Fast construction on a restricted budget

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Abstract

We introduce a model of a controlled random graph process. In this model, the edges of the complete graph K_n are ordered randomly and then revealed, one by one, to a player called Builder. He must decide, immediately and irrevocably, whether to purchase each observed edge. The observation time is bounded by parameter t, and the total budget of purchased edges is bounded by parameter b. Builder's goal is to devise a strategy that, with high probability, allows him to construct a graph of purchased edges possessing a target graph property $\mathcal{P}$, all within the limitations of observation time and total budget. We show the following:

- Builder has a strategy to achieve k-vertex-connectivity at the hitting time for this property by purchasing at most $c_k n$ edges for an explicit $c_k < k$; and a strategy to achieve minimum degree k (slightly) after the threshold for minimum degree k by purchasing at most $(1+\varepsilon)kn/2$ edges (which is optimal).
- Builder has a strategy to create a Hamilton cycle at the hitting time for Hamiltonicity by purchasing at most Cn edges for an absolute constant $C > 1$; this is optimal in the sense that C cannot be arbitrarily close to 1. This substantially extends the classical hitting time result for Hamiltonicity due to Ajtai–Komlós–Szemerédi and Bollobás.
- Builder has a strategy to create a perfect matching by time $(1+\varepsilon)n\log n/2$ while purchasing at most $(1+\varepsilon)n/2$ edges (which is optimal).
- Builder has a strategy to create a copy of a given k-vertex tree if $t \geq b \gg \max\{(n/t)^{k-2}, 1\}$, and this is optimal;
- For $\ell = 2k+1$ or $\ell = 2k+2$, Builder has a strategy to create a copy of a cycle of length ℓ if $b \gg \max\{n^{k+2}/t^{k+1}, n/\sqrt{t}\}$, and this is optimal.

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1 Introduction

1.1 The model

The random graph process, introduced by Erdős and Rényi [22, 23], is a stochastic process that starts with an empty n-vertex graph and, at each step, gains a new uniformly selected random edge. At any fixed time t, the process is distributed as the uniform random graph $G(n,t)$. A **graph property** is a family of graphs that is closed under isomorphisms. It is **monotone** if it is closed under addition of edges. A vast body of literature is concerned with finding *thresholds* for various monotone graph properties in the random graph process, namely, with finding time t_c such that the random graph process belongs to $\mathcal{P}$ with high probability (**whp**; that is, with probability tending to 1 as $n \to \infty$) whenever t is much (or somewhat) larger than t_c and does not belong to $\mathcal{P}$ **whp** if t is much (or somewhat) smaller than t_c.

In many cases, if one observes the random graph process at time t above the threshold, the graph has the desired property but, in fact, contains a much sparser subgraph that has the property. For example, one of the outstanding results in this model regards a threshold for Hamiltonicity: Komlós and Szemerédi [37] and, independently, Bollobás [17] showed that if $2t/n - \log n - \log\log n$ tends to infinity, then the random graph process, at time t, contains a Hamilton cycle **whp**. Evidently, not all $\sim n\log n/2$ observed edges must be included in the resulting graph for it to be Hamiltonian (as any Hamilton cycle only uses n edges). Nevertheless, when an edge arrives, it is generally hard to determine whether it will be crucial for Hamiltonicity.

The above motivates the following "online" version of building a subgraph of the random graph process. We think of it as a one-player game, where the player ("**Builder**") has a limited "budget". Edges "arrive" one at a time in random order and are presented to Builder. Whenever he *observes* an edge, he must immediately and irrevocably decide whether to *purchase* it. A non-purchased edge is thrown away and never reappears. The **time** (total number of presented edges) and the **budget** (maximum number of edges Builder can purchase) are both capped. The question is as to whether or not, under the given time and budget constraints, Builder has a strategy that allows him to obtain, **whp**, a particular monotone graph property in the graph of purchased edges.

To demonstrate the model, consider the property of connectedness. Let τ_C be the (random) time at which the random graph process becomes connected. Evidently, if Builder wishes to construct a connected subgraph of the random graph process, he must purchase at least $n-1$ edges. However, in this case purchasing $n-1$ edges suffices: Builder's strategy would be to purchase an edge if and only if it decreases the number of connected components in his graph. That way, Builder maintains a forest, which becomes connected exactly at time τ_C. Therefore, in this example, Builder does not have to "pay" for having to make decisions online. However, this is not always the case. For example, if Builder wants to purchase a triangle and wishes to do so while observing $o(n^2)$ edges, he must purchase (many) more than three edges (see Theorem 1.6).

We denote the underlying random graph process at time t (namely, after t edges have been presented to Builder) by G_t (where the number of vertices, n, is implicit). As we mentioned earlier, G_t is distributed as the uniform random graph $G(n,t)$. The **hitting time** for a monotone graph property $\mathcal{P}$ is the (random) minimum time t for which G_t has $\mathcal{P}$. Builder's graph at time t, denoted B_t, is a subgraph of G_t on the same vertex set that consists of the edges purchased by Builder by time t. A (t,b)**-strategy** of Builder is a (potentially random) function that, for any $s \le t$, decides whether to purchase the edge presented at time s, given B_{s-1}, under the limitation that B_s has at most b edges.

1.2 Our results

Our first result discusses a strategy for constructing a subgraph with a given minimum degree and, for $k \geq 3$, of a given vertex-connectivity, as quickly as possible. For every positive integer k, denote by τ_k the hitting time for minimum degree k in the random graph process. It was proved by Erdős and Rényi [24] that $\tau_k = \frac{n}{2}(\log n + (k-1)\log\log n + O(1))$ **whp** (see Lemma 2.8). Let κ denote vertex-connectivity.

Theorem 1.1 (Minimum degree and vertex-connectivity at the hitting time)**.** *For every positive integer k there exists a constant $\mathfrak{o}_k \in (k/2, 3k/4]$ such that the following holds. If $b \geq \mathfrak{o}_k n + \omega(\sqrt{n}\log n)$ then there exists a (τ_k, b)-strategy B of Builder such that*

$$\lim_{n\to\infty} \mathbb{P}(\delta(B_{\tau_k}) \geq k) = 1.$$

If $k \geq 3$, the same strategy guarantees

$$\lim_{n\to\infty} \mathbb{P}(\kappa(B_{\tau_k}) \geq k) = 1.$$

For $k = 1$, as we mentioned in the introduction, a sufficient and necessary budget for connectivity at the hitting time is $n-1$. For $k = 2$, Theorem 1.3 below implies that a budget of $O(n)$ suffices for 2-connectivity at the hitting time for minimum degree 2. The constant $\mathfrak{o}_k$ in Theorem 1.1 is explicit and computable (see Eq. (1) in Section 3; the first three values in the sequence $\mathfrak{o}_k$ are $\frac{3}{4}$, $\frac{11}{8}$ and $\frac{63}{32}$), and is roughly $k/2$ when k is large (see Corollary 3.2). We believe that at the hitting time τ_k, that constant is optimal (see Conjecture 6.1 and the discussion following it). However, if we allow the time to be optimal only asymptotically, then an asymptotically optimal budget suffices.

Theorem 1.2 (Minimum degree)**.** *Let k be a positive integer and let $\varepsilon > 0$. If $t \geq (1+\varepsilon)n\log n/2$ and $b \geq (1+\varepsilon)kn/2$ then there exists a (t, b)-strategy B of Builder such that*

$$\lim_{n\to\infty} \mathbb{P}(\delta(B_t) \geq k) = 1.$$

The above theorem is tight in the following sense: if $t \leq (1-\varepsilon)n\log n/2$ then the underlying random graph process has, **whp**, isolated vertices; and if $b < kn/2$ then every strategy will fail since Builder will not be able to purchase enough edges to obtain the required minimum degree. We remark that after an earlier version of this paper appeared online, a vertex-connectivity version of Theorem 1.2 that holds for every $k \geq 2$ has been proved by Lichev [42] (see Section 6). Theorem 1.2 follows from Theorem 1.4 below (for $k = 1$) and from Lichev's result (for $k \geq 2$). We keep it here due to its short and simple proof (see Section 4.1). Note that Theorem 1.1 is *not* implied by Lichev's result, as Theorem 1.1 implies a strategy that succeeds **whp** *at the hitting time* τ_k, whereas Lichev's strategy typically requires $\omega(n)$ additional steps (which can, as we demonstrate later in Theorem 1.3 and the discussion following it, make a big difference). An asymptotic version of Lichev's result, according to which for every $\varepsilon > 0$ there exists k_0 such that for every $k \geq k_0$ there exists a strategy to build a k-vertex-connected graph in time $(1+\varepsilon)n\log n/2$ and with budget $(1+\varepsilon)kn/2$ follows from Theorem 1.1 together with the observation that $\mathfrak{o}_k \sim k/2$ when k is large (Corollary 3.2).

We continue to Hamiltonicity. The classical *hitting-time* result of Bollobás [17] and independently of Ajtai, Komlós, and Szemerédi [1] states that the random graph process, **whp**, becomes Hamiltonian as soon as its minimum degree reaches 2. As we reminded earlier, $\tau_2 = \Theta(n\log n)$, while a Hamilton cycle has only n edges. The following result states that there exists an online algorithm choosing only linearly many edges from the first τ_2 edges of the random graph process that still obtains a Hamilton cycle **whp**. Moreover, as we show in the proof, this algorithm is *extremely* simple, deterministic, and requires only local information. The theorem recovers – and substantially strengthens – the aforementioned classical result.

Theorem 1.3 (Hamiltonicity at the hitting time)**.** *There exists $C > 1$ for which there exists a (τ_2, Cn)-strategy B of Builder with*

$$\lim_{n\to\infty} \mathbb{P}(B_{\tau_2} \text{ is Hamiltonian}) = 1.$$

The constant C we obtain in Theorem 1.3 is large, and we made no serious effort to optimise it. A natural question is whether it could be as low as $1+\varepsilon$ for every $\varepsilon > 0$. The answer, found by Anastos [3] (see the concluding remarks there), is negative. As part of the proof of Theorem 1.3, we show that the so-called *random k-nearest neighbour graph* (see Section 3) – for large enough constant k – is **whp** Hamiltonian (Theorem 3.4). This partially solves an open problem of the first author [29, Problem 45]. To prove Theorem 1.3, we prove a stronger hitting time version of Theorem 3.4 (Theorem 3.5) which states that the k-nearest neighbour graph, considered as a process and *stopped as the minimum degree becomes* 2, is **whp** Hamiltonian.

We remark that an immediate corollary of Theorem 1.3 (using Lemma 2.8) is that if $t \geq (1+\varepsilon)n\log n/2$ (namely, slightly after the Hamiltonicity threshold) and $b \geq Cn$ (namely, when the budget is inflated by a constant), then there exists a (t,b)-strategy of Builder that succeeds (**whp**) in building a Hamilton cycle. In an earlier version of this paper that appeared online, we also proved a complementary statement: if $t \geq Cn\log n/2$ and $b \geq (1+\varepsilon)n$, then there exists a "successful" (t,b)-strategy (see Theorem 4.1). After that early version appeared online, Anastos [3] proved, using different and more involved techniques, that Hamiltonicity can be achieved in asymptotically optimal time *and* budget. Namely, he showed that in the above statement, one could take $C = C(\varepsilon) = 1+\varepsilon$. He states this result as a special case in a model that generalises both the budget model discussed here and the so-called Achlioptas process (see discussion in Section 1.4). We leave here our original proof due to its relative shortness and simplicity (see Section 4.2.2). Note that Theorem 1.3 does not follow from the work of Anastos.

For perfect matchings, we show that there exists an asymptotically-optimal-time asymptotically-optimal-budget strategy.

Theorem 1.4 (Perfect matchings)**.** *Suppose n is even. For every $\varepsilon > 0$ if $t \geq (1+\varepsilon)n\log n/2$ and $b \geq (1+\varepsilon)n/2$ then there exists a (t,b)-strategy B of Builder such that*

$$\lim_{n\to\infty} \mathbb{P}(B_t \text{ has a perfect matching}) = 1.$$

An earlier version of this paper that appeared online contained a non-optimal version of Theorem 1.4 (analogous to Theorem 4.1). Recently, Anastos [3] proved a more general version of Theorem 1.4 (see remark after Theorem 4.1), using different and more involved techniques. The proof we provide here is more elementary.

The next two theorems discuss optimal strategies for purchasing small subgraphs. We resolve the problem whenever the target subgraph is a fixed tree or a cycle.

Theorem 1.5 (Small trees)**.** *Let $k \geq 3$ be an integer and let T be a k-vertex tree. If $t \geq b \gg \max\{(n/t)^{k-2}, 1\}$ then there exists a (t,b)-strategy B of Builder such that*

$$\lim_{n\to\infty} \mathbb{P}(T \subseteq B_t) = 1$$

and if $b \ll (n/t)^{k-2}$ then for any (t,b)-strategy B of Builder,

$$\lim_{n\to\infty} \mathbb{P}(T \subseteq B_t) = 0.$$

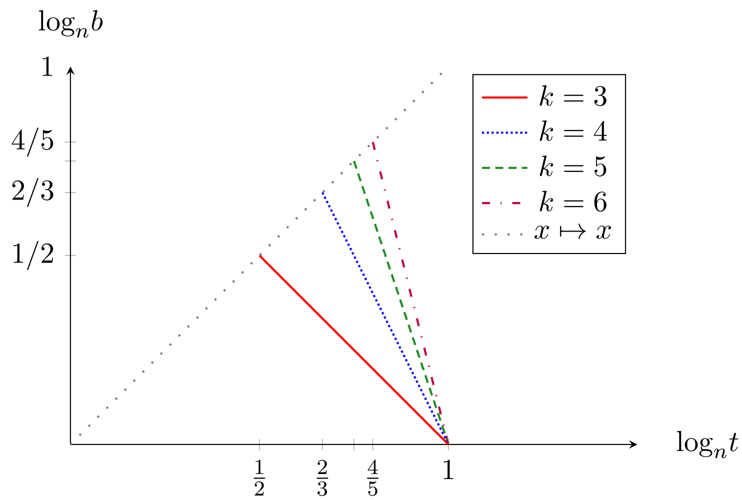

Figure 1: Budget thresholds for k-vertex trees.

In simple (and slightly imprecise) terms, Theorem 1.5 states that the *budget threshold* for a k-vertex tree (for $k \geq 3$) is $b^* = (n/t)^{k-2}$. On a logarithmic scale (with base n), this corresponds to $\log_n b^* = (k-2)(1 - \log_n t)$. This, along with the bound $b \leq t$, is illustrated in Fig. 1.

Remark. We show in the proof of Theorem 1.5 that if $t \gg n$ then, in fact, there exists a (t,b)-strategy for $b = k-1$ that succeeds **whp** in building a copy of a k-vertex tree.

Theorem 1.6 (Short cycles)**.** *Let $k \geq 1$ be an integer and let $H = C_{2k+1}$ or $H = C_{2k+2}$. Write $b^* = b^*(n,t,k) = \max\{n^{k+2}/t^{k+1}, n/\sqrt{t}\}$. If $t \gg n$ and $b \gg b^*$ then there exists a (t,b)-strategy B of Builder such that*

$$\lim_{n\to\infty} \mathbb{P}(H \subseteq B_t) = 1,$$

and if $t \ll n$ or $b \ll b^*$ then for any (t,b)-strategy B of Builder,

$$\lim_{n\to\infty} \mathbb{P}(H \subseteq B_t) = 0.$$

Again, in simple (and slightly imprecise) terms, Theorem 1.6 states that the *budget threshold* for an ℓ-cycle, where $\ell = 2k+1$ or $\ell = 2k+2$ (for $k \geq 1$), is $b^* = \max\{n^{k+2}/t^{k+1}, n/\sqrt{t}\}$. On a logarithmic scale (with base n), this corresponds to $\log_n b^* = \max\{k+2-(k+1)\log_n t, 1 - \frac{1}{2}\log_n t\}$. This is illustrated in Fig. 2. For discussion on the difficulty arising in handling graphs with more than one cycle see Section 6.

1.3 Tools and techniques

Expanders *Expanders* are graphs in which (sufficiently) small sets expand. Namely, these are graphs in which the neighbourhood of each small set is larger than that set by a constant factor[1]. It is well known that connected expanders are helpful in finding Hamilton cycles (see Lemma 2.12); more concretely, connected non-hamiltonian expanders have many “boosters”, namely, non-edges whose addition to the graph creates a graph that is either Hamiltonian or whose longest path is longer (for a comprehensive account, we refer the reader to [38]). Thus, expanders will play a crucial role in the proof of Theorem 1.3 (and also in the proof of Theorem 1.4): Builder will attempt

[1] We need, and hence use, a rather weak notion of expansion; see Section 2.3.

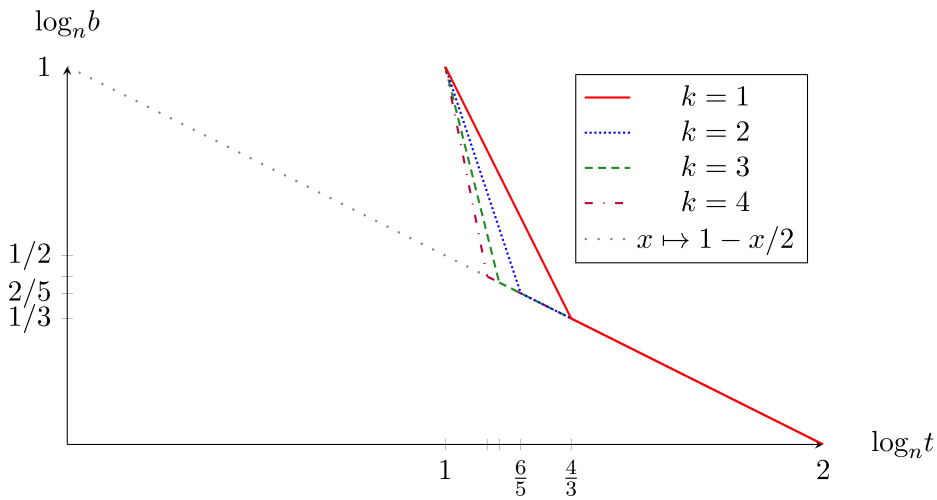

Figure 2: Budget thresholds for C_{2k+1}, C_{2k+2}.

to construct (sparse, and therefore cheap) connected expanders within the random graph process. A standard method for obtaining sparse expanders in random graphs is choosing an appropriate random (sub)graph model. Natural candidates are discussed in the following paragraphs.

Random k-out graphs In the goals described in Theorems 1.1 to 1.4, Builder must achieve a certain minimum degree in his (spanning) graph. The (standard) random graph process is quite wasteful in this regard, as to avoid isolated vertices, a superlinear number of edges must arrive. Thus, a wise Builder would instead construct a much sparser subgraph of the random graph process with the desired minimum degree. A classical sparse graph with a given minimum degree is the *random regular graph* (see, e.g., in [30]). However, such a graph is generally very hard to construct online. A much simpler alternative is the so-called *random k-out graph* (see, e.g., in [30]). In the k-out graph, each vertex chooses, random, independently, and without repetitions, k neighbours to connect to. Thus, the total number of edges in a k-out graph is at most kn, and the minimum degree is at least k. Unfortunately, k-out graphs are not generally subgraphs of the random graph process at the hitting time for minimum degree k. We will therefore analyse a different model, which can be considered the undirected counterpart of the k-out graph (see the next paragraph). Nevertheless, we will exploit the simplicity of the k-out graph to analyse its (slightly more complicated) undirected counterpart.

Random k-nearest neighbour graphs Suppose the edges of the complete graph are endowed with (potentially random; distinct) "lengths". The graph made up only of the k shortest edges incident with each vertex is called the *(random) k-nearest neighbour graph*, and has been studied in various, primarily geometric, contexts (see, e.g., [21]). When the weights are independent uniform random variables supported on $[0, 1]$, the model becomes a symmetric random graph[2], which we denote by $\mathcal{O}_k$ (see Section 3). To prove Theorem 1.1, we devise a simple strategy that emulates $\mathcal{O}_k$. Since the minimum degree of $\mathcal{O}_k$ is obviously k, the statement would follow from a theorem of Cooper and Frieze [20] according to which the number of edges of $\mathcal{O}_k$ is at most cn for some concrete constant $c = c(k) < k$ which is roughly $k/2$ when k is large (see Theorem 3.1 and Corollary 3.2).

[2]The choice of uniform distribution supported on $[0, 1]$ is arbitrary; any distribution without atoms would yield the same random graph.

By observing that $\mathcal{O}_k$ is stochastically dominated by the random k-out graph (Observation 3.7), we prove that it is a connected (Corollary 3.12) expander (Corollary 3.11), and use that to prove that it is Hamiltonian (Theorems 3.4 and 3.5). We use that helpful fact in several places: when aiming to create a Hamilton cycle at the hitting time and with inflated budget, we construct a subgraph of $\mathcal{O}_k$ that is already Hamiltonian; when creating a Hamilton cycle under an asymptotically optimal budget, we emulate copies of $\mathcal{O}_k$ in small parts of the graph, to allow us to *absorb* a long path into a Hamilton cycle; finally, we use a similar approach for the construction of perfect matchings.

High-level arguments for the containment of fixed graphs We give brief proof outlines for Theorems 1.5 and 1.6. The proof for the upper bound on the budget threshold for trees is essentially inductive: given a fixed k-vertex tree T, we let T' be obtained from T by removing a leaf. The inductive argument is that Builder can construct sufficiently many copies of T' while leaving enough time to extend some of them into copies of T. The proof for the lower bound is based on a similar idea: we show that Builder cannot construct a connected component of size at least k. In order to construct such a component, he needs to construct enough smaller components so that in the remaining time, he will, **whp**, connect two of them so that the resulting component will be large enough.

The strategy for obtaining a cycle of length $\ell = 2k + 1$ goes through the construction of many "traps", namely, non-edges whose addition to Builder's graph would create the desired cycle. An optimised way to construct many such traps quickly is by constructing r d-ary trees of depth k (for a correct choice of r, d). If d is large, most pairs of leaves in such a tree are connected by a path of length $2k$, and thus form a trap. The argument for a cycle of length $\ell = 2k + 2$ is similar. We complement the upper bound with two lower bounds; each matches the upper bound in a different regime. We first show a "universal" lower bound, based on the straightforward observation that the number of traps Builder has in his graph is bounded by $O(b^2)$, as he has at most $2b$ vertices of positive degree in his graph. A more involved lower bound fits the earlier regimes (lower values of t). To this aim, we observe that Builder's graph is typically $\Theta(1)$-degenerate. Then, we use this observation, together with an estimate on the number of paths of a fixed length that contain a vertex of "high" degree, to bound from above the number of paths of length ℓ (and hence the number of traps).

1.4 Related work

The study of random graph processes is at the heart of the theory of random graphs, providing a dynamic point of view on their evolution. One of the main questions is to determine the thresholds of monotone increasing graph properties or their (more refined) hitting times. Classical results of this sort include the thresholds of minimum degree and vertex-connectivity k [24], the appearance of a "giant" component [23], Hamiltonicity [37, 17] and the containment of fixed subgraphs [23, 16]. For a comprehensive coverage of the topic, we refer the reader to the books of Bollobás [18], Janson, Łuczak and Ruciński [36], and Frieze and Karoński [30]. In our context, an obvious necessary condition for the existence of a winning strategy for Builder is that the "time" is above the threshold (or at least at the hitting time), as otherwise, the underlying graph process is not guaranteed to have the desired property **whp**. Evidently, if $b = t$ (namely, if the budget equals the time), and t is (sufficiently) above the threshold of the target property, then Builder has a winning strategy: the naive one that purchases each observed edge. Since in many cases Builder can do better, the model can be seen as an extension of the "standard" random graph process.

In the last couple of decades, partly inspired by the remarkable work of Azar, Broder, Karlin and Upfal [4] on *balanced allocations*, there has been a growing interest in *controlled* random processes.

In the context of graph processes, an algorithm is provided with a random flow of edges (usually, but not always, the random graph process) and with (the *offline* version) or without (the *online* version) "peeking into the future", makes a decision that handles that flow by accepting/rejecting edges, by colouring them, or by other means. We mention several related models that fall into this category, mainly in the online version.

The *Achlioptas process*, proposed by Dimitris Achlioptas in 2000, is perhaps the most studied controlled random graph process. In this process, the algorithm is fed by a stream of random k-sets of edges (with or without repetitions) and should pick, immediately and irrevocably ("online"), a single edge to accept, rejecting the others. The algorithm's goal is to make the graph of accepted edges satisfy some monotone increasing (decreasing) graph property while minimising (maximising) the total number of rounds. Early work on this model [10,11,13] treated the original question posed by Achlioptas of *avoiding* a giant component for as long as possible. Other works [27,15] considered the opposite of the original question, namely, the question of accelerating the appearance of the giant (see also [47] for a general framework). The model has also been studied for other objectives, such as avoiding [39,45] or creating [41] a fixed subgraph or obtaining a Hamilton cycle [40]. Several variants have also been studied, such as an *offline* version, in which the algorithm sees all k-sets of edges at the beginning of the process (see, e.g., [14]); and a *memoryless* version, in which the algorithm's decision may not depend on its previous decisions (see [9]). A common generalisation of the Achlioptas process and the process considered in this paper, where an algorithm should pick *at most* a single edge to accept/purchase subject to a restricted "budget", was recently studied by Anastos in [3] (see remarks after Theorem 4.1 and Theorems 1.3 and 1.4).

The *semi-random process*, proposed by the third author in 2016, is a variation of the Achlioptas process in which the algorithm is fed by a stream of random spanning stars (with repetitions) instead of k-sets of edges. As with the Achlioptas process, the algorithm must immediately and irrevocably pick a single edge to accept. Work on this model treats monotone properties such as containment of fixed subgraphs [8,5], minimum degree and vertex-connectivity [8], perfect matchings [34], Hamilton cycles [33,35] and bounded degree spanning graphs [7].

Another model that fits in this setting was studied by Frieze and Pegden [31,32] and by Anastos [2] under the name "purchasing under uncertainty". In their model, whenever an edge arrives, it is given an independent random "cost", and the algorithm has to decide whether to purchase that edge, aiming to pay the minimum total cost required to obtain a desired graph property.

In a Ramsey-type version of controlled random processes, incoming edges are coloured (irrevocably) by an online algorithm. The algorithm aims to avoid, or to create, a monochromatic property. In [28], the triangle-avoidance game for up to 3 colours is discussed. This was extended to any fixed cyclic graph and any number of colours in [43,44]. The model was further studied in the contexts of the giant component [12,48] and Hamilton cycles [19].

Finally, we would like to mention a related adaptation of the *two-stage optimisation with recourse* framework for the *minimum spanning tree* model [26]. Here, every edge of the complete graph is given an independent random "Monday" cost and another independent random "Tuesday" cost. The algorithm sees all Monday costs and decides (immediately and irrevocably) which edges to incorporate into its (future) spanning tree. Then, Tuesday costs are revealed, and the algorithm uses them to complete its (current) forest into a spanning tree. The algorithm's goal is to minimise the total cost of edges in its constructed tree.

1.5 Paper organisation and notation

The rest of the paper is organised as follows. In Section 2 we introduce some preliminaries. Section 3 is devoted to the random k-nearest neighbours graph and to the k-out graph. Section 4 contains the proofs of Theorems 1.1 to 1.4 and Theorem 4.1, and Section 5 contains the proofs of Theorems 1.5 and 1.6. In the final section, Section 6, we mention a few relevant open problems.

Throughout the paper, all logarithms are in the natural basis, unless stated otherwise. If f, g are functions of n we write $f \preceq g$ if $f = O(g)$, $f \ll g$ if $f = o(g)$, $f \asymp g$ if $f = \Theta(g)$, and $f \sim g$ if $f = (1 + o(1))g$. For two vertices u, v of a graph we write $u \sim v$ to denote that they are neighbours. For simplicity and clarity of presentation, we often make no particular effort to optimise the constants obtained in our proofs and omit floor and ceiling signs whenever they are not crucial.

2 Preliminaries

2.1 Concentration inequalities

We will make use of the following version of Chernoff bounds (see, e.g., in, [36, Chapter 2]).

Theorem 2.1 (Chernoff bounds)**.** *Let $n \geq 1$ be an integer and let $p \in [0, 1]$, let $x \sim \mathsf{Bin}(n, p)$, and let $\mu = \mathbb{E}x = np$. Then, for every $a > 0$,*

$$\mathbb{P}(x \leq \mu - a) \leq \exp\left(-\frac{a^2}{2\mu}\right), \qquad \mathbb{P}(x \geq \mu + a) \leq \exp\left(-\frac{a^2}{2(\mu + a/3)}\right).$$

The next proposition is often helpful in combination with Theorem 2.1.

Proposition 2.2. *Let $n \geq 1$ be an integer and let $p \in [0, 1]$, write $\mu = np$, and let $x_1, \ldots, x_n$ be (not necessarily independent) Bernoulli random variables. Suppose further that for every $(y_1, \ldots, y_n) \in \{0, 1\}^n$ and for every $i = 1, \ldots, n$, $\mathbb{P}(x_i = 1 \mid x_j = y_j,\ \forall 1 \leq j < i) \geq p$. Then, $x = \sum_{i=1}^n x_i$ stochastically dominates a binomial random variable with n attempts and success probability p.*

Similarly, if for every $(y_1, \ldots, y_n) \in \{0, 1\}^n$ and for every $i = 1, \ldots, n$, $\mathbb{P}(x_i = 1 \mid x_j = y_j,\ \forall 1 \leq j < i) \leq p$, then x is stochastically dominated by a binomial random variable with n attempts and success probability p.

The following are trivial yet useful bounds.

Claim 2.3. *Let $n \geq 1$ be an integer and let $p \in [0, \frac{1}{2}]$, and let $x \sim \mathsf{Bin}(n, p)$. Write $q = 1 - p$ and let $1 \leq k \leq np/q$ be an integer. Then*

$$\mathbb{P}(x \leq k) \leq \left(\frac{enp}{kq}\right)^k e^{-np}.$$

Proof. By the binomial theorem, for every $\alpha \in (0, p/q]$,

$$(1 + \alpha)^n = \sum_{i=0}^n \binom{n}{i} \alpha^i \geq \sum_{i=0}^k \binom{n}{i} \left(\frac{p}{q}\right)^i \left(\frac{\alpha q}{p}\right)^k,$$

hence

$$\mathbb{P}(x \leq k) \;= \sum_{i=0}^k \binom{n}{i} \left(\frac{p}{q}\right)^i q^n \;\leq \frac{(1 + \alpha)^n p^k}{\alpha^k q^k} \cdot (1 - p)^n \;\leq \frac{e^{\alpha n} p^k}{\alpha^k q^k} \cdot e^{-np}.$$

Taking $\alpha = k/n \leq p/q$ we obtain

$$\mathbb{P}(x \leq k) \;\leq \left(\frac{enp}{kq}\right)^k \cdot e^{-np}. \qquad \square$$

2.2 The random graph process

Recall that G_t denotes the random graph process at time t. The next (pretty standard) lemma essentially states that in G_t, where t is superlinear (but not too large), there are no small dense sets.

Lemma 2.4. *Suppose that $R \gg 1$ and $n^{-5}R^2t^3 \leq \alpha$ for some constant $\alpha < 1/8$. Then, in G_t,* ***whp****, every vertex set of size $r \leq R$ spans at most $3r$ edges.*

Proof. Let $N = \binom{n}{2}$ and $p = t/N \sim 2tn^{-2}$. For a set S of s edges the probability that S is contained in G_t is at most p^s. By the union bound, the probability that there exists a set U of size r that spans at least $3r$ edges is at most

$$\binom{n}{r}\binom{\binom{r}{2}}{3r}p^{3r} \leq \left(nr^2p^3\right)^r.$$

We note that $nR^2p^3 \sim 8n^{-5}R^2t^3 \leq 8\alpha$, and hence $nRp^3 \sim 8\alpha/R \ll 1$. By the union bound over $2 \leq r \leq R$, the probability that there exists a set violating the promise of the lemma is at most

$$\sum_{r=2}^{R}\left(nr^2p^3\right)^r \leq \sum_{r=2}^{\sqrt{R}}\left(nRp^3\right)^r + \sum_{r=\sqrt{R}}^{R}\left(nR^2p^3\right)^r \leq \sum_{r=2}^{\sqrt{R}}(o(1))^r + \sum_{r=\sqrt{R}}^{R}((1+o(1))8\alpha)^r = o(1),$$

(here the $o(1)$ terms in the penultimate expression are uniform in r) and the claim follows. $\square$

The lemma easily implies the following useful fact.

Claim 2.5. *Suppose $1 \ll b^2 \leq n^5/(40t^3)$. Then,* ***whp****, every subgraph of G_t with at most b edges is 6-degenerate.*

Proof. Let B be a subgraph of G_t with at most b edges. Thus, B has at most $2b$ non-isolated vertices. Apply Lemma 2.4 with $R = 2b$ and t (we can do so as $b \gg 1$ and $n^{-5}(2b)^2t^3 \leq 1/10$). We therefore obtain that, with high probability, for every $q \geq 0$, every subgraph of B on q vertices spans at most $3q$ edges, and hence has minimum degree of at most 6, and the result follows. $\square$

We introduce more notation. For $d \geq 1$, let X_d^t be the set of vertices in G_t with degree less than d, and let Y_d^t be its complement. The next lemmas state that in time linear in n there is a transition between "most vertices are of degree less than d" and "most vertices are of degree at least d".

Lemma 2.6. *Let $d \geq 1$ be an integer. Then, deterministically, $|Y_d^t| \leq 2t/d$.*

Proof. It follows from $t = |E(G_t)| \geq |Y_d^t| \cdot d/2$. $\square$

Lemma 2.7. *For every integer $d \geq 1$, if $t \geq 6dn$ then $|X_d^t| \leq n/100$* ***whp****.*

Remark. When proving statements about G_t it is often convenient to prove them for $G \sim G(n,p)$ with $p \sim t/\binom{n}{2}$ instead. We could then translate the results back to G_t using standard methods (see, e.g., in [30]).

Proof. Let $t = 6dn$ (the result would follow for $t \geq 6dn$ due to monotonicity). Write $p = 12d/n \sim t/\binom{n}{2}$ and $m = n/100$. If $|X_d^t| > m$ then there exists a vertex set V_0 with $|V_0| = m$ such that $|E(V_0, \overline{V_0})| < md$. Let $x = |E(V_0, \overline{V_0})|$ and note that $x \sim \mathsf{Bin}(m(n-m), p)$. By Claim 2.3 and the union bound over the choice of V_0, the probability that the statement of the lemma does not hold is at most

$$\binom{n}{m}\mathbb{P}(x < md) \leq \left(\frac{en}{m}\right)^m \left(\frac{em(n-m)p}{md(1-p)}\right)^{md} e^{-m(n-m)p}$$
$$\leq \left(\frac{en}{m}\right)^m \left(\frac{enp}{d}\right)^{md} e^{-0.99mnp} \leq \left(100e \cdot \left(12e \cdot e^{-11}\right)^d\right)^m = o(1),$$

where in the last computation we used the fact that $e^{10} > 1200$. $\square$

Recall that for every positive integer k, τ_k is the hitting time for minimum degree k in the random graph process. We will use the next two known results about τ_k.

Lemma 2.8 ([24]). *For every fixed $k \geq 1$, $\tau_k = n(\log n + (k-1)\log\log n + x)/2$, where $x = O(1)$* ***whp****.*

For a proof of the next lemma see, e.g., [38].

Lemma 2.9. *Let $k \geq 1$ be an integer. Then,* ***whp****, $|X_k^{\tau_2}| \leq n^{0.5}$, the distance between two vertices in $X_k^{\tau_2}$ is at least 5, and no vertex in $X_k^{\tau_2}$ lies in a cycle of length at most 4.*

Finally, we would need the following lemma on the size of the k-core of a random graph. The **k-core** of a graph is its (unique) maximal subgraph of minimum degree at least k.

Lemma 2.10. *For every $k \geq 1$ the size of the k-core of G_t, for $t = 8kn$, is* ***whp*** *at least $n/2$.*

Proof. We prove the statement for $G(n,p)$ with $p = 16k/n \sim t/\binom{n}{2}$. First we show that **whp** the number of edges between any set of size $n/2$ and its complement is at least kn. Indeed, let V_0 be a fixed vertex set of size $n/2$. Then, $x = |E(V_0, \overline{V_0})| \sim \mathsf{Bin}(n^2/4, p)$, hence $\mathbb{E}x = 4kn$. By Chernoff bounds (Theorem 2.1), $\mathbb{P}(x \leq kn) \leq \exp(-\frac{9}{8}kn)$. By the union bound over all choices of V_0, the probability that there exists such a set is at most $2^n \cdot \exp(-\frac{9}{8}kn) = o(1)$.

Now, suppose the k-core is smaller than $n/2$. Consider the process of removing vertices of degree smaller than k one by one. By definition of the k-core, this process lasts more than $n/2$ steps. Stop the process exactly after $n/2$ steps. At this point, the remaining set $|V_0|$ is of size $n/2$. On the other hand, we have removed less than $k \cdot n/2$ edges, hence $|E(V_0, \overline{V_0})| < kn$, and we have seen that **whp** there is no such set. $\square$

2.3 Rotations, expanders, and boosters

In our proofs we shall repeatedly use the so-called *rotation–extension* technique of Pósa [46]. Given a longest path $P = (v_0, \ldots, v_j)$ we say that P' obtained from P by an **elementary rotation** of P (with v_0 fixed) if $P' = (v_0, \ldots, v_{i-1}, v_i, v_j, v_{j-1}, \ldots, v_{i+1})$ $(i < j)$. We let $\mathcal{R}(P)$ denote the set of endpoints of paths obtained from P by a (finite) sequence of elementary rotations. For a vertex set A, denote by $N(A)$ the *external* neighbourhood of A, that is, the set of all vertices which are not in A that have a neighbour in A. We will use the following classical lemma of Pósa:

Lemma 2.11 (Pósa's lemma [46]). *Let G be a graph and let P be a longest path in G. Then $|N(\mathcal{R}(P))| \leq 2|\mathcal{R}(P)| - 1$.*

Say that a graph $G = (V, E)$ is an **R-expander** if every set $U \subseteq V$ with $|U| \le R$ has $|N(U)| \ge 2|U|$. For an n-vertex graph G denote by $\lambda(G)$ the length of a longest path in G, or n if G is Hamiltonian. A non-edge e of G is called a **booster** if $\lambda(G+e) \ge \min\{\lambda(G)+1, n\}$. The following lemma is a corollary of Lemma 2.11. For a proof see, e.g., [38].

Lemma 2.12. *Let G be a connected R-expander which contains no Hamilton cycle. Then G has at least $(R+1)^2/2$ boosters.*

In our arguments below, we combine Lemma 2.12 with the *sprinkling* technique, standard for many Hamiltonicity arguments in random graphs. We begin with an expander, and add random edges. Typically, the first few random edges turn it into a connected graph (if it was not connected to begin with), and then the remaining random edges hit boosters (which are numerous, according to Lemma 2.12), advancing the expander towards Hamiltonicity. We summarize this in the following lemma:

Lemma 2.13 (Sprinkling)**.** *For every $\beta > 0$ there exists $C > 0$ such that the following holds. For every n-vertex βn-expander, the addition of Cn random non-edges to G makes it **whp** Hamiltonian.*

Proof. Since G is a βn-expander, each of its connected components has size at least $3\beta n$, so there are at most $1/(3\beta) = O(1)$ components. Thus, in the first $t_1 \to \infty$ rounds, there will be, **whp**, an edge (observed and purchased) between every pair of components. Thus, **whp**, G_{t_1} is connected.

Now, using Lemma 2.12 and the monotonicity of expansion, we know that G_{t_1} is either Hamiltonian (in which case we are done) or it has at least $\beta^2\binom{n}{2}$ boosters. Thus, the probability that a random edge hits a booster is at least β^2. Let $t_2 = \tau n$ be an integer with $\tau > \beta^{-2}$, and consider the next t_2 random edges. For $i \in \{t_1+1, \dots, t_1+t_2\}$ let y_i be the indicator of the event that the i'th edge hits a booster, or that G_{i-1} is already Hamiltonian. Let $y = \sum_{i=1}^{t_1} y_i$, and note that $y \ge n$ implies that $G_{t_1+t_2}$ is Hamiltonian. Since y stochastically dominates a binomial random variable with t_2 attempts and success probability β^2, it follows from Theorem 2.1 that the probability that $y \le n$ is $o(1)$. Thus, after adding t_2 further random non-edges, the graph becomes **whp** Hamiltonian. □

We also use Lemma 2.11 to show that in Hamiltonian expanders, the set of endpoints of Hamilton paths whose other endpoint is fixed is large.

Lemma 2.14. *Let G be a Hamiltonian R-expander and let v be a vertex of G. Then, the number of endpoints of Hamilton paths of G whose other endpoint is v is at least R.*

Proof. Since G is Hamiltonian, there exists a Hamilton path P with v as an endpoint. By Lemma 2.11, $|N(\mathcal{R}(P))| \le 2|\mathcal{R}(P)| - 1$; thus $|\mathcal{R}(P)| > R$. □

2.4 Greedy k-matchings and k-cores

We show that the greedy strategy works "well" for purchasing a large k-matching; namely, a subgraph of maximum degree k in which all but a few vertices are of degree k. This will be useful in the proofs of Theorem 1.2 and Theorem 1.4.

Lemma 2.15. *Let k be a positive integer and let $\varepsilon > 0$. Then, there exists $C = C(k, \varepsilon) > 0$ such that if $t \ge Cn$ and $b \ge (k-\varepsilon)n/2$ then there exists a (t, b)-strategy of Builder (that succeeds **whp**) that purchases a graph with maximum degree k in which all but at most εn vertices are of degree k.*

Proof. Builder follows the greedy strategy; that is, he purchases every edge both of whose endpoints are of degree below k. Observe that this strategy ensures that the maximum degree of Builder's graph is at most k at any stage. Let U denote the (dynamic) set of vertices of degree below k. Let $C = k\varepsilon^{-2}$ and let $t \geq Cn$. For $i = 1, \dots, t$ let x_i be the indicator of the event that the i'th edge arriving is contained in U (and thus purchased by Builder) *or* $|U| \leq \varepsilon n$. The probability for this event is at least $\sim \varepsilon^2$. Thus, by Proposition 2.2, $x := \sum x_i$ stochastically dominates a binomial random variable with mean $\sim \varepsilon^2 Cn = kn$. Therefore, the probability that $x < (k-\varepsilon)n/2$ is, by Chernoff bounds (Theorem 2.1), $o(1)$. On the event that $x \geq (k-\varepsilon)n/2$, either $|U| \leq \varepsilon n$, or Builder has purchased at least $(k-\varepsilon)n/2$ edges, in which case it follows that $|U| \leq \varepsilon n$. Obviously, by that time Builder has purchased at most $(k-\varepsilon)n/2$ edges. □

Given a vertex set we may use Theorem 3.1 below to construct a spanning graph of an arbitrary fixed minimum degree (see Section 4.1). That requires, however, that the time will be long enough so that every vertex will have the desired degree in the underlying graph process. If we are satisfied with a *small* graph of a given minimum degree, we can do it in linear time. The following corollary follows directly from Lemma 2.10.

Corollary 2.16. *Let $k \geq 1$ be an integer and let $b = t = 8kn$. Then, by purchasing every observed edge, Builder has a (t, b)-strategy resulting (**whp**) in a graph whose k-core is of size at least $n/2$.*

3 The random k-nearest neighbour graph

Suppose the edges of the complete graph are endowed with independent uniform random "lengths" in $[0, 1]$. The (random) graph obtained by retaining (only) the k shortest edges incident with each vertex is called the **random k-nearest neighbour graph** (see [20, 30]). We denote it by $\mathcal{O}_k$ (or by $\mathcal{O}_k(n)$ if we wish to emphasize that it is on n vertices). As we remarked earlier, the choice of uniform distribution supported on $[0, 1]$ is arbitrary; any distribution without atoms would yield the same random graph (see [20]); we use this fact in the proof of Observation 3.7. The following theorem of Cooper and Frieze [20] (stated here in a weaker form) will be useful for us in several places in this paper.

Theorem 3.1 ([20, Theorem 1.1])**.** *For every constant k, the number of edges of $\mathcal{O}_k$ is **whp***

$$kn - \frac{n(n-1)}{2n-3} \sum_{1 \leq i \leq j \leq k} \frac{\binom{n-2}{i-1}\binom{n-2}{j-1}}{2^{\delta(i,j)}\binom{2n-4}{i+j-2}} + O(\sqrt{n}\log n),$$

where $\delta(i, j)$ is the Kronecker delta.

It would be useful for us to state Theorem 3.1 in an "asymptotic" form. To this end, we complement the analysis of Cooper and Frieze by obtaining an asymptotic estimate of the expected size of $\mathcal{O}_k$. Let

$$\mathfrak{o}_k = k - \frac{1}{4} \sum_{i,j=0}^{k-1} \binom{i+j}{i} 2^{-(i+j)}. \tag{1}$$

Corollary 3.2. *For every integer $k \geq 1$ the number of edges of $\mathcal{O}_k$ is **whp** $\mathfrak{o}_k n + O(\sqrt{n}\log n)$. Moreover, $k/2 < \mathfrak{o}_k \leq 3k/4$, and $\lim_{k\to\infty} \mathfrak{o}_k/k = 1/2$.*

Proof. First note that for every nonnegative integer k,

$$\sum_{i=0}^{k} \binom{i+k}{i} 2^{-(i+k)} = 1. \tag{2}$$

Indeed, consider a sequence of fair coin flips that stops whenever $k+1$ heads *or* $k+1$ tails are encountered. Let x be the number of tails if the sequence stopped by encountering $k+1$ heads, or the the number heads otherwise. Thus, x is supported on $\{0,\dots,k\}$ and $\mathbb{P}(x=i) = \binom{i+k}{i}2^{-(i+k)}$. For an integer $k \ge 0$ write $f(k) = \sum_{i,j=0}^{k} \binom{i+j}{i} 2^{-(i+j)}$ (so $\mathfrak{o}_k = k - f(k-1)/4$). Note that for $k \ge 1$, by Eq. (2),

$$f(k) = f(k-1) + 2\sum_{i=0}^{k} \binom{i+k}{i} 2^{-(i+k)} - \binom{2k}{k} 2^{-2k} = f(k-1) + 2 - \Theta(k^{-1/2}).$$

Thus, $f(0) = 1$, $f(k) \le 1 + 2k$ for $k \ge 0$ and

$$\lim_{k\to\infty} \frac{f(k)}{k} = 2. \tag{3}$$

It follows from Theorem 3.1 that the number of edges of $\mathcal{O}_k$ is, **whp**,

$$\begin{aligned} &\left(k - \frac{1}{2}\sum_{1\le i\le j\le k} \binom{i+j-2}{i-1} 2^{-(i+j-2)-\delta(i,j)}\right) n + O(\sqrt{n}\log n) \\ &= \left(k - \frac{1}{4}\sum_{i,j=0}^{k-1} \binom{i+j}{i} 2^{-(i+j)}\right) n + O(\sqrt{n}\log n) = \mathfrak{o}_k n + O(\sqrt{n}\log n). \end{aligned}$$

Evidently, $f(k)$ is increasing (hence $\mathfrak{o}_k$ is decreasing); thus, $\frac{k}{2} + \frac{1}{4} \le \mathfrak{o}_k \le \mathfrak{o}_1 = 3k/4$. In addition, Eq. (3) implies that $\mathfrak{o}_k/k \to 1/2$ as $k \to \infty$. □

One of the main results of [20], that we conveniently use here, is that for $k \ge 3$, $\mathcal{O}_k$ is **whp** k-vertex-connected. Cooper and Frieze also showed (there) that the probability that $\mathcal{O}_2$ is connected is bounded away from 0 and 1.

Theorem 3.3 ([20, Theorem 1.4])**.** *For $k \ge 3$, $\mathcal{O}_k$ is **whp** k-vertex-connected.*

Our main goal in this section is to prove that for large enough k, $\mathcal{O}_k$ is **whp** Hamiltonian.

Theorem 3.4. *There exists $k_H > 0$ such that for every $k \ge k_H$, $\mathcal{O}_k$ is **whp** Hamiltonian.*

As mentioned in the introduction, this partially resolves [29, Problem 45]. It is conjectured that $k_H = 3$ or $k_H = 4$.

In fact, we need (and hence prove) a stronger result that immediately implies Theorem 3.4. To state it, consider the following equivalent way of generating $\mathcal{O}_k$ (see [20]). Given a uniform random ordering of the edges of the complete graph K_n, we consider them one by one. When an edge arrives, we add it to $\mathcal{O}_k$ if and only if one of its endpoints is incident to a vertex that is — at this point — of degree less than k. When we only add edges that arrived by time t for some fixed $0 \le t \le \binom{n}{2}$, we obtain a subgraph of $\mathcal{O}_k$ that we denote by $\mathcal{O}_k^t$. This process yields a coupling between the random graph process at time t and $\mathcal{O}_k^t$. Under this coupling, which we call the **natural coupling**, $\mathcal{O}_k^t$ is a subgraph of G_t. For convenience, we denote $\mathcal{O}_{k,d} = \mathcal{O}_k^{\tau_d}$ for every $0 \le d \le k$. Note that as long as the minimum degree of the random graph process is at most 1, it cannot contain a Hamilton cycle; thus, for $t < \tau_2$, $\mathcal{O}_k^t$ is not Hamiltonian.

Theorem 3.5. *There exists $k_H > 0$ such that for every $k \geq k_H$, $\mathcal{O}_{k,2}$ is **whp** Hamiltonian.*

As mentioned in the introduction, a classical result due to Bollobás [17] and independently Ajtai, Komlós, and Szemerédi [1] is a hitting-time result, which states that, **whp**, the random graph process will contain a Hamiltonian cycle as soon as its minimum degree reaches 2. Theorem 3.5 refines that result by showing that there exists an *online* algorithm that can choose a small number of edges (linear in the number of vertices) from the first τ_2 edges of the random graph process that induce a Hamiltonian cycle **whp**. Furthermore, this algorithm is deterministic, simple, and only requires local information, namely, the degrees of the endpoints of the considered edge.

The rest of this section is devoted to the proof of Theorem 3.5, which implies, in turn, Theorem 1.3 (see Section 4.2.1). While the proof shares some ideas with previous proofs of the hitting time result for Hamiltonicity of the random graph process (see, e.g., [38]), it still has its twists and turns to achieve the goal. In particular, we find a way around the common method of *random sparsification* accompanied by *sprinkling* (which is not available to us in this setting) by casting our argument directly on the (already sparse) graph and finding boosters inside it using a sophisticated argument. We begin with a couple of helpful observations. The following is immediate:

Observation 3.6. *For integers $0 \leq d < k$, time t and a vertex v, under the natural coupling between G_t and $\mathcal{O}_k^t$ one has $d_{\mathcal{O}_k^t}(v) = d$ if and only if $d_{G_t}(v) = d$.*

The next observation that we will need for this goal is that $\mathcal{O}_k$ is stochastically dominated by the **random k-out graph** (see, e.g., in [30]): the random graph whose edges are generated independently for each vertex by a random choice of k distinct edges incident to that vertex. For brevity, denote it by $\mathcal{G}_k$. Note that $\mathcal{G}_k$ has roughly kn edges, whereas, as stated in Corollary 3.2, $\mathcal{O}_k$ is nearly twice as sparse (for large k).

Observation 3.7. *$\mathcal{O}_k$ is stochastically dominated by $\mathcal{G}_k$.*

Proof. For every ordered pair of distinct vertices u, v let $\mathbf{w}(u, v)$ be a uniform $[0, 1]$ random weight assigned to the ordered pair (u, v). Then, $\mathcal{G}_k$ is obtained by adding the edge $\{u, v\}$ whenever $\mathbf{w}(u, v)$ is one of the k smallest weights in the family $\{\mathbf{w}(u, w)\}_w$. For every (unordered) pair of distinct vertices u, v let $\mathbf{x}\{u, v\} = \min\{\mathbf{w}(u, v), \mathbf{w}(v, u)\}$. Observe crucially that while $\mathbf{x}$ is not uniform, it has a continuous density function on $[0, 1]$ and, more importantly, is independent for distinct edges. Thus, $\mathcal{O}_k$ is obtained by including the edge $\{u, v\}$ whenever $\mathbf{x}\{u, v\}$ is one of the k smallest weights in the family $\{\mathbf{x}\{u, w\}\}_w$. Let us now show that under this coupling, $\mathcal{O}_k \subseteq \mathcal{G}_k$. Let $\{u, v\} \in \mathcal{O}_k$ and assume without loss of generality that $\mathbf{x}\{u, v\} = \mathbf{w}(u, v)$. Pick a vertex w for which $\mathbf{x}\{u, v\} < \mathbf{x}\{u, w\}$ (there are at least $n-1-k$ such vertices). Then, $\mathbf{w}(u, v) = \mathbf{x}\{u, v\} < \mathbf{x}\{u, w\} \leq \mathbf{w}(u, w)$. It follows that there are at least $n - 1 - k$ vertices w for which $\mathbf{w}(u, v) < \mathbf{w}(u, w)$, hence $\{u, v\} \in \mathcal{G}_k$. $\square$

We now show that for $k \geq 12$, $\mathcal{O}_{k,2}$ is **whp** an expander[3].

Lemma 3.8. *There exists $\beta > 0$ such that for $k \geq 12$, $\mathcal{O}_{k,2}$ is **whp** a βn-expander.*

Proof. We prove it for $k = 12$ and derive the statement for every $k \geq 12$ due to monotonicity. Fix $b = k/11 > 1$. We first prove that there exists $\gamma > 0$ for which, **whp**, no vertex set S in $\mathcal{O}_{k,2}$ of cardinality $s \leq \gamma n$ spans more than bs edges. Since $\mathcal{O}_{k,2}$ is stochastically dominated by $\mathcal{O}_k$ and (by Observation 3.7) $\mathcal{O}_k$ is stochastically dominated by $\mathcal{G}_k$, we may prove that statement for $\mathcal{G}_k$. Note that the probability that a given edge appears in $\mathcal{G}_k$ is at most twice the probability that a given

[3] We made no effort to optimise the constant 12 in the lemma.

vertex chooses a given incident edge (which is $k/(n-1)$). By the union bound over the choice of S and the choices of the spanned edges, the probability that there exists a vertex set S in $\mathcal{G}_k$ of cardinality $s \le \gamma n$ that spans more than bs edges is at most

$$\sum_{s=2}^{\gamma n} \binom{n}{s} \binom{\binom{s}{2}}{bs} \left(\frac{3k}{n}\right)^{bs} \le \sum_{s=2}^{\gamma n} \left[\frac{en}{s}\left(\frac{es}{2b}\cdot\frac{3k}{n}\right)^{b}\right]^{s} = \sum_{s=2}^{\gamma n}\left(\Theta\left(\left(\frac{s}{n}\right)^{b-1}\right)\right)^{s}.$$

If $n/\log n \le s \le \gamma n$ then the s'th summand is at most $c^{n/\log n}$ for some $c<1$ (for small enough $\gamma>0$). Otherwise, it is $(o(1))^{s}$. Thus, the sum is $o(1)$.

Back to $\mathcal{O}_{k,2}$, let $X = X_k^{\tau_2}$ be the set of vertices of degree smaller than k at time τ_2. Suppose first that there exists a set B of size at most βn for $\beta = \gamma/5$ with $B \cap X = \varnothing$ and $|N(B)| \le 4|B|$. Set $S = B \cup N(B)$ and note that $s = |S| \le 5\beta n = \gamma n$ while $|E(S)| \ge k\beta n/2 > bs$, contradicting the above. Now, let A be a vertex set with $|A| \le \beta n$. Let $A_X = A \cap X$ and $A_Y = A \smallsetminus X$. Let further $S_X = A_X \cup N(A_X)$ and $S_Y = A_Y \cup N(A_Y)$. By the above, $|N(A_Y)| \ge 4|A_Y|$. By Lemma 2.9 and Observation 3.6, $|N(A_X)| \ge 2|A_X|$ and every vertex in A_Y (or in general) has at most one neighbour in S_X. Hence, in particular, $|N(A_Y) \smallsetminus S_X| \ge |N(A_Y)| - |A_Y| \ge 3|A_Y|$. Thus,

$$|N(A)| = |N(A_X) \smallsetminus A_Y| + |N(A_Y) \smallsetminus S_X| \ge (2|A_X| - |A_Y|) + 3|A_Y| \ge 2|A|,$$

as required. □

Our next step is to show that $\mathcal{O}_{k,2}$ is also typically connected (for large enough k). For that, we need the following lemma. As we mentioned earlier, Cooper and Frieze [20] showed that $\mathcal{O}_2$ is connected with probability that is bounded away from 0 and 1. Since $\mathcal{O}_3 \not\subseteq \mathcal{O}_{k,2}$ for any k, we cannot use Theorem 3.3.

Lemma 3.9. *Let $\alpha > 0$ and let F be a fixed set of edges in K_n with $|F| \ge \alpha n^2$. Then, there exist $k_0 = k_0(\alpha)$, $\eta = \eta(\alpha)$ and $c = c(\alpha) > 0$ such that for every $k \ge k_0$, if $t = \eta k n$ then $\mathbb{P}(F \cap E(\mathcal{O}_k^t) = \varnothing) \le \exp(-ckn)$.*

Proof. Let $\gamma = \sqrt{\alpha}$, and $\eta = \gamma/3$. Take $k > 6(1-\log\gamma)/\gamma^2$, and $t_0 = \eta k n$. Let $E_0 = \{e_1, \dots, e_{t_0}\}$ be the first t_0 edges of the underlying random graph process (in this order) and let $F_0 = F \cap E_0$. Let $V_0 = \bigcup F_0$ be the set of vertices covered by the edges of F_0. We observe that typically V_0 cannot be too small; indeed, for a given set S to be $\bigcup F_0$ we need every edge of F that is not contained in S to appear after time t_0. By the union bound over all choices of V_0 of size at most γn,

$$\begin{aligned}\mathbb{P}(|V_0| \le \gamma n) &\le \binom{n}{\gamma n}\left(1 - \frac{t_0}{\binom{n}{2}}\right)^{|F| - \binom{\gamma n}{2}} \le \exp((\gamma - \gamma\log\gamma - \eta k(2\alpha - \gamma^2))n) \\ &\le \exp((\gamma - \gamma\log\gamma - \gamma^3 k/3)n) \le \exp(-ckn),\end{aligned}$$

for $c = \gamma^3/6 > 0$. Let $F_1 = F_0 \cap E(\mathcal{O}_k^{t_0})$. Note that if $F_1 = \varnothing$ then when an edge in F_0 arrived both its endpoints had degree at least k; hence all vertices of V_0 have degree at least k in $\mathcal{O}_k^{t_0}$. But in this case, $|V_0| \ge \gamma n$ implies that $\gamma n k/2 \le |E(\mathcal{O}_k^{t_0})| \le t_0 = \eta k n = \gamma k n/3$, a contradiction. Thus,

$$\mathbb{P}(F \cap E(\mathcal{O}_k^t) = \varnothing) \le \mathbb{P}(F \cap E(\mathcal{O}_k^{t_0}) = \varnothing) \le \mathbb{P}(F_1 = \varnothing) = \exp(-ckn).$$ □

Lemma 3.10. *For every $\beta > 0$ there exists $k_0 = k_0(\beta)$ such that for every $k \ge k_0$, **whp**, every two disjoint vertex sets in $\mathcal{O}_{k,2}$ with $|A|, |B| \ge \beta n$ are connected by an edge.*

Proof. Let $\alpha = \beta^2$ and let k_0, η, c be the constants guaranteed by Lemma 3.9. Let $\mathrm{H}(\beta) = -\beta \log \beta - (1-\beta)\log(1-\beta)$, and set $k_1 = 3\,\mathrm{H}(\beta)/c$. Take $k \geq \max\{k_0, k_1\}$ and $t = \lceil \eta k n \rceil$. Fix two disjoint vertex sets A, B with $|A|, |B| = \beta n$. Let $F = E(A, B)$, so $|F| \geq \alpha n^2$. By Lemma 3.9, the probability that in $\mathcal{O}_k^t$ does not contain an edge from F (and hence the probability that in $\mathcal{O}_k^t$ the sets A, B are not connected by an edge) is at most $\exp(-ckn)$. By the union bound over all sets A, B, the probability that in $\mathcal{O}_k^t$ there are two disjoint sets of size at least βn that are not connected by an edge is at most

$$\binom{n}{\beta n}^2 e^{-ckn} = \exp(((2+o(1))H(\beta) - ck)n) = \exp(-\Omega(n)) = o(1).$$

The result follows since **whp** $t \leq \tau_2$, implying $\mathcal{O}_k^t \subseteq \mathcal{O}_{k,2}$. ☐

Corollary 3.11. *There exists $k_0 > 0$ such that for every $k \geq k_0$, $\mathcal{O}_{k,2}$ is **whp** an $\frac{n}{4}$-expander.*

Proof. Let β be the constant guaranteed by Lemma 3.8 (we may (and will) assume that $\beta < 1/4$), and let $k_0 = k_0(\beta)$ be the constant guaranteed by Lemma 3.10. Let $k \geq \max\{k_0, 12\}$. Thus, by Lemma 3.8, $\mathcal{O}_{k,2}$ is **whp** a βn-expander; and, by Lemma 3.10, $\mathcal{O}_{k,2}$ has **whp** the property that any two disjoint vertex sets of size at least βn are connected by an edge. Assume both properties hold and let A be a vertex set with $|A| \leq n/4$. If $|A| \leq \beta n$ then $|N(A)| \geq 2|A|$ since $\mathcal{O}_{k,2}$ is a βn-expander. Otherwise, there is an edge between A and any other set of size at least βn, hence $|N(A)| \geq n - \beta n - |A| \geq (3/4 - \beta)n \geq n/2 \geq 2|A|$. ☐

Corollary 3.12. *With the same $k_0 > 0$ as in Corollary 3.11, for every $k \geq k_0$, $\mathcal{O}_{k,2}$ is **whp** connected.*

Proof. Let k_0 be the constant guaranteed by Corollary 3.11 and let $k \geq k_0$. From Corollary 3.11 it follows that every connected component of $\mathcal{O}_{k,2}$ is, **whp**, of size at least $3n/4$, implying that the graph is connected. ☐

We note that Corollary 3.12 can be obtained directly from Lemmas 3.8 and 3.10. However, we state Corollary 3.11 separately, as it will be used in the proof of Theorem 3.5.

Proof of Theorem 3.5. Let k_0 be the constant from Corollary 3.11, let $\ell = \max\{k_0, 500\}$ and let $k = 100\ell$ (this k will be the k_H from the statement of the theorem). Note that by proving the statement of the theorem for k, due to monotonicity of $\mathcal{O}_{k,2}$ in the first parameter, we prove it for every $k' \geq k$. Let $t_0 = 6\ell n$ and write $a = 6\ell/k$, so $t_0 = akn$. Consider the first t_0 edges of the random graph process (and recall from Lemma 2.8 that $t_0 \ll \tau_2$ **whp**). When an edge e arrives we colour it *red* if it is incident to a vertex of degree $d < \ell$, or *black* otherwise.

Recall (from Section 2.2) that Y_ℓ^t is the set of vertices in G_t with degree at least ℓ. Let $L = Y_\ell^{t_0}$, and note that by Lemma 2.7, $|L| \geq 0.99n$ **whp**. We now consider the next edges of the random graph process, until time τ_2. When an edge e arrives we colour it *red* if it is incident to a vertex of degree $d < \ell$ (note that these red edges are *not* contained in L), or *black* if it is any other edge not contained in L. If the arriving edge is contained in L, we do not record its identity, but rather its arrival time. In particular, we count these edges. We observe crucially that the exact identity of such an edge does not affect the rest of the colouring process; indeed, any permutation of the edges of the random graph process that fixes the coloured edges (by time τ_2) and the edges not in L is feasible. Let m be the number of edges we counted in L and let $t_1, \ldots, t_m$ be their arrival times ($t_0 < t_1 < \cdots < t_m < \tau_2$). Note that the probability of an edge to fall inside L before $O(n)$ edges fell there (without conditioning on the rest of the process) is at least $p := 0.98$ (since the number of existing edges in L is $o(n^2)$). Thus, for any $b = O(1)$, the number of edges falling into L between t_0 and $t_0 + bn$ stochastically dominates a binomial random variable with bn attempts and success probability p. Take $b = 500$, and $m' = \lfloor bnp/2 \rfloor$. It follows that $m' < m$ and $t_i < t_0 + bn$ for every $1 \le i \le m'$ **whp**.

Let F be the set of non-edges in L at time τ_2. Conditioning on the entire colouring process (including m and the times $t_1, \dots, t_m$), the set of edges that fall into L at times $t_1, \dots, t_m$ is distributed as a uniformly sampled edge set of size m of F. Let $f_1, \dots, f_m$ be a random ordering of this random set of edges. Write $F_i = \{f_1, \dots, f_i\}$ for $0 \le i \le m$. We add the random edges in F_m one by one (edge f_i at time t_i), so the distribution of f_i is uniform over $F \setminus F_{i-1}$. Some of these edges will be coloured *blue* according to a rule explained below. For $i \in [m]$, let $K_i = Y_k^{t_i - 1} \cap L$ be the set of vertices of degree at least k in L just before time t_i. We note by Lemma 2.6, Observation 3.6 and the discussion above that $|K_i| \le 2t_i/k \le 2(a + b/k)n$ for every $i \in [m']$, **whp**. Note that the graph that consists of the red edges at time τ_2 is $\mathcal{O}_{\ell,2}$. For every $0 \le i \le m$, denote by H_i the graph that consists of the red edges (at time τ_2) and the blue edges among F_i. In particular, $H_0 = \mathcal{O}_{\ell,2}$, and for every $i \ge 0$, H_i is a supergraph of $\mathcal{O}_{\ell,2}$. By Corollaries 3.11 and 3.12 and Lemma 2.12, and since expansion (and connectivity) is monotone, we know that **whp**, for all i, either H_i is Hamiltonian, or there are at least $n^2/32$ boosters with respect to it. Suppose H_i is not Hamiltonian, and let Z_i be the set of boosters with respect to H_i. Let $Z_i' = Z_i \cap \binom{L}{2} \setminus \binom{K_i}{2} \setminus F_i$. These are the boosters that are (a) contained in L, (b) have at least one endpoint of degree less than k at the time of arrival, and (c) have not been revealed earlier. It follows that

$$|Z_i'| \ge |Z_i| - |E(X_\ell^{t_0}, V)| - \left|\binom{K_i}{2}\right| - |F_i| \ge \left(\frac{1}{32} - \frac{1}{100} - 2\left(a + \frac{b}{k}\right)^2 - o(1)\right)n^2.$$

This is at least $n^2/100$ for $\ell \ge 500$ and $k = 100\ell$. When considering the edge f_{i+1} we colour it blue if it is in Z_i', or if it is in $\binom{L}{2} \setminus \binom{K_i}{2}$ and H_i is already Hamiltonian. We observe that since $H_0 = \mathcal{O}_{\ell,2} \subseteq \mathcal{O}_{k,2}$, and since every blue edge is in $\mathcal{O}_{k,2}$, we have that $\mathcal{H}_i \subseteq \mathcal{O}_{k,2}$. Note that the probability that it is in Z_i' is at least $|Z_i'|/(|F| - |F_i|) \ge q := 1/100$, independently of the past. Thus, using Proposition 2.2, the number of blue edges stochastically dominates a binomial random variable with $m' \sim bnp/2$ attempts and success probability q, and hence mean $\sim qbnp/2 > 2n$. By Chernoff bounds (Theorem 2.1), the number of blue edges is at least n **whp**. We finish the proof by noting that if H_i is not Hamiltonian and f_{i+1} is coloured blue then f_{i+1} is a booster with respect to H_i that is contained in $\mathcal{O}_{k,2}$. Thus, $H_{m'}$ (and thus H_m; and thus $\mathcal{O}_{k,2}$) is **whp** Hamiltonian. □

The proof above yields the following corollary.

Corollary 3.13. *Let k_H be the constant from Theorem 3.5. Then, for every $k \ge k_H$, $\mathcal{O}_{k,2}$ is **whp** a Hamiltonian $\frac{n}{4}$-expander.*

4 Spanning subgraphs

4.1 Minimum degree and vertex-connectivity

It follows from Corollary 3.2 that Builder can construct a graph with minimum degree k by purchasing (sufficiently) more than $\mathfrak{o}_k n$ edges, while observing only enough edges to guarantee a sufficient minimum degree in the underlying random graph process.

Proof of Theorem 1.1. We describe Builder's strategy. Builder purchases any edge touching at least one vertex of degree less than k. Since by time τ_k the random graph process has, by definition, minimum degree of at least k, the obtained graph is exactly $\mathcal{O}_k$. We conclude by observing that by Corollary 3.2 Builder purchased, **whp**, at most $\mathfrak{o}_k n + O(\sqrt{n}\log n)$ edges, and that for $k \geq 3$, by Theorem 3.3, $\mathcal{O}_k$ is k-vertex-connected. □

Proof of Theorem 1.2. Let $\varepsilon' = \varepsilon/2$. We describe Builder's strategy in stages.

Stage I (Constructing a k-matching) Time: Cn — Budget: $(k-\varepsilon')n/2$

During the first Cn steps, for $C = C(\varepsilon', k)$, Builder constructs a k-matching, in which all but at most $\varepsilon' n$ vertices are of degree k. This is possible, **whp**, by Lemma 2.15.

Stage II (Handling low-degree vertices) Time: $(1+\varepsilon')n\log n/2$ — Budget: $\varepsilon' kn$

Denote by V_0 the set of vertices of degree smaller than k in Builder's graph at the end of Stage I. By the above, **whp**, $|V_0| \leq \varepsilon' n$. Builder now emulates the $\mathcal{O}_k$ model with respect to these vertices; namely, Builder purchases any edge contained in V_0 that is incident to at least one vertex of degree less than k in $G[V_0]$. Since in the random graph process, by time t, the minimum degree is logarithmic in n, Builder will observe at least k additional edges incident to each vertex of V_0. He will purchase a subset of these (thus, at most $\varepsilon' kn$ edges). □

4.2 Hamilton cycles

4.2.1 Hitting time, inflated budget

Proof of Theorem 1.3. Let k be a large enough constant so that $\mathcal{O}_{k,2}$ is **whp** Hamiltonian (such k is guaranteed to exist by Theorem 3.5). Builder emulates $\mathcal{O}_{k,2}$ by purchasing every edge that is incident to a vertex of degree less than k. He completes this in time τ_2 while purchasing at most kn edges. □

4.2.2 Inflated time, optimal budget

In this section, we prove the following theorem:

Theorem 4.1 (Hamiltonicity)**.** *For every $\varepsilon > 0$ there exists $C > 1$ such that the following holds. If $t \geq Cn\log n$ and $b \geq (1+\varepsilon)n$ then there exists a (t,b)-strategy B of Builder such that*

$$\lim_{n\to\infty} \mathbb{P}(B_t \text{ is Hamiltonian}) = 1.$$

Proof of Theorem 4.1 We describe Builder's strategy in (four) stages. Set $\varepsilon' = \varepsilon/40$. We describe a (t,b)-strategy for $b = (1+\varepsilon)n$ and $t < n\log n/\varepsilon'$.

Stage I (Growing disjoint paths) Time: $\Theta(n\log^{1/3} n)$ — Budget: $(1-\varepsilon')n$

Let C be a large constant to be chosen later. In the first stage, Builder grows many sublinear paths, covering together $(1-\varepsilon')n$ vertices. Let $s' \sim n/\log^{1/3} n$ be an integer and let $s_0 = n/\log^{1/2} n \ll s'$. Builder grows simultaneously paths $P_1, \ldots, P_{s'}$ as follows. He begins by letting each P_i be a (distinct) vertex v_0^i. He then claims every edge that extends one of the paths without intersecting with other paths. Formally, let $\nu = (1-\varepsilon')n/s'$. If at a given stage Builder has the paths $(v_0^1, \ldots, v_{\ell_1}^1), \ldots, (v_0^{s'}, \ldots, v_{\ell_{s'}}^{s'})$ on the vertex set V_P then he claims an observed edge if and only if it is of the form $\{v_{\ell_j}^j, w\}$ for some $j = 1, \ldots, s'$, $\ell_j < \nu$ and $w \notin V_P$. Builder stops if all but at most s_0 of the paths are of length at least ν (in which case the stage is "successful"), or when he has observed $t_1 = \frac{C}{\varepsilon'} n \log^{1/3} n$ edges (in which case the stage "fails"), whichever comes first. For $1 \le j \le s'$ and $1 \le i \le t_1$, let y_i^j be the indicator that Builder has purchased the i'th observed edge, and that this edge extends the path P_j. For convenience, if $|V_P| \ge (1-\varepsilon')n$ we set y_i^j to 1. Observe that for every j, i, $\mathbb{P}(y_i^j = 1) \ge (n - |V_P|)/\binom{n}{2} > \varepsilon'/n$. If the stage fails then there is an s_0-subset $S_0 \subseteq [s']$ such that for every $j \in S_0$, the path P_j ends up with length less than ν. Write $y_0 = \sum_{j \in s_0} \sum_{i=1}^{t_1} y_i^j$. Thus, if the stage fails then $y_0 < \nu s_0$. But y_0 stochastically dominates a binomial random variable with t_1 attempts and success probability $p_1 = \varepsilon' s_0/n$. Set $\mu = t_1 p_1 \sim C s_0 \log^{1/3} n \gg s'$. By the union bound over the choices of an s_0-subset of the paths and using Chernoff bounds (Theorem 2.1), for large enough C, the probability that the stage fails is at most

$$\binom{s'}{s_0} \mathbb{P}(y_0 < \nu s_0) \le 2^{s'} \exp(-\mu/10) = o(1).$$

Stage II (Connecting the paths) Time: $n \log^{1/2} n$ — Budget: $o(n)$

At this stage Builder tries to connect most of the $s = s' - s_0 \sim s'$ complete paths to each other, eventually obtaining an almost-spanning path in his graph. For convenience, we re-enumerate the complete paths by $[s]$. Let $t_2 = n \log^{1/2} n$. During the next t_2 observed edges, Builder purchases an edge connecting the last vertex of one of his paths to the first vertex of another. Formally, he purchases every observed edge of the form $\{v_{\ell_i}^i, v_0^j\}$ for $i \ne j$. We will need the following lemma:

Lemma 4.2 ([6, Lemma 4.4])**.** *Let $s, k \ge 1$ and let D be an s-vertex digraph in which for every two disjoint $A, B \subseteq V(D)$ with $|A|, |B| \ge k$, D contains an edge between A and B. Then, D contains a directed path of length $s - 2k + 1$.*

We use the lemma on an auxiliary digraph D on the vertex set $[s]$ that is defined as follows. We observe t_2 random edges. Whenever Builder observes (and purchases) an edge of the form $\{v_{\ell_i}^i, v_0^j\}$, we add the edge (i, j) to D. We now have to claim that D satisfies the requirements of the lemma for $k = \varepsilon' s$.

Claim 4.3. ***Whp**, every two disjoint $A, B \subseteq [s]$ with $|A|, |B| \ge \varepsilon' s$ satisfy $E_D(A, B) \ne \varnothing$.*

Proof. Fix disjoint A, B with $|A|, |B| \ge \varepsilon' s$. The event that $E_D(A, B) = \varnothing$ implies that none of the t_2 observed edges hits a pair $\{v_{\ell_i}^i, v_0^j\}$ for $i \in A$ and $j \in B$. There are at least $\varepsilon'^2 s^2$ such edges, hence that probability is at most

$$\binom{\binom{n}{2} - \varepsilon'^2 s^2}{t_2} \Big/ \binom{\binom{n}{2}}{t_2} \le \exp\left(-\varepsilon'^2 s^2 t_2 \Big/ \binom{n}{2}\right) = \exp(-\Theta(n \log^{-1/6} n)).$$

Here we used the general bound $\binom{a-b}{c}/\binom{a}{c} \le \exp(-bc/a)$. By the union bound over the choices of A, B, the probability that the event in the statement does not hold is at most

$$\binom{n}{\varepsilon' s}^2 \exp(-\Theta(n \log^{-1/6} n)) \le \exp(\varepsilon' s \log\log n - \Theta(n \log^{-1/6} n)) = o(1). \qquad \square$$

Claim 4.3 and Lemma 4.2 imply that D has, **whp**, a directed path of length $(1 - 2\varepsilon')s$. By the way we have defined D, this implies the existence of a path Q of length $(1 - 3\varepsilon')n$ in Builder's graph. To analyse the number of purchased edges at this stage, we note that at any stage an edge is purchased with probability $\asymp s^2/n^2$, hence the expected number of purchased edges is $\asymp t_2 s^2/n^2 \asymp n/\log^{1/6} n$. By Chernoff bounds (Theorem 2.1), the number of purchased edges is sublinear **whp**.

Stage III (Preparing the ground) Time: $\frac{1}{\varepsilon'} n \log n$ — Budget: $36\varepsilon' n$

Let q_1, q_2 be the endpoints of Q. Let $V_1 \cup V_2 = V \setminus V(Q)$ be a partition of the vertices outside Q (so $|V_i| = 3\varepsilon' n/2$ for $i = 1, 2$). Builder performs the next two tasks simultaneously.

(Connecting the endpoints of Q to V_1, V_2) Builder claims the first observed edge from q_i to V_i for each $i = 1, 2$ (unless such an edge already exists). After $cn \log n$ steps, for any $c > 0$, there will be **whp** a neighbour of each of q_i in V_i. Call this neighbour w_i.

(Constructing expanders on V_1, V_2) Builder purchases a copy of $\mathcal{O}_{12}$ in V_i. Observing that every new edge falls inside V_i with probability at least $2\varepsilon'^2$, we conclude, using Theorem 2.1, Lemma 2.8, and Corollary 3.2, that this could be done **whp** by observing at most $\frac{1}{\varepsilon'} n \log n$ edges and purchasing at most $36\varepsilon' n$ of them. By Lemma 3.8, the obtained graphs are both βn-expanders for some $\beta > 0$. For $i = 1, 2$ let $B_i = B[V_i]$ denote Builder's graph on the vertex set V_i.

Stage IV (Sprinkling) Time: $O(n)$ — Budget: $3\varepsilon' n$

By Lemma 2.13 and Theorem 2.1 there exists $C_4 = C_4(\beta, \varepsilon') > 0$ such that by observing at most $C_4 n$ edges and purchasing at most $\frac{3}{2}\varepsilon' n$ edges among those landing inside each V_i, both B_1, B_2 become **whp** Hamiltonian. Conditioning on that event, denote the sets of endpoints of Hamilton cycles in B_i whose other endpoint is w_i by Y_i for $i = 1, 2$. By Lemma 2.14, $|Y_i| \geq \beta n$.

Stage V (Closing a Hamilton cycle) Time: $\log n$ — Budget: 1

To close a Hamilton cycle, Builder purchases an edge between Y_1 and Y_2. The probability of failing to do so after observing $\log n$ steps is $o(1)$. □

4.3 Perfect matchings

In this section we prove Theorem 1.4. Recall the constant k_H from Theorem 3.4, and let $\varepsilon' = \varepsilon / k_H$. To aid in understanding the proof, we refer the reader to Fig. 3, which illustrates the inclusion relationships and relative sizes of sets constructed during the first three stages.

Stage I (Constructing a k-core) Time: $\Theta(n)$ — Budget: $\varepsilon' n$

Let k be some large (even) constant to be determined later (in Stage VI). Choose $\varepsilon_1 = \varepsilon'/(10k)$ and let V_1 be an arbitrary vertex set of size $\sim \varepsilon_1 n$. During the first $t_1 \sim 9\varepsilon_1^{-1} kn$ steps Builder purchases every edge that lands inside V_1. By that time, **whp**, at least $8k\varepsilon_1 n$ (and at most $10k\varepsilon_1 n = \varepsilon' n$) edges are being purchased. Thus, by Corollary 2.16, **whp**, the obtained graph contains a k-core on vertex set $U_1 \subseteq V_1$ of size at least $\varepsilon_1 n/2$. Write $\eta_1 = |U_1|/n$ and note that $\eta_1 \leq \varepsilon_1 \leq \varepsilon'/10$.

Stage II (Constructing a large matching) Time: $\Theta(n)$ — Budget: $n/2$

Let $\varepsilon_2 = \varepsilon'/3$. Let $Y = V \setminus U_1$; so $|Y| = (1 - \eta_1)n$. Let $C_2 = C_2(1, \varepsilon_2)$ be the constant guaranteed by Lemma 2.15. Builder follows a (t_2, b_2)-strategy proposed by Lemma 2.15 to construct a matching M_2 with vertex set $X_2 \subseteq Y$ where $|X_2| \geq |Y|(1 - \varepsilon_2) \sim (1 - \varepsilon_2)(1 - \eta_1)n$. The lemma guarantees that this is doable, **whp**, for $t_2 \sim C_2 n$ and $(1 - \varepsilon)n/2 \leq b_2 \leq n/2$. Write $\eta_2 = 1 - |X_2|/n$, and note that $\eta_1 \leq \eta_2 \leq \eta_1 + \varepsilon_2$.

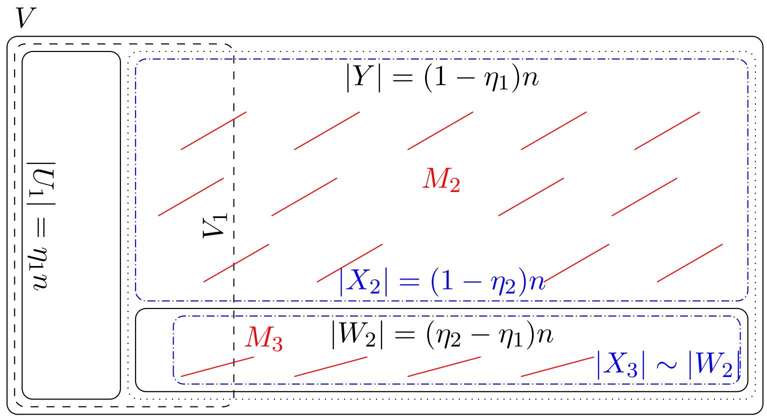

Figure 3: A diagram illustrating the inclusion relationships and relative sizes of the sets $V, V_1, U_1, Y, X_2, W_2, X_3$ constructed during Stages I–III in the proof of Theorem 1.4, in addition to the location of the matchings M_2, M_3.

Stage III (Extending the large matching) Time: $\varepsilon' n \log n$ — Budget: $\varepsilon n/3$

Denote the vertices outside $X_2 \cup U_1$ by W_2. Recall that $|W_2| = n - (|X_2| + |U_1|) = (\eta_2 - \eta_1)n$. Write $\eta_3 = \eta_2 - \eta_1 \le \varepsilon_2$. During the next $t_3 \sim \varepsilon' n \log n$ steps Builder (partially) constructs $\mathcal{O}_{k_H}$ on W_2. The number of purchased edges is **whp** at most $k_H \eta_3 n \le k_H \varepsilon_2 n \le \varepsilon n/3$. Let S be the set of vertices of degree less than k_H in W_2 at the end of this process. We now show that, **whp**, $|S| \le n^{1-\delta_3}$ for $\delta_3 = \eta_3\varepsilon'/2$. Indeed, let $w \in W_2$. The probability that the next edge contains w and is contained in W_2 is $p_3 \sim 2\eta_3/n$. Thus, the total number of observed edges incident to w stochastically dominates a binomial random variable with t_3 attempts and success probability p_3. Consequently, the probability that edges containing w were observed less than k_H times is, by Claim 2.3, at most $n^{-\eta_3\varepsilon'}$. Hence, by Markov's inequality, $\mathbb{P}(|S| \ge n^{1-\delta_3}) \le \eta_3 n^{1-\eta_3\varepsilon'}/n^{1-\delta_3} \ll 1$. We argue that $B[W_2]$ contains a matching that covers all but $2k_H n^{1-\delta_3}$ vertices. Indeed, if we had continued constructing the copy of $\mathcal{O}_{k_H}$ without any time restrictions, we would have had – according to Theorem 3.4 – a (nearly) perfect matching there. But, deterministically, we would have purchased at most $k_H|S|$ additional edges to achieve that goal. Thus, **whp**, the current largest matching covers all but at most $2k_H n^{1-\delta_3}$ vertices in W_2. Denote this matching by M_3, and let $X_3 = V(M_3)$.

Stage IV (Building stars) Time: $(1+\varepsilon')n \log n/2$ — Budget: $o(n)$

We append the matching constructed in Stage III to the matching constructed in Stage II; namely, we set $M = M_2 \cup M_3$, $X = X_2 \cup X_3$ and $W = V \setminus (X \cup U_1)$. We recall that $|W| \le n^{1-\delta_4}$ for some $0 < \delta_4 < \delta_3$ and $|X| \sim (1-\eta_1)n$. Let $K > 1/(2\eta_1\varepsilon')$ be an integer. In this stage Builder attempts to construct disjoint K-leaf stars at each vertex of W, with leaves inside X, where each edge of M contains at most one leaf. For $w \in W$ let x_w denote the number of leaves in the star that is rooted at w that Builder managed to purchase by the end of the stage. We show that **whp** for every $w \in W$, $x_w = K$, hence the stage is successful. Indeed, whenever an edge arrives it has probability $2/n$ to be incident to w. The other end of such an edge is incident to an available edge of M (i.e., an edge which does not already contain a leaf of a star constructed at this stage) with probability at least $(|X| - 2K|W|)/n \sim 1 - \eta_1$. Thus, x_w stochastically dominates a binomial random variable with $t_4 = (1+\varepsilon')n \log n/2$ attempts and success probability $p_4 = (1-2\eta_1) \cdot 2/n$. Therefore, by

Claim 2.3 (since $\varepsilon' < 1$ and $2\eta_1 < \varepsilon'$),

$$\mathbb{P}(x_w < K) \leq \left(\frac{et_4p_4}{K(1-p_4)}\right)^K e^{-t_4p_4} = (\Theta(\log n))^K \cdot n^{-(1+\varepsilon')(1-2\eta_1)} \ll n^{-1}.$$

By the union bound over all $o(n)$ vertices of W, the probability that any x_w is below K is $o(1)$.

Stage V (Building 3-paths) Time: $\varepsilon' n \log n$ — Budget: $o(n)$

For every $w \in W$ let $a_1^w, \ldots, a_K^w \in X$ be the leaves of the stars rooted at w that were constructed in the previous stage. For $i = 1, \ldots, K$, let $b_i^w \in X$ be the neighbour of a_i^w in M. Builder now purchases every edge that connects b_i^w (for some $w \in W$ and $i \in [K]$) to a vertex $u \in U_1$, as long as none of $b_1^w, \ldots, b_K^w, u$ was touched earlier during this stage. Fix $w \in W$. The probability that in a given step an edge between (an untouched vertex of) U_1 and $\{b_1^w, \ldots, b_K^w\}$ appears is at least $p_5 = K(|U_1| - |W|)/\binom{n}{2} \sim 2K\eta_1/n$. Thus, the probability that in $t_5 = \varepsilon' n \log n$ steps such an edge does not appear is at most $e^{-p_5t_5} = n^{-2K\eta_1\varepsilon'}$. Noting that $K > (2\eta_1\varepsilon')^{-1}$ we get that this probability is $o(n^{-1})$, hence, by the union bound over all $o(n)$ vertices of W, we match one leaf in the star rooted at w with a unique vertex in U_1 for every $w \in W$ **whp**.

Stage VI (Sprinkling) Time: $\Theta(n)$ — Budget: $\varepsilon' n$

We note that each of the 3-paths constructed in the previous stage can be used as an augmenting path. Namely, we let M' be the matching obtained by M by replacing every edge $e \in M$ that is contained in such a 3-path (as a middle edge) by the other two edges of that path. Let U' be the set of vertices in U_1 matched in M', and write $U^* = U_1 \setminus U'$. Thus, M' matches all vertices but those in U^*. We now show that $B[U^*]$ is **whp** an expander. For that, note that U' is a uniformly chosen subset of size $|W| \leq n^{1-\delta_4}$ of U_1. We first argue that $\delta(B[U^*]) \geq k/2$ **whp**. Indeed, fix $u \in U_1$, and let $u_1, \ldots, u_k$ be arbitrary neighbours of u in $B[U_1]$. The probability that $d(u, U^*) \leq k/2$ is at most the probability that $k/2$ of these k neighbours end up in U'. By the union bound, this is at most $\binom{k}{k/2}n^{-\delta_4}$

$$\binom{k}{k/2}\frac{\binom{|U_1|-k/2}{|W|-k/2}}{\binom{|U_1|}{|W|}} \leq \left(4 \cdot \frac{|W|}{|U_1|}\right)^{k/2} \asymp n^{-\delta_4 k/2}.$$

Making sure $k > 2/\delta_4$, this is $o(n^{-1})$, and hence, by the union bound, **whp**, $\delta(B[U^*]) \geq k/2$. We remark that there is no circular dependency of the constants here.

We recall that by Lemma 2.4 (taking $t = t_1 \sim 9kn$ and $R = 3\beta_6 n$ for $\beta_6 = 3^{-5}k^{-3/2}$) the random graph process G_t, and hence also $B[U^*]$, typically does not have a set of size $r \leq 3\beta_6 n$ that spans more than $3r$ edges. This, together with the minimum degree of $B[U^*]$, implies expansion; indeed, let $A \subseteq U^*$ be a vertex set with $|A| = a \leq R/3 = \beta_6 n$. Suppose $|N(A)| < 2|A|$ and write $S = A \cup N(A)$ (so $r = |S| < 3a \leq R$). But the number of edges spanned by S is at least $\delta(B[U^*])|A|/2 \geq ka/4 > kr/12 \geq 3r$ (making sure $k \geq 36$), a contradiction to Lemma 2.4. Thus, $B[U^*]$ is a $\beta_6 n$-expander. By Lemma 2.13 there exists $C_6 = C_6(\beta_6)$ such that adding to $B[U^*]$ $C_6\eta_1 n$ random non-edges makes it, **whp**, Hamiltonian. Letting $t_6 = C_6' n/\eta_1$ for $C_6' > C_6$, it follows from Theorem 2.1 that after observing t_6 edges, at least $C_6\eta_1 n$ of them land, **whp**, inside U^* (whose size is $\sim \eta_1 n$). Builder purchases each of these edges. Thus, by Lemma 2.13, Builder's resulting graph on U^* contains, **whp**, a Hamilton cycle. That Hamilton cycle contains a perfect matching M_6. Appending M_6 to M, we get a perfect matching in Builder's graph. □

5 Small subgraphs

5.1 Trees

1-statement

Proof of the 1*-statement of Theorem 1.5.* Let $T_1 \subseteq \ldots \subseteq T_k$ be a sequence of subtrees of T, T_i having i vertices (so $T_k = T$). We first note that if $t \gg n$ then a budget $b = k - 1$ suffices with high probability. Indeed, suppose Builder has constructed T_i (note that he builds T_1 with a zero budget). To build T_{i+1}, Builder waits for one of at least $\sim n$ edges whose addition to his current copy of T_i would create a copy of T_{i+1}. By time $t/(k-1)$ Builder observes (and purchases) such an edge **whp**. Thus, by time t Builder constructs, **whp**, a copy of T.

We may thus assume from now on that $t \le cn$ for some $0 < c < 1/5$ and $t \ge b \gg (n/t)^{k-2}$; by monotonicity, the result extends to the remaining range. In particular, $t \gg n^{1-1/(k-1)}$. For $i = 1, \ldots, k$ define

$$s_i = \frac{b}{k-1} \cdot \left(\frac{t}{(k-1)n}\right)^{i-2},$$

and note that s_i is a strictly monotone decreasing sequence with $n \ge s_1 > s_k \gg 1$. We describe Builder's strategy in stages (for convenience, the first stage will be indexed by 2). We assume that at the beginning of stage i, $i = 2, \ldots, k$, Builder has built s_{i-1} vertex-disjoint copies of T_{i-1}. The assumption trivially holds for $i = 2$, since $s_1 \le n$. At stage $i \ge 2$, Builder purchases any edge extending one of the s_{i-1} copies of T_{i-1} he currently has to a copy of T_i, so that the resulting trees are still vertex-disjoint. He continues doing so only as long as he does not have s_i copies of T_i (in which case this stage is successful) or if he had observed more than $t/(k-1)$ edges during this stage (in which case this stage fails, and thus the entire strategy fails). For stage i and $1 \le j \le t/(k-1)$ let y^i_j denote be the event that the j'th observed edge of stage i is being purchased, or that s_i copies of T_i have already been built. Being at stage i, there are at least $s_{i-1} - s_i$ potential trees to extend, each can be extended by observing one of at least $n - 2b$ edges. Thus, noting that $s_{i-1} - s_i \ge s_{i-1}(1 - c/(k-1))$ and $n - 2b \ge n - 2t \ge n(1 - 2c)$, and conditioning on all possible values of $y^i_{j'}$ for $1 \le j' < j$, we have

$$\mathbb{P}(y^i_j) \ge \frac{(s_{i-1} - s_i)(n - 2b)}{\binom{n}{2}} \ge \frac{2(1 - \frac{c}{k-1})s_{i-1} \cdot (1 - 2c)}{n}.$$

Write $c' = 2(1 - c/(k-1))(1 - 2c)$ and note that for small enough $c > 0$ ($c < 1/5$ suffices) we have $c' > 1$. Therefore (by Proposition 2.2), $y^i = \sum_j y^i_j$ stochastically dominates a binomial random variable with mean $\frac{t}{k-1} \cdot \frac{c' s_{i-1}}{n} = c' s_i$. By Chernoff bounds (Theorem 2.1), the stage is successful **whp**. Thus, Builder builds, **whp**, $s_k \ge 1$ copies of $T_k = T$ by the end of $k-1$ stages, that is, by time t. □

0-statement

Proof of the 0*-statement of Theorem 1.5.* We begin by assuming that $b \ge k - 1 \ge 2$; otherwise, the statement is trivial. We also assume that $b \ll (n/t)^{k-2}$, which implies that $t \ll n$. Moreover, we may assume that $b \ll t$. Indeed, suppose that $b \asymp t$. Then, from $b \ll (n/t)^{k-2}$, it follows that $t \ll (n/t)^{k-2}$, leading to $t \ll n^{1-1/(k-1)}$. This implies that **whp** no tree on k vertices exists in the graph of the observed edges.

We prove that any (t, b)-strategy of Builder fails, **whp**, to build a connected component of size k. Define a sequence $(r_i)_{i=1}^{k-1}$ by $r_1 = n$ and $r_i = b(3kt/n)^{i-2}$ for $i = 2, \ldots, k-1$. We argue that

whp Builder cannot build more than r_i connected components of size i. The proof is by induction on i. The case $i = 1$ is trivial. The case $i = 2$ follows deterministically by the budget restriction, as $r_2 = b$. To build a connected component of size $2 < i < k$, Builder must observe an edge connecting a connected component of size $1 \leq j < i$ with a connected component of size $i - j \geq j$. By induction, the probability that the next observed edge is such, conditioning on any previous sequence of failures and successes, is at most

$$\frac{\sum_{j=1}^{\lfloor i/2 \rfloor}(jr_j \cdot (i-j)r_{i-j})}{\binom{n}{2} - t} \leq \frac{inr_{i-1} + \Theta\big(b^2(t/n)^{i-4}\big)}{\binom{n}{2} - t} \leq \frac{2.5kr_{i-1}}{n}.$$

Here we used the fact that $b \ll t$, hence $b^2(t/n)^{i-4} \ll nr_{i-1}$. Thus, by Chernoff bounds (Theorem 2.1 and Proposition 2.2), during t rounds the number of such edges is, **whp**, smaller than $r_{i-1} \cdot 3kt/n = r_i$. To build a tree of order k, Builder must construct a connected component of size at least k. For that, Builder must observe an edge connecting a connected component of size $1 \leq i \leq k-1$ with a connected component of size $\max\{i, k-i\} \leq j \leq k-1$ (in his graph). The probability that the next observed edge is such is at most

$$\frac{\sum_{i=1}^{k-1}\sum_{j=\max\{i,k-i\}}^{k-1}(ir_i \cdot jr_j)}{\binom{n}{2} - t} \preceq \frac{r_{k-1}}{n}.$$

Hence, during t rounds the expected number of edges Builder observes that create such a connected component is $\preceq r_{k-1}t/n \ll 1$; thus, Builder fails to construct such a component **whp**. $\square$

5.2 Cycles

For a fixed graph H we call a non-edge e in Builder's graph an ***H*-trap** (or, simply, a trap) if $B + e$ contains a copy of H.

1-statement

Proof of the 1-statement of Theorem 1.6. Let $b \gg b^* = b^*(n, t, k)$. As $b^* \ll n$, we may assume $b \ll n$. Choose $n \ll t' \ll t$ such that it still holds that $b \gg b' := b^*(n, t', k)$. Set $s = n^2/(b't') = \min\{(t'/n)^k, n/\sqrt{t'}\}$, and note that $1 \ll s \ll b$. Indeed, $b/s \gg b'/s = n^2/(s^2t') \geq 1$. Denote $d = cs^{1/k}$ for $c < 1/(k+5)$ and note that $d \gg 1$. Note further that $dn = cs^{1/k}n \leq ct'$ and that $c^k < 1/6$.

Stage I (Growing d-ary trees) Time: t' — Budget: $b/2$

Builder performs Stage I in k rounds, each lasting dn steps. Set $r = b/s \gg 1$. Builder chooses $2^k r$ vertices arbitrarily, call them *roots*. Denote by $P_0 = \{v_1, \ldots, v_{2^k r}\}$ the set of these roots, and let $L_0 = R_0 = P_0$ (we think of each root as being both on the right side of a tree and on the left side of a tree). Suppose at the beginning of round $i \in [k]$ we have a set $J_{i-1} \subseteq [P_0]$ with $|J_{i-1}| = 2^{k-(i-1)}r$ and trees $(T_j)_{j \in J_{i-1}}$, where the tree T_j is rooted at v_j and is of depth $i - 1$ (so at the beginning of round 1 these trees are isolated roots). Write L_i^j for the set of vertices in L_i which are in tree j, and define R_i^j analogously. Builder's goal at round i is to "extend" *half* of these trees, by attaching $d|L_{i-1}^j|$ leaves to vertices in L_{i-1}^j and $d|R_{i-1}^j|$ (distinct) leaves to vertices in R_{i-1}^j. Each of the new leaves should not be already covered by Builder's graph. A leaf attached to L_{i-1}^j will be placed in L_i^j, and a leaf attached to R_{i-1}^j will be placed in R_i^j. Note that if $i = 1$ then $L_{i-1}^j = R_{i-1}^j = \{v_j\}$, in which case we attach *two* d-leaf stars at each root v_j. Let $J_i \subseteq J_{i-1}$ be the subset of indices for which

T_j has been successfully extended. Set $L_i = \bigcup_{j\in J_i} L_i^j$ and $R_i = \bigcup_{j\in J_i} R_i^j$. Note that if Builder's strategy has not failed by the beginning of round i, $|J_{i-1}| = 2^{k-(i-1)}r$ and $|L_{i-1}^j| = |R_{i-1}^j| = d^{i-1}$ for every $j \in J_{i-1}$. For $i \in [k]$ and $j \in J_{i-1}$ let $x_i^{(j)}$ be the number of edges observed at round i with one endpoint in L_{i-1}^j and the other outside the set of vertices covered by the edges of Breaker's graph at the time of observation. If $|x_i^{(j)}| \geq d|L_{i-1}^j|$ then Builder purchases the first $|L_{i-1}^j|$ of those observed edges and this stage is "left-successful" for tree j. Similarly, let $y_i^{(j)}$ be the number of edges observed at round i between R_{i-1}^j and the uncovered part of the graph. If $|y_i^{(j)}| \geq d|R_{i-1}^j|$ then Builder purchases the first $|R_{i-1}^j|$ of those observed edges and this stage is "right-successful" for tree j. If a stage is both left-successful and right-successful for a tree j then T_j is successfully extended. As we remarked earlier, after extending successfully $2^{k-i}r$ trees, Builder stops.

Let us analyse this strategy. For $0 \leq i \leq k$, denote by b_i the number of covered vertices by the end of step i. Observe that

$$b_i \leq \sum_{i'=0}^{i} 2^{k+1-i'} r d^{i'} \leq (1+o(1)) 2^{k-i+1} r d^i$$

(since $d \gg 1$). In particular, $b_k < 3rd^k = 3c^k b \ll n$. Take $i \in [k]$ and assume the strategy was successful so far. Thus, $x_i^{(j)}$ (and also $y_i^{(j)}$) stochastically dominates a binomial random variable with dn attempts and success probability at least $|L_{i-1}^j| \cdot (n - b_i)/\binom{n}{2} \sim 2d^{i-1}/n$. Thus, by Chernoff bounds (Theorem 2.1), $\mathbb{P}(x_i^{(j)} < d|L_{i-1}^j|) = \mathbb{P}(x_i^{(j)} < d^i) < \exp(-d^i/5) \ll 1$. By Markov's inequality, **whp** for at least half of the trees indexed by J_{i-1} Builder succeeds in extending the tree. The total time that passes until the end of this stage is at most $kdn \leq ckt' < t' \ll t$, and the total budget is at most $3c^k b < b/2$.

Stage II (Connecting leaves) Time: $t - t'$ — Budget: 1

Note that for every $j \in J_k$, if $u \in L_k^j$ and $v \in R_k^j$ then u, v are both leaves of T_j of distance $2k$ from each other. Thus, by adding the edge $\{u, v\}$, a cycle of length ℓ is formed. Thus, the number of traps is $|J_k| \cdot |L_k^j| \cdot |R_k^j| = rd^{2k} \asymp rs^2 = bs \gg n^2/t'$. Thus, after time $t - t' \gg t'$ we hit a trap **whp**.

Even cycles We briefly discuss the modifications needed to obtain the result for cycles of length $2k+2$. Instead of choosing P_0, Builder greedily constructs a matching M_0 of size $2^k r$; this is doable (**whp**) since the first Cr observed edges (recalling that $r \ll n$) are mostly disjoint, and, in particular, for large enough C, contain a matching of size $2^k r$. Then, for each edge in M_0, Builder puts one vertex in L_0 and one vertex in R_0. Builder continues in the same fashion as in the case of odd cycles, where the only difference is that instead of $L_0 = R_0$ we have $L_0 \cap R_0 = \varnothing$. We finish by noting that an edge between $u \in L_k^j$ and $v \in R_k^j$ now closes a cycle of length $2k+2$. □

0-statement

Let $H = C_\ell$ for $\ell = 2k+1$ or $\ell = 2k+2$. The 0-statement in Theorem 1.6 follows from combining the next two claims together with the known fact that if $t \ll n$ then $G_{n,t}$, **whp**, contains no copy of H.

Claim 5.1. *If $b \ll n/\sqrt{t}$ then for any (t, b)-strategy B of Builder,*

$$\lim_{n\to\infty} \mathbb{P}(H \subseteq B_t) = 0.$$

Claim 5.2. *If $b \ll n^{k+2}/t^{k+1}$ then for any (t,b)-strategy B of Builder,*

$$\lim_{n\to\infty} \mathbb{P}(H \subseteq B_t) = 0.$$

Proof of Claim 5.1. We may assume $t \ll n^2$. Consider a strategy for building H. Let B' be Builder's graph before obtaining the first copy of H. We show that at this point, the time it takes Builder to hit an H-trap (and hence construct a copy of H) is typically much larger than t. Let X denote the number of H-traps at this point. Observe that every H-trap is a non-edge of B', both whose endpoints are vertices of positive degree. Evidently, the number of vertices of positive degree is at most $2b$, thus $X \le 4b^2$. Thus, the probability of hitting a trap in t steps is at most $tX/(\binom{n}{2} - t) \preceq tb^2/n^2 \ll 1$. □

For the proof of Claim 5.2 we will need a couple of lemmas. In what follows, we may assume that $t \le n^{(k+1)/(k+1/2)}$, as otherwise $n^{k+2}/t^{k+1} \le n/\sqrt{t}$ and the claim follows from Claim 5.1.

Claim 5.3. *Let $\ell \ge 1$ be an integer, write $d = 2t/n \gg 1$ and let $C > 1$. Then, **whp**, the number of paths of length ℓ in G_t that contain a vertex whose degree is at least Cd is $o(n/d)$.*

Proof. Let L be the set of all vertices of degree at least Cd, and let Q be the set of all paths containing a vertex from L. By Chernoff bounds (Theorem 2.1),

$$\mathbb{E}|Q| \le n^{\ell+1}\left(\frac{t}{n^2}\right)^{\ell}(\ell+1)\cdot \mathbb{P}(d(v) \ge Cd - 2) \le nd^{\ell}e^{-cd} \ll n/d,$$

for some constant $c = c(C) > 0$, and the result follows from Markov's inequality. □

We turn to count the number of copies of a path of length ℓ in a z-degenerate graph G_0 with maximum degree Δ.

Lemma 5.4. *Let $z, q, \Delta, \ell \ge 1$ be integers. Let G_0 be a z-degenerate q-vertex graph with maximum degree Δ. Then, the number of paths of length ℓ in G_0 is at most $q \cdot 2^{\ell} z^{\lceil \ell/2 \rceil} \Delta^{\lfloor \ell/2 \rfloor}$.*

Proof. Let $v_1, \dots, v_q$ be an ordering of the vertices of G_0 such that for $1 \le i \le q$, v_i has at most z neighbours among $v_1, \dots, v_{i-1}$. Given a path P of length ℓ, one of its consistent orientations has at least k edges going backwards (w.r.t. the orientation). Thus, to count the number of copies of P in G_0, it suffices to count the number of *directed* paths of length ℓ with at least $\lceil \ell/2 \rceil$ edges going backwards. To count those, first choose a starting vertex (n options), then choose the locations of the edges going back (at most 2^{ℓ} options), and finally choose from at most z options for an edge pointing back, or at most Δ options otherwise. Altogether, there are at most $q \cdot 2^{\ell} z^{\lceil \ell/2 \rceil} \Delta^{\lfloor \ell/2 \rfloor}$ such paths. □

Proof of Claim 5.2. Consider a strategy for building H. Let B' be Builder's graph before obtaining the first copy of H. We show that at this point, the time it takes Builder to hit an H-trap (and hence construct a copy of H) is typically much larger than t. Let X denote the number of H-traps at this point. Observe that X is bounded from above by the number of paths of length $\ell - 1$. Write $d = 2t/n$ and let B'' be obtained from B' by deleting every vertex of degree at least Cd. By Claim 5.3, the number of $(\ell - 1)$ paths that we removed when obtaining B'' is **whp** $o(n/d)$. Evidently, $\Delta(B'') \le Cd$. By Claim 2.5, since $b^2 n^{2k+4}/t^{2k+2} \le n^5/(40t^3)$ (here we use $t \ge 40n$), and since we may assume $b \gg 1$, B'' is, **whp**, 6-degenerate. Thus, by Lemma 5.4, B' has, **whp**, at most $2b \cdot 6^{k+1} \cdot (3t/n)^k + o(n/d) \ll n^2/t$ paths of length $\ell - 1$. Thus, **whp**, $X \ll n^2/t$. Therefore, **whp**, the probability of hitting a trap in t steps is at most $tX/(\binom{n}{2} - t) \ll 1$. □

6 Concluding remarks and open problems

We have proposed a new model for a controlled random graph process that falls into the broader category of online decision-making under uncertainty. The question we considered is the following: given a monotone graph property $\mathcal{P}$, is there an online algorithm that decides whether to take or leave any arriving edge that obtains, **whp**, the desired property within given time and budget constraints. We analysed the model for several natural graph properties: connectivity, minimum degree, Hamiltonicity, perfect matchings and the containment of fixed-size trees and cycles. Our focal point was the investigation of the inevitable trade-off between the total number of edges Builder observes (maximum time) and the number of edges he might need to purchase (minimum budget). In some cases, the trade-off is substantial (for example, to construct a triangle in a close-to-optimal time, namely, when these just appear in the random graph process, Builder must purchase an almost-linear number of edges). For containment of fixed subgraphs, we have quantified that trade-off precisely.

Although we have made some sizable progress, there are a few challenging open problems to consider. Returning to our first result, we have shown (Theorem 1.1) that the budget needed to obtain minimum degree k at the hitting time is at most $\mathfrak{o}_k n$. We believe that this result is (asymptotically) optimal in the following sense:

Conjecture 6.1. *For every $k \geq 2$ and $\varepsilon > 0$, if $b \leq (\mathfrak{o}_k - \varepsilon)n$ then for any (τ_k, b)-strategy B of Builder,*

$$\lim_{n\to\infty} \mathbb{P}(\delta(B_{\tau_k}) \geq k) = 0.$$

We continued by showing (Theorem 1.2) that if we allow the time to be only asymptotically optimal, then an asymptotically optimal budget suffices. For large enough k, a similar statement for k-vertex-connectivity follows from Theorem 1.1 and Corollary 3.2. For $k = 1$, as demonstrated in the introduction, this is trivially true. Recently, after an earlier version of this paper appeared online, Lichev [42] proved that this is true for every $k \geq 2$:

Theorem 6.2 ([42])**.** *Let $k \geq 2$ be an integer and let $\varepsilon > 0$. If $t \geq (1+\varepsilon)n\log n/2$ and $b \geq (1+\varepsilon)kn/2$ then there exists a (t, b)-strategy B of Builder such that*

$$\lim_{n\to\infty} \mathbb{P}(B_t \text{ is } k\text{-connected}) = 1.$$

For $k = 2$, we do not have a guess for the correct budget threshold at the hitting time for 2-connectivity, and believe it might be larger than $\mathfrak{o}_2 n = \frac{11}{8}n$.

For Hamilton cycles, we showed (in Theorem 1.3) that at the hitting time and with an inflated budget, Builder has a strategy that typically constructs a Hamilton cycle. Together with Theorem 4.1 it follows that if time is (asymptotically) optimal and the budget is inflated (by a constant factor), or when the budget is (asymptotically) optimal and the time is inflated, then Builder has such a strategy. As we mentioned in the introduction, Anastos [3] recently proved that this is doable in asymptotically optimal time *and* budget (but *not* at the hitting time). We remark that for any $k = k(n) \geq 1$, if $t \ll n^2/k^2$ and $b \leq n + k$, then any (t, b)-strategy of Builder fails **whp**; this follows almost directly from [25] (see Section 5 there). This implies that if t is asymptotically optimal (or even $t = O(n \log n)$) then, in particular, a budget of $n + o(\sqrt{n}/\log n)$ does not suffice.

It might be interesting to extend either the result of Anastos or Theorem 1.3 to other spanning structures. For example, can Builder construct (**whp**) the square of a Hamilton cycle at its hitting time (or after its threshold) using only a linear budget? More generally, can Builder construct (**whp**) any given bounded-degree spanning graph slightly after its threshold using only a linear budget? It might also be interesting to optimise the constant C in Theorem 1.3, or find the minimum k for which $\mathcal{O}_{k,2}$ (or $\mathcal{O}_k$) is **whp** Hamiltonian.

Theorems 1.5 and 1.6 considered the budget thresholds for trees and cycles. It is not hard to extend our result on cycles to any fixed unicyclic graph: after obtaining the cycle, the remaining forest can be constructed quickly and with a constant budget. The smallest graph not covered by our results is therefore the diamond (a K_4 with one edge removed). One of the difficulties in this particular problem (or, possibly, in the case of any non-unicyclic graph) is that several quite different natural strategies, some are naive and some more sophisticated, turn out to give different upper bounds; each is superior in a different time regime. We handled a similar difficulty when discussing cycles but with lesser apparent complexity. It would be interesting to develop tools and approaches allowing to tackle the case of a generic fixed graph.

Finally, while *the* random graph process might be the most natural underlying graph process, other processes may be considered. One compelling example is the so-called *semi-random graph process* [8] mentioned in the introduction. The semi-random process has already exhibited some intriguing phenomena that significantly distinguish it from the (standard) random graph process. In our context, instead of observing a flow of random edges, Builder sees a flow of random vertices and, at each round, decides whether to connect the observed vertex to any other vertex. Evidently, Builder can achieve connectivity with $t = b = n-1$; answering questions concerning, e.g., Hamilton cycles or fixed subgraphs would be interesting.

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