On the expected efficiency of branch and bound for the asymmetric TSP

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Abstract

Let the costs $C(i, j)$ for an instance of the asymmetric traveling salesperson problem be independent uniform $[0,1]$ random variables. We consider the efficiency of branch and bound algorithms that use the assignment relaxation as a lower bound. We show that w.h.p. the number of steps taken in any such branch and bound algorithm is $e^{\Omega(n^a)}$ for some small absolute constant $a > 0$.

1 Introduction

Given an $n \times n$ matrix $C = (C(i, j))$ we can define two discrete optimization problems. Let $S_n$ denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$. Let $T_n \subseteq S_n$ denote the set of cyclic permutations i.e. those permutations whose cycle structure consists of a single cycle. The Assignment Problem (AP) is the problem of minimising $C(\pi) = \sum_{i=1}^{n} C(i, \pi(i))$ over all permutations $\pi \in S_n$. We let $Z_{AP} = Z_{AP}^{(C)}$ denote the optimal cost for AP. The Asymmetric Traveling-Salesperson Problem (ATSP) is the problem of minimising $C(\pi) = \sum_{i=1}^{n} C(i, \pi(i))$ over all permutations $\pi \in T_n$. We let $Z_{ATSP} = Z_{ATSP}^{(C)}$ denote the optimal cost for ATSP.

Alternatively, the assignment problem is that of finding a minimum cost perfect matching in the complete bipartite graph $K_{A,B}$ where $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ and the cost of edge $(a_i, b_j)$ is $C(i, j)$. The Asymmetric Traveling-Salesperson Problem is that of finding a minimum cost Hamilton cycle in the complete digraph $\vec{K}_n$ where the cost of edge $(i, j)$ is $C(i, j)$.

It is evident that $Z_{AP}^{(C)} \leq Z_{ATSP}^{(C)}$. The ATSP is NP-hard, whereas the AP is solvable in time $O(n^3)$. Several authors, e.g. Balas and Toth [3], Kalczynski [17], Zhang [20] have investigated whether the AP can be used effectively in a branch-and-bound algorithm to
We implicitly study a case where \( L(n) \rightarrow \infty \). We will assume that the costs \( C(i, j) \) are now independent copies of the uniform \([0, 1]\) random variable \( U[0, 1] \). This model was first considered by Karp [18]. He proved the surprising result that

\[
Z_{ATSP} - Z_{AP} = o(1) \text{ w.h.p.} \tag{1}
\]

Since w.h.p. \( Z_{AP} > 1 \) we see that this rigorously explained the observed quality of the assignment bound. Karp [18] proved (1) constructively, analysing an \( O(n^3) \) patching heuristic that transformed an optimal AP solution into a good ATSP solution. Karp and Steele [19] simplified and sharpened this analysis, and Dyer and Frieze [9] improved the error bound of a related more elaborate algorithm to \( O\left(\frac{\ln n}{n \ln \ln n}\right) \). Frieze and Sorkin [15] reduced the error bound to \( O\left(\frac{(\ln n)^2}{n}\right) \) w.h.p. One might think that with such a small gap between \( Z_{AP} \) and \( Z_{ATSP} \), that branch and bound might run in polynomial time w.h.p. Indeed one is encouraged by the recent results of Dey, Dubey and Molinaro [8] and Borst, Dadush, Huiberts and Tiwari [6] that with a similar integrality gap, branch and bound with LP based bounds solves random multi-dimensional knapsack problems in polynomial time w.h.p.

The algorithm with the best known worst-case time for solving the ATSP exactly is the \( O(n^{22^n}) \) dynamic programming algorithm of Held and Karp [16]. Given that \( Z_{ATSP} - Z_{AP} \) is usually so small, it is clearly of interest to see if using \( Z_{AP} \) as a bound in a branch and bound algorithm will produce an optimum solution in polynomial time w.h.p. Frieze and Sorkin showed the following improvement over the worst-case:

W.h.p., a random instance of the ATSP can be solved exactly in time \( e^{\tilde{O}(\sqrt{n})} \).

Bellmore and Malone [4] claimed that the expected running time of a branch and bound algorithm for this problem was bounded by \( O(n^4) \). Lenstra and Rinnooy Kan [20] and Zhang [25] pointed out the failure to account for conditioning in this analysis.

Our main result shows that w.h.p. branch and bound does not run in polynomial time. Let \( D = \{(i, i) : i \in [n]\} \). These edges will be excluded from consideration i.e. we make \( C(i, i) = \infty \) for \( i \in [n] \). In any use of \( Z_{AP} \) as a lower bound to ATSP, it is natural to avoid using loops \((i, i)\). Unfortunately it is NP-hard to avoid cycles of length two.

We now set up some notation for what we mean by a branch and bound algorithm. It is a rooted binary tree \( T \) and each node \( \nu \) is labeled with a triple of disjoint sets \( F(\nu) = (F_0(\nu), F_1(\nu), G(\nu)) \). \( F_0(\nu) \) is the set of edges which have been chosen to be excluded from tours corresponding to this node. \( F_1(\nu) \) is the set of edges which must be included in all tours.

\footnote{with high probability, i.e., with probability \( 1 - o(1) \) as \( n \rightarrow \infty \)}
corresponding to this node and \( G(\nu) = (A \times B) \setminus (F_0(\nu) \cup F_1(\nu) \cup D) \). \( \text{AP}(F) \) is the assignment problem with the additional restrictions imposed by \( F = F(\nu) \). The children \( \nu+, \nu- \) of \( \nu \) must be such that for some edge \( e \) we have \((F_0(\nu+), F_1(\nu+)) = (F_0(\nu), F_1(\nu) \cup \{e\}) \) and \((F_0(\nu-), F_1(\nu-)) = (F_0(\nu) \cup \{e\}, F_1(\nu)) \). In this way, one child adds \( e \) to every solution and the other excludes \( e \). There are some obvious restrictions placed on the set of edges in \( F_1 \) that make them extendable to a tour. The digraph induced by \( F_1 \neq \emptyset \) must have maximum in- and out-degree one and cannot contain a cycle \( C, |C| < n \). So, sometimes the child \( \nu+ \) is not created. We say that the associated edge \( e \) is inadmissible and we let \( F_0 \) denote the set of inadmissible edges.

Making the branch and bound tree binary does not lose generality, as we may replace any rooted tree by a binary tree in a natural way, see Figure 1.

In summary

- \( F_1 \) denotes the edges that are forced to be in the solution to \( \text{AP}(F) \) by branching.
- \( F_0 \) denotes the edges that are forced to be out of the solution to \( \text{AP}(F) \) by branching.
- \( \widehat{F}_0 \) denotes the edges that forced to be out of the solution to \( \text{AP}(F) \) through the inclusion of \( F_1 \).
- \( D = \{(i,i) : i \in [n]\} \).
- The optimal value for \( \text{AP}(F) \) will be denoted by \( Z_{\text{AP}}(F) \).

**Theorem 1** If \( \xi > 0 \) is a sufficiently small absolute constant then w.h.p. any branch and bound algorithm for the ATSP that uses \( Z_{\text{AP}} \) as a lower bound and branches by including and excluding edges from the solution, must explore at least \( e^{\Omega(n^\xi)} \) sub-problems.

(In the following analysis, one should equate \( \xi \) with \( \epsilon/3 \).)

**2 Outline Proof of Theorem 1**

The result rests on two lemmas. The first lemma proves a high probability lower bound on \( Z_{\text{ATSP}} - Z_{\text{AP}} \). Throughout the paper \( \epsilon \) is a sufficiently small positive constant. We prove the following lemma in Section 4.
Lemma 2

\[ \Pr \left( Z_{\text{ATSP}} - Z_{\text{AP}} \leq \frac{1}{n^{3/2}} \right) = o(1). \]

We will find that w.h.p. there are many nodes of the branch and bound tree with restrictions \( F \), for which the optimal solution to \( \text{AP}(F) \) will have cost less than \( Z_{\text{AP}} + \frac{1}{n^{3/2}} \). These nodes will not be eliminated by bounding.

We let \( M_F \) denote the set of \( n \) edges in the perfect matching that solves \( \text{AP}(F) \), this is unique with probability one. In the subsequent analysis, we concentrate on the case where

\[ |F_1| \leq n^{3e/8} \text{ and } |F_0| \leq n^{3e/4}. \]  \( \text{(2)} \)

We prove the following lemma in Section \( 3 \)

Lemma 3 W.h.p, simultaneously for every \( F \) that satisfies \( (2) \) there exist distinct perfect matchings \( M_i = \mu_i(F) \), \( i = 1, 2, \ldots, d = \lceil n^{1/3} \rceil \) which are feasible for \( \text{AP}(F) \) and whose costs \( C(M_i) \), \( i = 1, 2, \ldots, d \) satisfy

\[ C(M_i) - C(M_F) \leq \frac{1}{n^{3/2+2e}}. \]

Furthermore, there exist edges \( e_1, e_2, \ldots, e_d \notin F \) such that \( e_i \in M_i \setminus M_j \) for all \( 1 \leq i \neq j \leq d. \)

Assume the truth of Lemmas \( 2 \) \( 3 \) for the moment and let us see how to prove Theorem \( 8 \). W.h.p. we can generate a large number of distinct perfect matchings that have a lower cost than \( Z_{\text{ATSP}} \). We construct a \( d \)-ary tree \( T_\nu \) of depth at most \( d \), where each node \( \nu \) is labelled by a set of restrictions \( F^{(\nu)} = (F^{(\nu)}_1, F^{(\nu)}_0, F^{(\nu)}_\emptyset) \) and a perfect matching \( M^{(\nu)} \). Our construction will be such that if node \( \mu \) is a child of node \( \nu \) then

\[ F^{(\mu)}_1 \supseteq F^{(\nu)}_1 \text{ and } |F^{(\mu)}_1| = |F^{(\nu)}_1| + 1 \text{ and } F^{(\mu)}_0 \supseteq F^{(\nu)}_0 \text{ and } |F^{(\mu)}_0| = |F^{(\nu)}_0| + d - 1. \]  \( \text{(3)} \)

Let \( e_i, M_i(\emptyset), i = 1, 2, \ldots, d \) be the edges and matchings promised by Lemma \( 3 \) (Here we are using the notation of Lemma \( 3 \) with \( F = \emptyset \) at the root.) Then for \( i = 1, 2, \ldots, d \), let \( F^{(i)}_1 = \{ e_i \} \) and \( F^{(i)}_0 = \{ e_j : j \neq i \} \). The child \( \rho_i \) of the root \( \rho \) will be labelled with \( (F^{(i)}_1, F^{(i)}_0, F^{(i)}_\emptyset) \) and \( M_i(\emptyset) \) for \( i = 1, 2, \ldots, d \).

Suppose now that we have constructed \( 1 \leq k < d \) levels of \( T_\nu \) and that \( \nu \) is a node at level \( k \). We see that assuming \( (3) \), that \( F^{(\nu)} \) satisfies \( (2) \). Let now \( e_i, M_i(F^{(\nu)}), i = 1, 2, \ldots, d \) be the edges and matchings promised by Lemma \( 3 \). The children of \( \nu \) will be denoted \( \mu_1, \mu_2, \ldots, \mu_d \). Then for \( i = 1, 2, \ldots, d \), let \( F^{(\mu_i)}_1 = F^{(\nu)}_1 \cup \{ e_i \} \) and \( F^{(\mu_i)}_0 = F^{(\nu)}_0 \cup \{ e_j : j \neq i \} \) and \( M^{(\mu_i)} = M_i(F^{(\nu)}) \), verifying \( (3) \).

The tree \( T_\nu \) will have \( \lambda = d^d \) leaves \( L_\nu \). Furthermore, if \( \nu_1, \nu_2 \) are leaves of \( T_\nu \) then \( M^{(\nu_1)} \neq M^{(\nu_2)} \). Let \( \mathcal{M} = \{ M^{(\nu)} : \nu \in L_\nu \} \) and note that if \( \nu \in L_\nu \) then

\[ Z_{\text{AP}}(F^{(\nu)}) \leq C(M^{(\nu)}) \leq Z_{\text{AP}} + \frac{d}{n^{3/2+2e}} < Z_{\text{ATSP}} \]  \( \text{(4)} \)
Next let \( X_+ = \bigcap_{M \in \mathcal{M}} M \) and \( X_- = (A \times B) \setminus \bigcup_{M \in \mathcal{M}} M \) be the sets of edges in all or none of the matchings of \( \mathcal{M} \) respectively.

Now consider the actual branch and bound tree \( T \). We first construct a smaller tree \( \hat{T} \subseteq T \).

If at a node \( \nu \) of \( T \) we branch on \( e \in X_- \) then we remove \( \nu_+ \) and its descendants and replace \( \nu \) by \( \nu_- \). If at a node \( \nu \) of \( T \) we branch on \( e \in X_+ \) then we remove \( \nu_- \) and its descendants and replace \( \nu \) by \( \nu_+ \). Assume by induction on depth that \( |\hat{T}| \geq |\mathcal{M}| \).

For each node \( \nu \) of \( \hat{T} \) we let \( \mathcal{M}_\nu \) denote the matchings in \( \mathcal{M} \) that satisfy the constraints of \( \nu \). Suppose that the children of the root of \( \mathcal{T} \) include/exclude the edge \( e \). Let \( \mathcal{M}_+ = \{ M \in \mathcal{M} : e \in M \} \) and \( \mathcal{M}_- = \mathcal{M} \setminus \mathcal{M}_+ \). Both are non-empty and induction on depth tells us that \( \hat{T} \geq |\mathcal{M}_+| + |\mathcal{M}_-| = |\mathcal{M}| \).

The basis of the induction is at the leaves of \( \hat{T} \) where there are nodes \( \nu \) for which \( \mathcal{M}_\nu = \emptyset \). A node for which \( \mathcal{M}_\nu \neq \emptyset \) would cause a branch. This proves Theorem 1.

3 Analysis of the Assignment Problem

Let \( K_{A,B,F} \) be obtained from the complete bipartite graph \( K_{A,B} \) by deleting the edges in \( D \cup F_0 \cup \hat{F}_0 \). Recall that \( \text{AP}(F) \) is the problem of finding a minimum cost perfect matching from \( A \) to \( B \) with restrictions defined by \( F \). Our immediate aim is to show that w.h.p. the optimal matching \( M^*_F \) does not use expensive edges.

Given the bipartite graph \( K_{A,B,F} \), any permutation \( \pi : A \to B \) has an associated matching \( M_\pi = \{(x,y) : x \in A, y \in B, y = \pi(x)\} \), assuming that \( M_\pi \cap (D \cup F_0 \cup \hat{F}_0) = \emptyset \). Define the \( k \)-neighborhood of a vertex \( v \in A \cup B \) to be the \( k \) closest neighbors of \( v \), where distance is given by the matrix \( C \); let the \( k \)-neighborhood of a set be the union of the \( k \)-neighborhoods of its vertices. In particular, for the bipartite graph \( K_{A,B,F} \) and any \( S \subseteq A \), \( T \subseteq B \) and any permutation \( \pi \),

\[
N_k(S) = \{y \in B : \exists s \in S \text{ s.t. } (s,y) \text{ is one of the } k \text{ shortest edges of } K_{A,B,F} \text{ out of } s\}, \quad (5)
\]

\[
N_k(T) = \{x \in A : \exists t \in T \text{ s.t. } (x,t) \text{ is one of the } k \text{ shortest edges of } K_{A,B,F} \text{ into } t\}. \quad (6)
\]
All of the edges in $N_k(S), N_k(T)$ are oriented from $A$ to $B$ and do not belong to $M_\pi$. Given a cost matrix $C$ and permutation $\pi$ (perfect matching $M_\pi = \{(i, \pi(i)) : i \in [n]\}$), define the digraph

$$\vec{D}_F = \vec{D}_F(C, \pi) = (A \cup B, \vec{E}_{\pi})$$

consisting of backwards matching edges and forward “short” edges: Let

$$\zeta = \lceil n^c \rceil$$

and

$$\vec{E}_{\pi} = \{(y, x) : y \in B, x \in A, y = \pi(x)\} \cup \{(x, y) \notin F_0 \cup F_1 \cup \hat{F}_0 : x \in A, y \in N_\zeta(x)\} \cup \{(x, y) \notin F_0 \cup F_1 \cup \hat{F}_0 : y \in B, x \in N_\zeta(y)\}. \quad (8)$$

The edges of directed paths in $\vec{D}_F$ are alternately forwards $A \to B$ and backwards $B \to A$ and so they correspond to alternating paths with respect to the perfect matching $M_\pi$. The forward edges will replace the backward ones and so it helps to know (Lemma 4, next) that given $x \in A, y \in B$ we can find an alternating path from $x$ to $y$ with $O(1)$ edges. Let the unweighted/weighted $A:B$ diameter denote the maximum over $a \in A, b \in B$ of the minimum number/weight of edges in a path from $a$ to $b$ in $\vec{D}_F$.

**Lemma 4** Suppose that $F$ satisfies (2). Then over random cost matrices $C$, for every permutation $\pi$,

$$\Pr(\text{the unweighted } A:B \text{ diameter of } \vec{D}_F \geq 3/\epsilon) \leq e^{-\zeta/4}. \quad (9)$$

**Proof.** Let $\beta = \zeta/10$. We first estimate the probability that for all $S \subseteq A$ with $|S| \leq \left\lfloor \frac{2n}{3\beta} \right\rfloor$, $|N_\zeta(S)| \geq \beta|S|$. Note that only the cheap edges out of $S$, and not the backward matching edges into it, will be involved here. Note also, that because $|F_1| + |F_2| \ll \zeta$, at most $\zeta$ edges of $K_{A,B}$ are excluded in $K_{A,B,F}$ from those incident with a fixed vertex. (At most $o(\zeta)$ from $F_0$ and at most $o(\zeta)$ from $\hat{F}_0$.)

$$\Pr(\exists S : |S| \leq \left\lfloor \frac{2n}{3\beta} \right\rfloor, |N_\zeta(S)| < \beta|S|) \leq \sum_{s=1}^{\left\lfloor \frac{2n}{3\beta} \right\rfloor} \binom{n}{s} \left( \binom{\beta s}{n - s} \right)^s \leq \sum_{s=1}^{\left\lfloor \frac{2n}{3\beta} \right\rfloor} \left( \frac{ne}{s} \right)^s \left( \frac{ne}{\beta s} \right)^{\beta s} \left( \frac{\beta s}{n - \zeta} \right)^{\zeta s} \leq \sum_{s=1}^{\left\lfloor \frac{2n}{3\beta} \right\rfloor} \left( \frac{\beta s}{n} \right)^{\zeta - 1} e^{\beta + e^{2\zeta^2/n\beta}} s \leq e^{-\zeta/4}. \quad (9)$$

Similarly, with probability at least $1 - e^{-\zeta/4}$, for all $T \subseteq B$ with $|T| \leq \left\lfloor \frac{2n}{3\beta} \right\rfloor$, $|N_{\vec{D}_F}(T)| \geq \beta|T|$. (Again only non-matching edges, are involved.)
In the remainder of this proof, assume that we are in the high-probability “good” case, in which all small sets $S$ and $T$ have large vertex expansion.

Now, choose an arbitrary $x \in A$, and define $S_0, S_1, S_2, \ldots$, by
\[
S_0 = \{x\} \text{ and } S_i = \pi^{-1}(N_{\zeta}(S_{i-1})).
\]
Since we are in the good case, $|S_i| \geq \beta |S_{i-1}|$ provided $|S_{i-1}| \leq 2n/(3\beta)$, and so there exists a smallest index $i_S - 1 \leq \log_b(2n/(3\beta)) \leq \log \beta n - 1$ such that $|S_{iS} - 1| > n/\beta$. Arbitrarily discard vertices from $S_{iS} - 1$ to create a smaller set $S'_{iS} - 1$ with $|S'_{iS} - 1| = \lfloor n/\beta \rfloor$, so that $S'_{iS} = N_{\zeta}(S'_{iS} - 1)$ has cardinality $|S'_{iS}| \geq \beta |S'_{iS} - 1| \geq 2n/3$.

Similarly, for an arbitrary $y \in B$, define $T_0, T_1, \ldots$, by
\[
T_0 = \{y\} \text{ and } T_i = \pi(N_{\zeta}(T_{i-1})).
\]
Again, we will find an index $i_T \leq \log \beta n$ whose modified set has cardinality $|T'_{iT}| \geq 2n/3$.

With both $|S'_{iS}|$ and $|T'_{iT}|$ larger than $n/2$, there must be some $x' \in S'_{iS}$ for which $y' = \pi(x') \in T'_{iT}$. This establishes the existence of a walk and hence a path of length at most $2(i_S + i_T) \leq 2 \log \beta n \approx 2/\epsilon$ from $x$ to $y$ in $\overline{D}_F$.

Let
\[
\gamma_\epsilon = \frac{30\zeta}{\epsilon n}
\]

**Corollary 5** Suppose that $F$ satisfies $[2]$. Then over random cost matrices $C$, for every permutation $\pi$,
\[
\Pr \left( \text{the weighted diameter of } \overline{D}_F \geq \gamma_\epsilon \right) \leq e^{-\zeta/5}.
\]

**Proof.** The Chernoff bounds imply that with probability at least $1 - ne^{-\zeta}$, every edge $(x, y)$ where $y \in N_{\zeta}(x)$ satisfies $C(x, y) \leq 10\zeta/n$. The corollary now follows from Lemma 4.

It follows from this that the minimum cost matching $M_F^*$ only contains edges of cost at most $\zeta$.

**Lemma 6** Suppose that $F$ satisfies $[2]$. Then over random cost matrices $C$, for every permutation $\pi$,
\[
\Pr \left( \exists (a_i, b_j) \in M_F^* : C(i, j) \geq \gamma_\epsilon \right) \leq e^{-\zeta/5}.
\]

**Proof.** If there was an edge $e = (a_i, b_j)$ of cost greater than $\gamma_\epsilon$ in $M_F^*$ then we can reduce the cost of $M_F^*$ by deleting $e$ and using an alternating path from $a_i$ to $b_j$ of weight at most $\gamma_\epsilon$ to find a lower cost matching.

The number of choices for $F$ satisfying $[2]$ is at most $\binom{n^2}{n^{3\epsilon/8}}\binom{n^2}{n^{3\epsilon/4}}$ and so by the union bound
\[
\Pr \left( \exists F : \exists (a_i, b_j) \in M_F^* : C(i, j) \geq \gamma_\epsilon \right) \leq \binom{n^2}{n^{3\epsilon/8}}\binom{n^2}{n^{3\epsilon/4}} e^{-\zeta/5} = o(1).
\]
3.1 Proof of the lower bound

The assignment problem \( AP(F) \) has a linear programming formulation \( LP_F \). Let \( A_F = \{ x \in A : \exists y \in B \text{ such that } (x, y) \in F_1 \} \) and \( B_F = \{ y \in B : \exists x \in A \text{ such that } (x, y) \in F_1 \} \). Let \( \Omega_F = (A_F \times B_F) \setminus (F_0 \cup F_0^c \cup D) \). In the following \( z_{i,j} \) indicates whether or not \((i, j)\) is an edge of the optimal solution.

\[
\begin{align*}
\mathcal{L}P_F & \quad \text{Minimise } \sum_{(i,j) \in \Omega_F} C(i, j) z_{i,j} \\
\text{subject to } & \sum_{j : (i,j) \in \Omega_F} z_{i,j} = 1, \forall i \in A_F. \quad (10) \\
& \sum_{i : (i,j) \in \Omega_F} z_{i,j} = 1, \forall j \in B_F. \\
& 0 \leq z_{i,j} \leq 1, \forall (i, j) \in \Omega_F.
\end{align*}
\]

This has the dual linear program:

\[
\mathcal{D}L\mathcal{P}_F \quad \text{Maximise } \sum_{i \in L_F} u_i + \sum_{j \in R_F} v_j \\
\text{subject to } u_i + v_j \leq C(i, j), \forall (i, j) \in \Omega_F. \quad (11)
\]

We now let

\[
n_F = n - |F_1| \text{ and } m_F = |\Omega_F|.
\]

**Remark 7** Condition on an optimal basis for \((10)\). We may w.l.o.g. take \( u_1 = 0 \) in \((11)\), whereupon with probability 1 the other dual variables are uniquely determined. Furthermore, the reduced costs of the non-basic variables \( \bar{C}(i, j) = C(i, j) - u_i - v_j \) are independently and uniformly distributed, with \( \bar{C}(i, j) = U[\max\{0, -u_i - v_j\}, 1 - u_i - v_j] \). Note also that this implies that with probability one, \( \bar{C}(i, j) > 0 \) for all non-basic \((i, j)\).

**Proof.** The \( 2n_F - 1 \) dual variables are unique with probability 1 because they satisfy \( 2n_F - 1 \) full rank linear equations. The only conditions on the non-basic edge costs are that \( C(i, j) \in [0, 1] \) (equivalently \( \bar{C}(i, j) \in [-u_i - v_j, 1 - u_i - v_j] \)) and \( \bar{C}(i, j) \geq 0 \); intersecting these intervals yields the last claim. \( \square \)

3.2 Trees and bases

An optimal basis of \( \mathcal{L}P_F \) can be represented by a spanning tree \( T_F^* \) of \( K_{A,B} \) that contains the perfect matching \( M^* \), see for example Ahuja, Magnanti and Orlin [1], Chapter 11. We have that for every optimal basis \( T_F^* \),

\[
C(i, j) = u_i + v_j \text{ for } (a_i, b_j) \in E(T_F^*) \quad (12)
\]
and
\[ C(i, j) \geq u_i + v_j \text{ for } (a_i, b_j) \notin E(T_F^*). \] (13)

Note that if \( \lambda \) is arbitrary then replacing \( u_i \) by \( \widehat{u}_i = u_i - \lambda, i = 1, 2, \ldots, n \) and \( v_i \) by \( \widehat{v}_i = v_i + \lambda, i = 1, 2, \ldots, n \) has no affect on these constraints. We say that \( \mathbf{u}, \mathbf{v} \) and \( \widehat{\mathbf{u}}, \widehat{\mathbf{v}} \) are equivalent. It follows that we can always fix the value of one component of \( \mathbf{u}, \mathbf{v} \). In the following, we fix \( u_{i\text{imin}} = 0 \) where \( \text{imin} = \arg \min \{ l : a_l \in A_F \} \).

**Lemma 8**
\[ \Pr \left( \max_{i,j} \{ |u_i|, |v_j| \} \geq 2\gamma_\epsilon \right) \leq ne^{-\epsilon}. \] (14)

**Proof.** For each \( a_i \in A_F \) there is some \( b_j \in B_F \) such that \( u_i + v_j = C(i, j) \). This is because of the fact that \( a_i \) meets at least one edge of \( T \) and we assume that (12) holds. We also know that if (13) holds then \( u_{i'} + v_j \leq C(i', j) \) for all \( i' \neq i \). It follows that \( u_i - u_{i'} \geq C(i, j) - C(i', j) \geq -\gamma_\epsilon \) for all \( i', j \). Since \( i \) is arbitrary, we deduce that \( |u_i - u_{i'}| \leq \gamma_\epsilon \) for all \( i, i' \in A_F \). Because \( u_{i\text{imin}} = 0 \), this implies that \( |u_i| \leq \gamma_\epsilon \) for \( i \in A_F \).

We deduce by a similar argument that \( |v_j| \leq \gamma_\epsilon \) for all \( j \in B_F \). Now because for the optimal matching edges \( (i, \phi(i)), i \in A_F \) we have \( u_i + v_{\phi(i)} = C(i, \phi(i)) \), we see that \( |v_j| \leq 2\gamma_\epsilon \) for \( j \in B_F \). The probability bound follows from Lemma 6. \( \square \)

Fix \( M_F^* \) and let \( K(\mathbf{u}, \mathbf{v}) \) be the subgraph of \( K_{A,B,F} \) induced by the edges \( (a_i, b_j) \) for which \( u_i + v_j \geq 0 \). We need to know that w.h.p. each vertex \( a_i \in A_F \) is connected in \( K_{A,B,F} \) to many \( b_j \in B_F \) for which \( u_i + v_j \geq 0 \).

We fix a tree \( T \) and condition on \( T_F^* = T \). Let an edge \( (a_i, b_j) \notin E(T) \) be non-degenerate if a simplex pivot on this edge leads to a change in perfect matching.

For \( i \in A_F \) let \( L_{i,+} = \{ j : (i, j) \text{ is non-degenerate} \} \), \( L_{i,-} = \{ i : (i, j) \text{ is non-degenerate} \} \). Then for \( i = 1, 2, \ldots, n \) let \( \mathcal{A}_{i,+} \) be the event that \( \{ j \in L_{i,+} : u_i + v_j > 0 \} \leq \eta n \) and let \( \mathcal{A}_{i,-} \) be the event that \( \{ i \in L_{i,-} : u_i + v_j > 0 \} \leq \eta n \) where \( \eta \) will be some small positive constant.

**Lemma 9** Fix a spanning tree \( T \) of \( K_{A,B,F} \).
\[ \Pr(\mathcal{A}_{i,+} \lor \mathcal{A}_{i,-} \mid T_F^* = T) = O(ne^{-\epsilon}) \text{ for } i, j = 1, 2, \ldots, n. \]

**Proof.** In the following analysis \( T \) is fixed. Throughout the proof we assume that the costs \( C(i, j) \) for \( (a_i, b_j) \in T \) are distributed as independent \( U[0, \gamma_\epsilon] \). Lemma 6 is the justification for this in that we can solve the assignment problem, only using edges of cost at most \( \gamma_\epsilon \). Furthermore, in \( K_{A,B,F} \), the number of edges of cost at most \( \gamma_\epsilon \) incident with a fixed vertex is dominated by \( \text{Bin}(n, \gamma_\epsilon) \) and so by the Chernoff bound and by the union bound over choices for \( F \), we see that w.h.p. the maximum degree of the trees we consider can be bounded by \( 2n^{2\epsilon} \).

We fix \( s \) and put \( u_s = 0 \). The remaining values \( u_i, i \neq s, v_j \) are then determined by the costs of the edges of the tree \( T \). Let \( \mathcal{B} \) be the event that \( C(i, j) \geq u_i + v_j \) for \( (a_i, b_j) \notin E(T) \). Note that if \( \mathcal{B} \) occurs then \( T_F^* = T \).
Let \( \mathcal{E} \) be the event that \(|u_i|, |v_j| \leq 2\gamma_\epsilon\) for all \(i, j\). It follows from Lemma 8 that \( \mathbb{P}(\mathcal{E}) \geq 1 - n e^{-\zeta} \).

We now condition on the set \( E_T \) of edges (and the associated costs) of \( \{(a_i, b_j) \notin E(T)\} \) such that \( C(i, j) \geq 2\gamma_\epsilon \). Let \( X_T = \{(a_i, b_j) \notin E(T)\} \setminus E_T \). Note that \(|X_T|\) is dominated by \( \text{Bin}(n^2, 2\gamma_\epsilon) \) and so \(|X_T| \leq 3n^2\gamma_\epsilon^2\) with probability \( 1 - e^{-O(n^2\gamma_\epsilon^2)} \).

Let \( Y = \{C(i, j) : (a_i, b_j) \in E(T)\} \) and let \( \delta_1(Y) \) be the indicator for \( \mathcal{A}_{s,+} \land \mathcal{E} \). We write

\[
\mathbb{P}(\mathcal{A}_{s,+} \mid \mathcal{B}) = \mathbb{P}(\mathcal{A}_{s,+} \land \mathcal{E} \mid \mathcal{B}) = \frac{\int \delta_1(Y) \mathbb{P}(\mathcal{B} \mid Y) d\mathbb{P}}{\int \mathbb{P}(\mathcal{B} \mid Y) d\mathbb{P}} \tag{15}
\]

Then we note that since \((a_i, b_j) \notin X_T \cup E(T)\) satisfies the condition (13),

\[
\mathbb{P}(\mathcal{B} \mid Y) = \prod_{(a_i, b_j) \in X_T} \exp \left\{ -(u_i(Y) + v_j(Y))^+ \right\} = e^{-W}, \tag{16}
\]

where \( W = W(Y) = \sum_{(a_i, b_j) \in X_T} (u_i(Y) + v_j(Y))^+ \leq 12n^2\gamma_\epsilon^2 = O(n^2\epsilon) \). Then we have

\[
\int_Y \delta_1(Y) \mathbb{P}(\mathcal{B} \mid Y) d\mathbb{P} = \int_Y e^{-W} \delta_1(Y) d\mathbb{P}
\leq \left( \int_Y e^{-2W} d\mathbb{P} \right)^{1/2} \times \left( \int_Y \delta_1(Y)^2 d\mathbb{P} \right)^{1/2}
= e^{-\mathbb{E}(W)} \left( \int_Y e^{-2(W - \mathbb{E}(W))} d\mathbb{P} \right)^{1/2} \times \mathbb{P}(\mathcal{A}_{s,+} \mid \mathcal{E})^{1/2}
\leq e^{-\mathbb{E}(W)} e^{O(n^2\epsilon)} \mathbb{P}(\mathcal{A}_{s,+} \mid \mathcal{E})^{1/2}, \tag{17}
\]

\[
\int \mathbb{P}(\mathcal{B} \mid Y) d\mathbb{P} = \mathbb{E}(e^{-W}) \geq e^{-\mathbb{E}(W)} \tag{18}
\]

It then follows from (15),(17) and (18) that

\[
\mathbb{P}(\mathcal{A}_{s,+} \mid \mathcal{B}) \leq e^{O(n^2\epsilon)} \mathbb{P}(\mathcal{A}_{s,+} \mid \mathcal{E}) \tag{19}
\]

Let \( b_j \) be a neighbor of \( a_s \) in \( K_{A,B,F} \) and let \( P_j = (i_1 = s, j_1, i_2, j_2, \ldots, i_k, j_k = j) \) define the path from \( a_s \) to \( b_j \) in \( T \). Then it follows from (12) that \( v_j = v_{j-1} - C(i_t, j_t-1) + C(i_t, j_t) \). Thus \( v_j \) is the final value \( S_k \) of a random walk \( S_t = X_0 + X_1 + \cdots + X_t, t = 0, 1, \ldots, k \), where \( X_0 \geq 0 \) and each \( X_t, t \geq 1 \) is the difference between two independent copies of \( U[0, \gamma_\epsilon] \). Given \( \mathcal{E} \) we can assume that the partial sums \( S_t \) satisfy \( |S_t| \leq 2\gamma_\epsilon \) for \( i = 1, 2, \ldots, k - 1 \). Assume for the moment that \( k \geq 4 \) and let \( x = w_{ik-3} \in [-2\gamma_\epsilon, 2\gamma_\epsilon] \). Given \( x \) we see that there is some positive probability \( p_0 = p_0(x) \) that \( S_k > 0 \). Indeed,

\[
p_0 = \mathbb{P}(S_k > 0 \mid \mathcal{E}) = \mathbb{P}(x + Z_1 - Z_2 > 0 \mid \mathcal{E}) \geq \mathbb{P}(x + Z_1 - Z_2 > 0) - n e^{-\zeta}, \tag{20}
\]

where \( Z_1 = Z_{1,1} + Z_{1,2} + Z_{1,3} \) and \( Z_2 = Z_{2,1} + Z_{2,2} \) are the sums of independent \( U[0, \gamma_\epsilon] \) random variables, each conditioned on being bounded above by \( \gamma_\epsilon \) and such that \( |x + \sum_{j=1}^t Z_{1,j} - \)
exists degree 2 \( Z_{2,j} \) for \( t = 1, 2 \) and that \( |x + Z_1 - Z_2| \leq 2\gamma \). The absolute constant \( \eta_0 = p_0(-2\gamma) > 0 \) is such that \( \min \{x \geq -2\gamma : p_0(x)\} \geq \eta_0 \).

We have so far demonstrated that if \( (a_i, b_j) \) is non-basic then there is at least a positive probability \( \eta_0 \) that \( u_i + v_j > 0 \). We need to be careful, because some pivots are degenerate. We avoid this problem as follows: if we delete \( (a_i, b_{\phi(i)}) \in M^* \) from \( T_F^* \) then we obtain two trees, one of which \( T_i \) say, will have at least \( n_F \) vertices. Assume without loss of generality that \( a_i \in T_i \) and let \( B_i = V(T_i) \cap B \) and note that \( |B_i| \geq n_F/2 \). The point of this partition is that adding the edge \((a_i, b)\) to \( T_F^* \) creates a cycle that alternates between edges in \( M^* \) and edges that are not in \( M^* \). So, if \( u_i + v_j > 0 \) then \((i, j)\) is non-degenerate and the corresponding pivot produces a new perfect matching with an increase in cost of \( \hat{C}(i, j) \).

We partition (most of the \( B_i \)-neighbors of \( a_s \)) into \( N_0, N_1, N_2, N_t = \{ b_j : k \geq 3, k \mod 3 = t \}, k \) being the number of edges in the path \( P_j \) from \( a_i \) to \( b_j \). Now because \( T \) has maximum degree \( 2n^2 \), as observed at the beginning of the proof of this lemma, we know that there exists \( t \) such that \( |N_t| \geq (n_F/2 - (2n^2)^3)/3 - |F_1| \geq n/7 \). It then follows from (20) that \( |L_{s,+}| \) dominates \( Bin(n/7, \eta_0) \) and then \( \Pr(|L_{s,+}| \leq \eta_0 n/10) = O(e^{-\Omega(n)}) \) follows from the Chernoff bounds. Similarly for \( L_{1,-} \). Applying the union bound over \( r \) choices for \( s \) and applying (19) gives the lemma with \( \eta = \eta_0/10 \).

Conditioning on events with probability of failure \( O(ne^{-c}) \), the number of non-degenerate non-basic edges dominates \( Bin(n^2/2 - o(n), n^{-3/(3+\epsilon)}) \) and each such edge yields an \( M_t \). Furthermore, the number of choices for \( F \) satisfying (2) is at most \( (n^2/n)(n^2/n) = o(n^{-1}e^c) \) and so we take a union bound over \( F \). Lemma 3 follows immediately.

4 Lower bound on TSP/Assignment gap — Proof of Lemma 2

This section deals with AP, ATSP without any restrictions, other than \( z_{i,i} = 0, i \in [n] \). Having solved \( LP_0 \), we will have \( n \) basic variables \( z_{i,j}, (i, j) \in I_1 \), with value 1 and \( n-1 \) basic variables \( z_{i,j}, (i, j) \in I_2 \), with value 0. The edges \((i, j) \in I = I_1 \cup I_2 \) form a tree \( T^* = T_0^* \) in \( K_{A,B} \). Let \( \pi \) be the permutation of \([n] \) associated with \( I_1 \) i.e. \( I_1 = \{(i, \pi(i))\}, i = 1, 2, \ldots, n \). Given \( T^* \) we obtain another tree \( T = \phi(T^*) \) on vertex set \([n] \) by contracting the \( n \) edges \((i, j) \in I_1 \). \( T \) has an edge \((i, j) \in I_2 \) for every pair \((i, \pi(j)) \in I_2 \). We orient this edge from \( i \) to \( j \) and let vertex \( i \) have out-degree \( d_i \) in \( T \) so that \( d_1 + d_2 + \ldots + d_n = n - 1 \).

**Lemma 10** Given \( T = \phi(T^*) \), the distribution of \( I = E(T^*) \) is

\[
\{(i, \rho \pi_0(i)) : i \in [n]\} \cup \bigcup_{i \in [n]} \{(i, \rho(\xi(i, t))) : t = 1, 2, \ldots, d_i\} \tag{21}
\]

where \( \pi_0 \) is a fixed permutation of \([n] \), \( \rho \) is a random permutation of \([n] \) and the edges \( \{(i, \pi_0(t)) : i \in [n]\} \cup \bigcup_{i \in [n]} \{(i, \xi(i, t)) : t = 1, 2, \ldots, d_i\} \) are those of some fixed tree \( T_0^* \) for which \( \phi(T_0^*) = T \).
We will condition on \( \phi \) and (23) we have
\[
\phi(T_1^*) = \phi(T_2^*) \iff T_2^* = \rho T_1^*
\]
where \( \rho = \pi_2 \pi_1^{-1} \).

We will condition on \( \phi(T^*) \) and consider the conditional distribution of the edges in \( I_2 \). Now because replacing \( C(i, j) \) by \( C(i, \rho(j)) \) for all \((i, j)\) does not change the distribution of \( C \), we have
\[
\Pr(T^* = T_1^*) = \Pr(\rho T^* = T_1^*) = \Pr(T^* = \rho T_1^*). \tag{23}
\]
Let \( T_1^*, T \) be fixed trees such that \( \phi(T^*) = T \) and let \( T^* \) be the random optimal basic tree and \( \rho = \pi_2 \pi_1^{-1} \) where \( \pi_2 \) is a random permutation of \([n]\) and \( \pi_1 \) is defined by \( T_1^* \). From (22) and (23) we have
\[
\Pr(T^* = T_1^* | \phi(T^*) = T) = \frac{\Pr(T^* = T_1^*)}{\Pr(\phi(T^*) = T)} = \frac{\Pr(T^* = \rho T_1^*)}{\Pr(\phi(T^*) = T)} = \Pr(T^* = \rho T_1^* | \phi(T^*) = T).
\]
The lemma follows. \( \square \)

It will be convenient to condition on the number of cycles of length \( i \) in the optimal assignment. Let \( \Pi \) denote the set of permutations of \( A \) with \( k_i \) cycles of length \( i = 2, 3, \ldots, n \). Let \( \pi \) be any fixed permutation with the given cycle structure. (For example, if \( t_1 = 0, t_{\sigma+1} = n \), and the multi-sets \( \{t_{j+1} - t_j : j \in [\sigma]\} \) and \( \{k_i \times i : i \in [n]\} \) coincide, then we may define \( \pi \) by: If \( x, y \in C_j \) and \( y = x + 1 \mod t_{j+1} - t_j \) then \( \pi(x) = y \).) Then given a bijection \( f : A \to A \) we define a permutation \( \pi_f \) on \( A \) by \( \pi_f = f^{-1} \pi f \). Each permutation \( \pi \in \Pi \) appears precisely \( \prod_{i=1}^{n} k_i! i^{k_i} \) times as \( \pi_f \). Thus choosing a random mapping \( f \), chooses a random \( \pi_f \) from \( \Pi \).

A natural way to look at this is to think of having oriented cycles on the plane whose vertices are at points \( A_1, A_2, \ldots, A_n \) and then randomly labelling these points with \( A \). Then if \( P' \) follows \( P \) on one of the cycles and \( P, P' \) are labelled \( x, x' \) by \( f \) then \( \pi_f(x) = x' \).

We now look at the probability that the gap \( \pi_C = Z_{\text{ATSP}}^{(C)} - Z_{\text{AP}}^{(C)} \) is at most \( \frac{1}{n^{3/2}} \). To go from the optimal assignment to a tour, we will, for some \( 2 \leq k \leq n \) have to:

1. Delete \( k \) edges from the optimal cycle cover, deleting at least one edge from each cycle.
2. Order the paths \( P_1, P_2, \ldots, P_k \) produced.
3. Add \( k \) edges to make a tour.

This must be done in such a way that the increased cost is at most \( \frac{1}{n^{3/2}} \). Let us call this a \( k \)-substitution.
Lemma 11

\[
\Pr(\exists k\text{-substitution}) \leq e^{2k/n^{1/2}}(k-1)! \sum_{S=\{i_1, i_2, \ldots, i_k\}}^{k} \prod_{t=1}^{k} d_{i_t}
\]

The statement “covers all cycles” refers to having at least one \(i_j\) in each cycle of the permutation \(\pi\). The \(d_i\) are the degrees as in (21).

Proof. Suppose that the path \(P_t\) joins \(y_t\) to \(z_t\) for \(t = 1, 2, \ldots, k\). We must add edges \((z_t, y_{t+1})\) for \(t = 1, 2, \ldots, k\) and for \(\pi_C\) to be less than \(1/n^{3/2}\) these edges will have to be either (i) basic (with value zero) or (ii) non-basic with reduced cost less than \(1/n^{3/2}\). Here basic/non-basic refers to the optimal solution to \(LP_F\) of Section 3.1.

Conditional on the event described in (14) we have

\[
\begin{align*}
\Pr(\bar{C}(i, j) \leq \frac{1}{n^{3/2}} | (i, j) \text{ is non-basic}) &\leq \frac{1}{n^{3/2}(1 - |u_i| - |v_j|)} \leq \frac{2}{n^{3/2}}. \\
\end{align*}
\]

Now let us consider the probability that we can join \(P_t\) to \(P_{t+1}\), given that we have joined up \(P_1, P_2, \ldots, P_t\). We need to estimate the probability that \((u, v) = (z_t, y_{t+1})\) is a basic edge since (25) deals with the possibility of a short non-basic edge. Having exposed the status of the edges \((z_i, y_{i+1}), 1 \leq i \leq t\) we see from (21) that

\[
\Pr((z_t, y_{t+1}) \text{ is a basic edge} | \text{we have joined up } P_1, P_2, \ldots, P_t) \leq \frac{d_{z_t}}{n - t + 1}.\]

To have a positive probability of creating a tour, the previous edge exposures must not concern edges with tail \(z_t\) or with head equal to an out-neighbor of \(z_t\). There are \(d_{z_t}\) random choices of out-neighbor \(\xi\) of \(z_t\) and at this point \(\rho(x_i)\) is random, subject only to \(t - 1\) previous selections.

Putting (25) and (26) together we see that

\[
\begin{align*}
\Pr(\text{we can join } P_1, \ldots, P_k) &\leq \prod_{t=1}^{k} \left( \frac{2}{n^{3/2}} + \frac{d_{z_t}}{n - t + 1} \right) \leq e^{2k/n^{1/2}} \prod_{t=1}^{k} \frac{d_{z_t}}{n - t + 1}.
\end{align*}
\]

Thus,

\[
\Pi_k = \Pr(\exists k\text{-substitution}) \leq e^{2k/n^{1/2}}(k-1)! \sum_{S=\{i_1, i_2, \ldots, i_k\}}^{k} \prod_{t=1}^{k} d_{i_t}.
\]

\[\square\]

Suppose now that there are \(a_r\) vertices for which \(d_i = r, 1 \leq r \leq n\). We need to argue that \(a_0 \geq \eta n\) w.h.p. for some small positive constant \(\eta\). Each leaf of \(T\) has out-degree zero and so we only need to show that w.h.p. \(T\) at least \(\eta n\) leaves.

Lemma 12 There exists a small positive constant \(\eta\) such that w.h.p. \(T\) has at least \(\eta n\) leaves.
Thus non-basic reduced cost is non-negative.  

Proof.  Note that each $T$ arises from exactly $2^{n-1}$ distinct $T^*$’s.  This is because we have two choices as to how to configure each edge that is not part of the matching.  (An edge $(i,j)$ in $T'$ can in $T^*$ be expanded to $(x_i,y_j)$ or to $(x_j,y_i)$.)  Let $b(T) = b(T^*)$ denote the number of branching nodes (degree $\geq 3$) of $T$ and $T^*$.  A tree $T$ is $\eta$-bushy if $b(T') \leq \eta n$.  Bohman and Frieze used this concept in \cite{BF} and showed that the number of $\eta$-bushy trees is at most $n!e^{\theta(\eta)n}$ where $\theta(\eta) \to 0$ as $\eta \to 0$.  It follows that the number of $\eta$-bushy trees of $K_{A,B}$ which have a perfect matching is at most $e^{\theta(\eta)n}2^{n-1}n!$.  Observe that the number of leaves in $T$ is at least $b(T)$.  We show that, for a sufficiently small constant $\eta$, 

$$\Pr(T^* \text{ is } \eta\text{-bushy}) = o(1).$$

(27)

This will prove the lemma.  For any tree $T^*$ with a perfect matching, we can put $u_i = 0$ and then solve the equations $u_i + v_j = C(i,j)$ for $(x_i,y_j) \in T^*$ to obtain the associated dual variables.  $T^*$ is optimal if $\bar{C}(i,j) = C(i,j) - u_i - v_j \geq 0$ for all $(x_i,y_j) \notin T^*$.  Let $Z_{T^*} = \sum_i u_i + \sum_j v_j$.  Now w.h.p. the optimal tree $T^*$ satisfies $Z_{T^*} \in [1.6, 1.7]$, because $Z_{T^*}$ is the optimal assignment cost.  We know both that the expectation of $Z_{T^*}$ is in the stated range and that the actual value is concentrated about the expectation, see Talagrand \cite{Ta1}.  Then if $E$ denotes the event \{\cite{BF} and $Z_{T^*} \in [1.6, 1.7]$\}, for any tree $T^*$, over random matrices $C(i,j)$,

$$\Pr(Z_{T^*} \in [1.6, 1.7] \text{ and } \cite{BF} \text{ and } \bar{C}(i,j) \geq 0, \forall \text{ non-basic } (i,j))$$

$$\leq \Pr(\bar{C}(i,j) \geq 0, \forall (i,j) \notin T^* \mid E) \times \Pr(Z_{T^*} \in [1.6, 1.7])$$

$$\leq \frac{1.7^n}{n!} E \left( \prod_{(x_i,y_j) \notin T} (1 - (u_i + v_j)^+) \mid E \right)$$

$$\leq \frac{1.7^n}{n!} E \left( \exp \left\{ - \sum_{(x_i,y_j) \notin T} (u_i + v_j) \right\} \mid E \right)$$

$$\leq \frac{1.7^n}{n!} \exp \left\{ -nZ_{T^*} \exp \left\{ \sum_{(x_i,y_j) \in T} (u_i + v_j) \right\} \mid E \right\}$$

$$\leq \frac{1.7^n}{n!} e^{-1.6n+O(\gamma n)}.$$

Explanation for \cite{BF} $\frac{1.7^n}{n!}$ bounds the probability that the sum of the lengths of the edges in the perfect matching of $T$ is at most 1.7.  The product term is the probability that each non-basic reduced cost is non-negative.

Thus 

$$\Pr(\exists \text{ an } \eta\text{-bushy tree } T^*: Z_{T^*} \in [1.6, 1.7] \text{ and } \cite{BF} \text{ and } \bar{C}(i,j) \geq 0 \forall (i,j) \notin I)$$

$$\leq n!2^n e^{\theta(\eta)n} \times \frac{1.7^n}{n!} e^{-1.6n+O(\gamma n)}$$

$$= o(1),$$

for $\eta$ sufficiently small.  This implies \cite{BF}.  \hfill \square

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4.1 Small $k$

In this section we assume that $k_0 \leq k \leq k_1$ where $k_0 = \frac{1}{2} \log n$ and $k_1 = n^{2/3}$. The lower bound follows from the fact that w.h.p. a random permutation has at least this many cycles.

Then, from (24),

$$\sum_{k=k_0}^{k_1} \Pi_k \leq \sum_{k=k_0}^{k_1} e^{2k/n^{1/2}} \frac{(k-1)!}{(n)_k} \sum_{k_1 + \cdots + k_n = k} \prod_{r=1}^{n} \left( \frac{a_r}{k_r} \right)^{k_r}$$

(29)

$$\leq \sum_{k=k_0}^{k_1} e^{2k(n^{1/2}+2k^2/n)} \frac{(k-1)!}{n^k} \sum_{k_1 + \cdots + k_n = k} \prod_{r=1}^{n} \left( \frac{r a_r}{k_r} \right)^{k_r}$$

$$\leq \sum_{k=k_0}^{k_1} e^{2k(n^{1/2}+2k^2/n)} \frac{(1-\eta)^k}{k} = o(1).$$

4.2 Large $k$

In this section we assume that $k > n^{2/3}$. Write

$$\Pi_k \leq e^{2kn^{1/2}} \frac{1}{k^n} \sum_{S=\{i_1,i_2,\ldots,i_k\}} \prod_{t=1}^{k} d_{i_t}.$$  (30)

Suppose now that we have $n$ bins, where bin $i$ contains $d_i$ distinguishable balls. The RHS of (30) (ignoring the term $e^{2kn^{1/2}}$) is $k^{-1}$ times the probability $P_k$ that if we choose $k$ of the balls at random, we never choose two balls from the same bin. (The sum is the number of allowed choices and $\left( \begin{array}{c} n \\ k \end{array} \right)$ is the total number of choices). Because at least $\eta n$ of the bins are empty, we can find at least $\eta n/2$ pairs of balls $\alpha_i, \beta_i$ are such that $\alpha_i$ and $\beta_i$ are in the same bin. Now choose the balls in two sets of size $k/2$ each. The probability that fewer than $\eta k/4$ of the $\alpha_i$ are chosen is, by the Chernoff bound, at most $e^{-\eta k^2/(32n)}$ and given that at least $\eta k/4$ are chosen, the probability that a corresponding $\beta_i$ is never chosen is at most $(1 - \frac{\eta k}{4n})^{k/2} \leq e^{-\eta k^2/(8n)}$. Thus,

$$\sum_{k \geq k_1} \Pi_k \leq \sum_{k \geq k_1} e^{2kn^{1/2}} k^{-1} \left( e^{-\eta k^2/(32n)} + e^{-\eta k^2/(8n)} \right)$$

$$\leq \frac{1}{k_1} \sum_{k \geq k_1} e^{-\eta k^2/(40n)}$$

$$\leq e^{-\eta^2 \log^2 n/50}.$$  

This completes the proof of Lemma 2. 

\[ \square \]
5 Summary and open questions

Theorem 1 answers the question as to whether or not the assignment problem is a good enough bound for branch and bound to run in expected polynomial time. It is not. One can strengthen this bound by replacing AP by the subtour elimination LP of Dantzig, Fulkerson and Johnson [7]. Perhaps this leads to a branch and bound algorithm that runs in polynomial time w.h.p.

Less is known probabilistically about the symmetric TSP. Frieze [10] proved that if the costs $C(i, j) = C(j, i)$ are independent uniform $[0, 1]$ then the asymptotic cost of the TSP and the cost $2F$ of the related 2-factor relaxation are asymptotically the same. The probabilistic bounds on $|TSP - 2F|$ are inferior to those given in [15]. Still, it is conceivable that the 2-factor relaxation or the subtour elimination constraints are sufficient for branch and bound to run in polynomial time w.h.p. Frieze and Pegden [14] and Pegden and Severaki [23] have studied branch and bound in the context of random instances of the Euclidean TSP. They show that adding sub-tour elimination inequalities do not make branch and bound run in polynomial expected time. Indeed branch and bound runs in exponential time w.h.p. The latter paper [23] even allows the addition of comb inequalities.

References


[23] W. Pegden and A. Sevekari, Comb inequalities for typical Euclidean TSP instances.
