min}^2 n/10^5$ and choose $X \subseteq S$ of size exactly $\alpha_{\min}\mu_{\min} n/200 \log n$. Then from (a), we have

$$|N_i(S)| \geq |N_i(X)| - |S| \geq |X|\mu_{\min}\alpha_{\min} \log n/100 - |S| \geq 3|S|.$$

(c) It follows from (b) that every component of G_i has size at least $s_0 = \mu_{\min}^2 \alpha_{\min}^2 n/10^5$. Now apply Property **B5** with $c = \alpha_{\min}$ to show that there cannot be two such large components.

6.4 Proof of Lemma 9

This follows from Part (b) of Lemmas 12 and 13.

6.5 Proof of Lemma 11

Connectivity follows as in the proof of Part (c) of Lemma 7 in Section 6.3. The other condition follows from Property **B5**.

7 Concluding remarks

The ultimate goal is to understand the thresholds for the existence of varying patterns in edge colored random graphs. The hardest question seems to be to find the threshold for the existence of arbitrary patterns. Periodic patterns were dealt with in [1] and [9].

Leaving this problem aside we can still ask for the likely value of $hcp(G_{n,p})$ for all values of p between the threshold for Hamiltonicity and the value in Theorem 2.

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