# Eigenvalues of Random Power Law Graphs
(DRAFT)

Fan Chung* Linyuan Lu* Van Vu †

November 1, 2002

**Abstract**

Many graphs arising in various information networks exhibit the "power law" behavior – the number of vertices of degree $k$ is proportional to $k^{-\beta}$ for some positive $\beta$. We show that if $\beta > 2.5$, the largest eigenvalue of a random power law graph is almost surely $(1+o(1))\sqrt{m}$ where $m$ is the maximum degree. When $2 < \beta < 2.5$, the largest eigenvalue is heavily concentrated at $cm^{3-\beta}$ for some constant $c$ depending on $\beta$ and the average degree. This result follows from a more general theorem which shows that the largest eigenvalue of a random graph with a given expected degree sequence is determined by $m$, the maximum degree, and $\tilde{d}$, the weighted average of the squares of the expected degrees. We show that $\lambda$ is almost surely $(1 + o(1)) \max\{\tilde{d}, \sqrt{m}\}$ provided some minor condition is satisfied. Our results have implications on the usage of spectral techniques in many areas related to pattern detection and information retrieval.

## 1 Introduction

Although graph theory has a history of more than 250 years, it is only very recently noted that the so-called "power law" is prevalent in realistic graphs arising in numerous arenas. Graphs with power law degree distribution are ubiquitous as observed in the Internet, the telecommunications graphs, email graphs and in various biological networks [2, 3, 4, 8, 11, 12, 13]. One of the basic problem concerns the distribution of the eigenvalues of power law graphs. In addition to theoretical interest, spectral methods are central in detecting clusters and finding patterns in various applications.

The eigenvalues of the adjacency matrices of various realistic power law graphs were computed and examined in [8, 9, 10]. Faloutsos et. al. [8] conjectured a power law distribution for eigenvalues of power law graphs. For a fixed value $\beta > 1$, we say that a graph is a power law graph with exponent $\beta$ if the number of vertices of degree $k$ is proportional to $k^{-\beta}$. We note that for most realistic graphs, their power law models usually have exponents $\beta$ falling between 2 and 3. For example, various Internet graphs [12] have exponents between 2.1 and 2.4. The Hollywood graph [4] has exponent $\beta \sim 2.3$. The telephone call graphs [1] has exponet $\beta = 2.1$. Recently, Mihail and

*Research supported in part by NSF Grants DMS 0100472 and ITR 0205061

†Department of Mathematics, UCSD, La Jolla, CA 92093. Research supported in part by NSF grant DMS-0200357 and by an A. Sloan fellowship. Email:vanvu@ucsd.edu; website: www.math.ucsd/∼vanvu.

Papadimitriou [15] showed that the largest eigenvalue of a random power law graph is the square root of the maximum degree (if the exponent $\beta$ of the power law graph satisfies $\beta > 3$.)

In this paper, we will show that the largest eigenvalue $\lambda$ of the adjacency matrix of a random power law graph is almost surely approximately the square root of the maximum degree $m$ if $\beta > 2.5$, and is almost surely approximately $cm^{3-\beta}$ if $2 < \beta < 2.5$. A phase transition occurs at $\beta = 2.5$. This result for power law graphs is an immediate consequence of a general result for eigenvalues of random graphs with arbitrary degree distribution.

We will use a random graph model from [5], which is a generalization of the Erdős-Rényi model, for random graphs with given expected degrees $w_1, w_2, \ldots, w_n$. The largest eigenvalue $\lambda$ of the adjacency matrix of a random graph in this model depends on two parameters – the maximum degree $m$ and the second order average degree $\tilde{d}$ defined by

$$\tilde{d} = \frac{\sum_{i=1}^n w_i^2}{\sum_{i=1}^n w_i}.$$

It has turned out that $\lambda$ is almost surely $(1 + o(1))\sqrt{m}$ if $\sqrt{m}$ is greater than $\tilde{d}$ by a factor of $\log^2 n$ and $\lambda$ is almost surely $(1 + o(1))\tilde{d}$ if $\sqrt{m}$ is smaller than $\tilde{d}$ by a factor of $\log n$. In other words, $\lambda$ is (asymptotically) the maximum of $\sqrt{m}$ and $\tilde{d}$ if the two values of $\sqrt{m}$ and $d$ are far apart (by a power of $\log n$). One might be tempted to conjecture that

$$\lambda = (1 + o(1)) \max\{\sqrt{m}, \tilde{d}\}.$$

This, however, is not true as shown by a counter example in the last section. Throughout the paper, the asymptotic notation is used under the assumption that $n \to \infty$. We say that an event holds almost surely, if the probability that it holds tends to 1 as $n$ tends to infinity.

Following the discussion in Mihail and Papadimitriou [15], our result has the following implications: The largest degree is a "local" aspect of a graph. If the largest eigenvalue depends only on the largest degree, spectral analysis of the Internet topology or spectral filtering for information retrieval can only be effective after high degree nodes have been normalized. Our result implies that such negative implications occurs only when the exponent $\beta$ exceeds 2.5.

## 2 Preliminaries

The primary model for classical random graphs is the Erdős-Rényi model $\mathcal{G}_p$, in which each edge is independently chosen with the probability $p$ for some given $p > 0$ (see [7]). In such random graphs the degrees (the number of neighbors) of vertices all have the same expected value. Here we consider the following extended random graph model for a general degree distribution.

For a sequence $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, We consider random graphs $G(\mathbf{w})$ in which edges are independently assigned to each pair of vertices $(i, j)$ with probability $w_i w_j \rho$, where $\rho = \frac{1}{\sum_{i=1}^n w_i}$. Notice that we allow loops in our model (for computational convenience) but their presence does not play any essential role. It is easy to verify that the expected degree of $i$ is $w_i$.

To this end, we assume that $\max_i w_i^2 < \sum_k w_k$ so that $p_{ij} \leq 1$ for all $i$ and $j$. This assumption assures that the sequence $w_i$ is graphical (in the sense that it satisfies the necessary and sufficient condition for a sequence to be realized by a graph [6]) except that we do not require the $w_i$'s to be integers). We will use $d_i$ to denote the actual degree of $v_i$ in a random graph $G$ in $G(\mathbf{w})$ where the weight $w_i$ denotes the expected degree.

For a subset $S$ of vertices, the volume $\mathrm{Vol}(S)$ is defined as the sum of weights in $S$. That is $\mathrm{Vol}(S) = \sum_{i\in S} w_i$. In particular, we have $\mathrm{Vol}(G) = \sum_i w_i$, and we denote $\rho = \frac{1}{\mathrm{Vol}(G)}$. The induced subgraph on $S$ is a random graph $G(\mathbf{w}')$ where the weight sequence is given by $w'_i = w_i \mathrm{Vol}(S)\rho$ for all $i \in S$. The second order average degree of $G(\mathbf{w}')$ is simply $\sum_{i\in S} w_i^2 \rho$.

The classical random graph $G(n,p)$ can be viewed as a special case of $G(\mathbf{w})$ by taking $\mathbf{w}$ to be $(pn, pn, \ldots, pn)$. In this special case, we have $\tilde{d} = d = m = np$. It is well known that the largest eigenvalue of the adjacency matrix of $G(n,p)$ is almost surely $(1+o(1))np$ provided that $np \gg \log n$. Here we will determine the first eigenvalue of the adjacency matrix of a random graph in $G(\mathbf{w})$.

There are two easy lower bounds for the largest eigenvalues $\lambda$, namely, $(1+o(1))\tilde{d}$ and $(1+o(1))\sqrt{m}$. (The proofs can be found in Section 4.) Our main result states that the maximum of the above two lower bounds is essentially an upper bound.

**Theorem 1** *If $\tilde{d} > \sqrt{m}\log n$, then the largest eigenvalue of a random graph in $G(\mathbf{w})$ is almost surely $(1+o(1))\tilde{d}$.*

**Theorem 2** *If $\sqrt{m} > \tilde{d}\log^2 n$, then almost surely the largest eigenvalue of a random graph in $G(\mathbf{w})$ is $(1+o(1))\sqrt{m}$.*

**Theorem 3** *The largest eigenvalue of a random graph in $G(\mathbf{w})$ is at most $7\sqrt{\log n}\cdot\max\{\sqrt{m}, \tilde{d}\}$.*

We remark that with more careful analysis the factor of $\log n$ in Theorem 1 can be replaced by $(\log n)^{1/2+\epsilon}$ and the factor of $\log^2 n$ can be replaced by $(\log n)^{3/2+\epsilon}$ for any positive $\epsilon$ provided that $n$ is sufficiently large. The constant "7" in theorem 3 can be improved. We made no effort to get the best constant coefficient here.

As an application of Theorems 1 and 2, we prove that the largest eigenvalue of the random power law graph is $(1+o(1))\sqrt{m}$ if $\beta > 2.5$, and $(1+o(1))\tilde{d}$ if $\beta < 2.5$. A transition happens when $\beta = 2.5$.

# 3 Basic facts

We will use the following concentration inequality for a sum of independent random variables (see [14]).

**Lemma A.** Let $X_i$ $(1 \le i \le n)$ be independent random variables satisfying $|X_i| \le M$. Let $X = \sum_i X_i$. Then we have

$$Pr(|X - E(X)| > a) \le e^{-\frac{a^2}{2(Var(X)+Ma/3)}}.$$

We will also use the following one-sided inequality [5] :

**Lemma B.** Let $X_1, \ldots, X_n$ be independent random variables with

$$Pr(X_i = 1) = p_i, \qquad Pr(X_i = 0) = 1 - p_i$$

For $X = \sum_{i=1}^n a_i X_i$, we have $E(X) = \sum_{i=1}^n a_i p_i$ and we define $\nu = \sum_{i=1}^n a_i^2 p_i$. Then we have

$$Pr(X < E(X) - t) \leq e^{-t^2/2\nu}.$$

The following lemma, due to Perron ([16], page 36) will also be very useful

**Lemma 1** *Suppose the entries of a $n \times n$ symmetric matrix $A$ are all non-negative. For any positive constants $c_1, c_2, \ldots, c_n$, the largest eigenvalue $\lambda(A)$ satisfies*

$$\lambda(A) \leq \max_{1 \leq i \leq n} \{\frac{1}{c_i} \sum_{j=1}^n c_j a_{ij}\}.$$

**Proof:** Let $C$ be the diagonal matrix $diag(c_1, c_2, \ldots, c_n)$. Both $A$ and $C^{-1}AC$ have the same eigenvalues. All entries of $C^{-1}AC$ are also non-negative. The first eigenvalue is bounded by the maximum row sum of $C^{-1}AC$. $\square$

Now we are ready to state our key lemma.

**Lemma 2** *For any given expected degree sequence $\mathbf{w}$, the largest eigenvalue $\lambda$ of a random graph in $G(\mathbf{w})$ is almost surely at most*

$$\tilde{d} + \sqrt{6\sqrt{m \log n}(\tilde{d} + \log n)} + 3\sqrt{m \log n}.$$

*In particular, we have $\lambda < 2\tilde{d} + 6\sqrt{m \log n}$.*

**Proof:** For a fixed value $x$ (to be chosen later), we define $c_i$, $1 \leq i \leq n$ as follows:

$$c_i = \begin{cases} w_i & \text{if } w_i > x \\ x & \text{otherwise.} \end{cases}$$

Let $A$ denote the adjacency matrix of $G$ in $G(\mathbf{w})$. The entries $a_{ij}$ of $A$ are independent random variables. Now we apply Lemma 1, choosing $X_i = \frac{1}{c_i} \sum_{j=1}^n c_j a_{ij}$. We have

$$\begin{aligned} E(X_i) &= \frac{1}{c_i} \sum_{j=1}^n c_j w_i w_j \rho \\ &= \begin{cases} \sum_{w_j > x} w_j^2 \rho + x \sum_{w_j \leq x} w_j \rho & \text{if } w_i > x \\ \frac{w_i}{x} \sum_{w_j > x} w_j^2 \rho + w_i \sum_{w_j \leq x} w_j \rho & \text{otherwise.} \end{cases} \\ &\leq \tilde{d} + x; \end{aligned}$$

$$
\begin{aligned}
Var(X_i) &\leq \frac{1}{c_i^2}\sum_{j=1}^{n} c_j^2 w_i w_j \rho \\
&= \begin{cases} \frac{1}{w_i}\sum_{w_j > x} w_j^3\rho + \frac{x^2}{w_i}\sum_{w_j\leq x} w_j\rho & \text{if } w_i > x \\ \frac{w_i}{x^2}\sum_{w_j>x} w_j^3\rho + w_i\sum_{w_j\leq x} w_j\rho & \text{otherwise.} \end{cases} \\
&\leq \frac{m}{x}\tilde{d} + x;
\end{aligned}
$$

By Lemma A, we have

$$Pr(|X_i - E(X_i)| > a) \leq e^{-\frac{a^2}{2(Var(X_i)+ma/3x)}}.$$

Here we choose $x = \sqrt{m\log n}$ and $a = \sqrt{6(\frac{m}{x}\tilde{d} + x)\log n} + 2\frac{m}{x}\log n$. With probability at least $1 - o(\frac{1}{n})$, we have $X_i < \tilde{d} + x + a$ for every fixed $1 \leq i \leq n$. So we can conclude that almost surely $X_i < \tilde{d} + x + a$ holds simultaneously for all $1 \leq i \leq n$.

By Lemma 1, we have (almost surely)

$$\lambda \leq \tilde{d} + \sqrt{6\sqrt{m\log n}(\tilde{d} + \log n)} + 3\sqrt{m\log n},$$

as desired. □

# 4 Proofs for the main theorems

This section presents the proofs of Theorems 1-3. We note that Theorem 1 is an easy consequence of Lemma 2. Theorem 2 requires some work and Theorem 3 is an immediate consequence of Lemma 2.

**Proof of Theorem 1.** We only need to prove the lower bound. Let $A$ be the adjacency matrix of a random graph $G$ in $G(\mathbf{w})$. We define

$$\alpha = \frac{1}{\sqrt{\sum_{i=1}^n w_i^2}}(w_1, w_2, \ldots, w_n)^*$$

where $\mathbf{x}^*$ denotes the transpose of $\mathbf{x}$. Let $X = \alpha^* A\alpha = \frac{1}{\sum_{i=1}^n w_i^2}(2\sum_{i<j} w_i w_j X_{i,j} + \sum_i w_i^2 X_{i,i})$. Here $X_{i,j}$ is the 0-1 random variable with $Pr(X_{i,j} = 1) = w_i w_j \rho$. We will use Lemma B to prove a lower bound on $X$. Notice that

$$
\begin{aligned}
E(X) &= \frac{1}{\sum_{i=1}^n w_i^2}(2\sum_{i<j} w_i^2 w_j^2\rho + \sum_i w_i^4\rho) \\
&= \sum_{i=1}^n w_i^2\rho \\
&= \tilde{d}
\end{aligned}
$$

and

$$\begin{aligned}
\nu &= \frac{1}{(\sum_{i=1}^n w_i^2)^2}(4\sum_{i<j} w_i^3 w_j^3\rho + \sum_i w_i^6\rho) \\
&\leq 2(\frac{\sum_{i=1}^n w_i^3}{\sum_{i=1}^n w_i^2})^2\rho \\
&\leq 2m^2\rho
\end{aligned}$$

Apply Lemma B with $t = \sqrt{2m^2\rho\log n}$, we have that with probability $1 - e^{-\log n/2} = 1 - o(1)$,

$$X > \tilde{d} - \sqrt{2m^2\rho\log n} = (1 + o(1))\tilde{d}.$$

Since $\lambda \geq X$, it follows that almost surely $\lambda \geq (1 + o(1))\tilde{d}$.

By the assumption of $\tilde{d} > \sqrt{m}\log n$, Lemma 2 implies that (almost surely) $\lambda \leq (1+o(1))\tilde{d}$. This and the previous fact complete the proof of Theorem 1. □

**Proof of Theorem 2.** The lower bound $\lambda > (1 + o(1))\sqrt{m}$ follows from the following simple observations:
(1). The maximal degree of $G$ is almost surely $(1 + o(1))m$.
(2). The largest eigenvalue of a star with size $k$ is $\sqrt{k-1}$.
(3). The largest eigenvalue of a graph is greater than or equal to the largest eigenvalue of a subgraph.

It remains to establish an upper bound of the same order under the assumptions of Theorem 2. In the following proof, we use a weaker assumption that $\sqrt{m} > (\tilde{d} + 1)\log^{1.5+\epsilon} n$ for any positive $\epsilon$.

Choose $s = \frac{m}{\log^{1+\epsilon/2} n}$ and $t = \tilde{d}\log^{1+\epsilon/2} n$. Let $S$ denote the set of vertices with weights greater than $s$, and let $T$ denote the set of vertices with weights less than or equal to $t$. Let $\bar{S}$ and $\bar{T}$ be the complements of $S$ and $T$, respectively.

Since $s > t$, $S$ and $T$ are disjoint sets. $G$ is covered by the following three subgraphs: $G(\bar{S})$–the induced subgraph on $\bar{S}$, $G(\bar{T})$–the induced subgraph on $\bar{T}$, and $G(S,T)$–the bipartite graph between $S$ and $T$. It is not hard to verify that

$$\mathrm{Vol}(S) \leq \frac{\tilde{d}}{s\rho}.$$

Both $G(\bar{S})$ and $G(\bar{T})$ are random graphs so that Lemma 2 can be applied. The maximum weight of $G(\bar{S})$ is at most $s$. We note that $\tilde{d}(G(\bar{S})) = \sum_{i\in\bar{S}} w_i^2\rho \leq \tilde{d}$. By Lemma 2, almost surely we have

$$\lambda(G(\bar{S})) \leq 2\tilde{d} + 6\sqrt{s\log n} = o(\sqrt{m}).$$

Similarly $\tilde{d}(G(\bar{T})) = \sum_{i\in\bar{T}} w_i^2\rho \leq \tilde{d}$. The maximum weight of $G(\bar{T})$ is at most

$$m\mathrm{Vol}(\bar{T})\rho \leq m\frac{\tilde{d}}{t} = \frac{m}{\log^{1+\epsilon/2} n}.$$

By Lemma 2, almost surely we have

$$\lambda(G(\bar{T})) \leq 2\tilde{d} + 6\sqrt{\frac{m}{\log^{1+\epsilon/2} n}\log n} = o(\sqrt{m}).$$

Next we consider the largest eigenvalue of $G(S,T)$.

*Claim 1.* The following holds almost surely. For any vertex $i \in S$, all but $\tilde{d}^2 \log^{2+\epsilon} n$ of its neighbors in $T$ have degree 1 in $G(S,T)$.

*Proof of Claim 1.* Fix a vertex $i \in S$ and expose its neighbors in $T$. With probability $1 - o(1/n)$, $i$ has at most $(1+o(1))m$ neighbors in $T$. For any neighbor $k \in T$ of $i$, the expected number of neighbors of $k$ (other than $i$) in $S$ is at most

$$\begin{aligned}
\mu &\leq E(\sum_{j \in S \setminus i} X_{kj}) \\
&\leq w_k \mathrm{Vol}(S)\rho \\
&\leq t\frac{\tilde{d}}{s} \\
&= \frac{\tilde{d}^2}{m} \log^{2+\epsilon} n \\
&< \frac{1}{\log n}.
\end{aligned}$$

It follows that the expected number of neighbors of $i$ with more than one neighbors in $S$ is at most $(1+o(1))m\mu \leq (1+o(1))\tilde{d}^2 \log^{2+\epsilon} n$. Using Lemma A, it is easy to show that with probability $1 - o(1/n)$, the number of neighbors of $i$ with more than one neighbors in $S$ is at most $(1+o(1))\tilde{d}^2 \log^{2+\epsilon} n$. The claim follows from the union bound.

*Claim 2.* Almost surely, the maximum degree of vertices in $T$ in $G(S,T)$ is at most $3 \log n$.

*Proof of Claim 2.* The expected degree of a vertex $i \in T$ in $G(S,T)$ is $w_i \mathrm{Vol}(S)\rho \leq \frac{t\tilde{d}}{s} < \frac{1}{\log n}$. A routine application of Lemma A shows (with room to spare) that with probability $1 - o(1/n)$, the degree of $i$ in $S$ is at most $3 \log n$. Again the union bound completes the proof.

Let $G_1$ be the subgraph of $G(S,T)$ consisting of all edges with degree 1 in $T$. Let $G_2$ be the subgraph of $G(S,T)$ consisting of all edges not in $G_1$. $G_1$ is an disjoint union of stars. The maximum expected degree is at most $(1+o(1))m$. We have

$$\lambda(G_1) \leq (1+o(1))\sqrt{m}.$$

The largest eigenvalue of $G_2$ is bounded above by $\sqrt{m_S m_T}$, where $m_S$ and $m_T$ are the maximum degrees in $S$ and $T$, respectively. Claims 1 and 2 show that $m_S \leq \tilde{d}^2 \log^{2+\epsilon} n$ and $m_T \leq \log n$. By Lemma 1, we have

$$\begin{aligned}
\lambda(G_2) &\leq \sqrt{m_S m_T} \\
&\leq \tilde{d} \log^{3/2+\epsilon/2} n \\
&= o(\sqrt{m})
\end{aligned}$$

Hence, we have

$$\lambda(G) \leq \lambda(G(\bar{S})) + \lambda(G(\bar{T})) + \lambda(G_1) + \lambda(G_2) \leq (1+o(1))\sqrt{m}.$$

The proof is complete. $\square$

# 5 Random power law graphs

In this section, we consider random graphs with power law degree distribution with exponent $\beta$. We choose the degree sequence $G(\mathbf{w}) = (w_1, w_2, \ldots, w_n)$ satisfying $w_i = ci^{-\frac{1}{\beta-1}}$ for $i_0 \le i \le n + i_0$. Here $c$ is determined by the average degree and $i_0$ depends on the maximum degree $m$, namely, $c = \frac{\beta-2}{\beta-1} dn^{\frac{1}{\beta-1}}, i_0 = n(\frac{d(\beta-2)}{m(\beta-1)})^{\beta-1}$. It is easy to verify that the number of vertices of degree $k$ is proportional to $k^{-\beta}$.

The second order average degree $\tilde{d}$ can be computed as follows:

$$\tilde{d} \;=\; \begin{cases} d\frac{(\beta-2)^2}{(\beta-1)(\beta-3)}(1+o(1)) & \text{if } \beta > 3. \\ \frac{1}{2} d \ln \frac{2m}{d}(1+o(1)). & \text{if } \beta = 3. \\ d\frac{(\beta-2)^2}{(\beta-1)(3-\beta)}(\frac{(\beta-1)m}{d(\beta-2)})^{3-\beta}(1+o(1)). & \text{if } 2<\beta<3. \end{cases}$$

We remark that for $\beta > 3$, the second order average degree is independent of the maximum degree. Consequently, the power law graphs with $\beta > 3$ are much easier to deal with. However, many massive graphs are power law graphs with $2 < \beta < 3$, in particular, Internet graphs [12] have exponents between 2.1 and 2.4.

**Theorem 4**

1. *For $\beta \ge 3$ and $m > d^2 \log^{3+\epsilon} n$, almost surely the largest eigenvalue of the random power law graph $G$ is $(1 + o(1))\sqrt{m}$.*

2. *For $3 > \beta > 2.5$ and $m > d^{\frac{\beta-2}{\beta-2.5}} \log^{\frac{3}{\beta-2.5}} n$, almost surely the largest eigenvalue of the random power law graph $G$ is $(1 + o(1))\sqrt{m}$.*

3. *For $2 < \beta < 2.5$ and $m > \log^{\frac{3}{2.5-\beta}} n$, almost surely the largest eigenvalue is $(1 + o(1))\tilde{d}$.*

**Proof:** If $\beta \ge 3$, clearly $\sqrt{m} > \tilde{d} \log^{2/3} n$. By Theorem 2, almost surely the largest eigenvalue of the random power law graph $G$ is $(1 + o(1))\sqrt{m}$.

If $3 > \beta > 2.5$, it is straightforward to verify $\sqrt{m} > \tilde{d} \log^3 n$. By Theorem 2, almost surely the largest eigenvalue of the random power law graph $G$ is $(1 + o(1))\sqrt{m}$.

When $\beta < 2.5$, we have $\tilde{d} > \sqrt{m} \log^{1+\epsilon} n$. The result follows from Theorem 1. □

# 6 Problems and remarks

In the case $\sqrt{m} \gg \tilde{d}$, our argument could also be used to determine eigenvalues other than the first one. Assume that $w_1 \ge w_2 \ge w_3 \ge \ldots$. For any index $k$ such that $\sqrt{w_k}$ is significantly larger than $\tilde{d}$, the $k^{th}$ largest eigenvalue is a.s. $(1 + o(1))\sqrt{w_k}$. We omit the details.

We have proved that the largest eigenvalue $\lambda$ of $G$ in $G(\mathbf{w})$ is roughly equal to $\tilde{d}$ or $\sqrt{m}$ if one of them is much larger than the other. What happens when $\tilde{d}$ and $\sqrt{m}$ are comparable? Is it true that $\lambda = (1 + o(1)) \max\{\sqrt{m}, \tilde{d}\}$? The following example shows that $\lambda(G)$ can be larger than $\tilde{d}$ and $\sqrt{m}$ by a constant factor.

**Example.** For given $m$ satisfying $m > \log^2 n$ and $d$ constant, we choose the expected degree sequence as follows: There are $n_1 = \frac{nd}{m^{3/2}} = o(n)$ vertices with weight $m$. The remaining vertices have weight $d$. We then have

$$\mathrm{Vol}(G) = n_1 m + (n - n_1)d \approx nd,$$

$$\tilde{d} = n_1 m^2 \rho + (n - n_1)d^2 \rho \approx \sqrt{m}.$$

Our random graph is defined with this special degree sequence.

*Claim 3.* The largest eigenvalue $\lambda$ of the adjacency matrix of $G(\mathbf{w})$ is almost surely at least $(1 - o(1))\frac{1+\sqrt{5}}{2}\sqrt{m} > 1.618\sqrt{m}$.

*Proof of Claim 3.* Let $S$ be the set of vertices with weight $m$, and $T$ be the remaining vertices. Since $\mathrm{Vol}(S) \approx \frac{1}{\sqrt{m}}\mathrm{Vol}(G)$ and $\mathrm{Vol}(T) \approx \mathrm{Vol}(G)$, the expected number of neighbors in $T$ for a vertex in $S$ is about $m$, while the expected number of neighbors in $S$ for a vertex in $T$ is about $\frac{d}{\sqrt{m}} = o(1)$. It can be shown that a random graph $G$ in $G(\mathbf{w})$ almost surely contains $n_1$ disjoint union of stars of size $m' = (1 + o(1))m$ with centers in $S$. As always, $A$ denotes the adjacency matrix of $G$.

Recall that $\lambda(A) \geq \alpha^* A\alpha$ for any unit vector $\alpha$. We next present a vector $\alpha$ such that the expectation of $\alpha^* A\alpha$ is significantly larger than $\sqrt{m}$. For any vertex $u$, the coordinates $\alpha_u$ is defined as follows:

$$\alpha_u = \begin{cases} \frac{\sqrt{c}}{\sqrt{n_1}} & \text{if } u \in S \\ \frac{\sqrt{1-c}}{\sqrt{n_1 m'}} & \text{if } u \text{ is a leaf of the stars.} \\ 0 & \text{otherwise.} \end{cases}$$

Here $c = (1 + 1/\sqrt{5})/2$ is a constant maximizing $E(\alpha^* A\alpha)$.

Clearly, $\alpha$ is a unit vector. We have

$$\begin{aligned} E(\alpha^* A\alpha) &\geq n_1^2 m^2 \rho \frac{c}{n_1} + 2n_1 m' \frac{\sqrt{c}}{\sqrt{n_1}} \frac{\sqrt{1-c}}{\sqrt{n_1 m'}} \\ &\approx \sqrt{m}(c + 2\sqrt{c(1-c)}) \\ &= \frac{1+\sqrt{5}}{2}\sqrt{m}. \end{aligned}$$

With the assumption $m > \log^2 n$, we conclude that almost surely $\alpha^* A\alpha$ is greater than $(\frac{1+\sqrt{5}}{2} + o(1))\sqrt{m}$, completing the proof. $\square$

We proved that the statement $\lambda = (1 + o(1)) \max\{\sqrt{m}, \tilde{d}\}$ is false. However, it looks plausible that $\lambda$ could be (almost surely) upper bounded by $(1 + o(1))(\tilde{d} + \sqrt{m})$ provided that $\tilde{d} + \sqrt{m}$ is sufficiently large (i.e, $\omega(\log n)$).

## References


[1] W. Aiello, F. Chung and L. Lu, A random graph model for massive graphs, *STOC* 2001, 171-180. The paper version appeared in *Experimental Math.* **10**, (2001), 53-66.

[2] W. Aiello, F. Chung and L. Lu, Random evolution in massive graphs, *Handbook of Massive Data Sets*, Volume 2, (Eds. J. Abello et al.), Kluwer Academic Publishers, (2002), 97-122. Extended abstract appeared in FOCS 2001, 510-519.

[3] R. Albert, H. Jeong and A. Barabási, Diameter of the world Wide Web, *Nature* **401** (1999), 130-131.

[4] Albert-László Barabási and Réka Albert, Emergence of scaling in random networks, Science **286** (1999) 509-512.

[5] Fan Chung and Linyuan Lu, Connected components in a random graph with given degree sequences, *Annals of Combinatorics*, to appear.

[6] P. Erdős and T. Gallai, Gráfok előírt fokú pontokkal (Graphs with points of prescribed degrees, in Hungarian), *Mat. Lapok* **11** (1961), 264-274.

[7] P. Erdős and A. Rényi, On random graphs. I, *Publ. Math. Debrecen* **6** (1959), 290-291.

[8] M. Faloutsos, P. Faloutsos and C. Faloutsos, On power-law relationships of the Internet topology, ACM SIG-COMM '99; *Comput. Commun. Rev.* **29** (1999) 251-263.

[9] I. J. Farkas, I. Derényi, A.-L. Barabási and T. Vicsek, Spectra of "Real-World" graphs: Beyond the semi-circle law, cond-mat/0102335.

[10] K.-I. Goh, B. Kahng and D. Kim, Spectra and eigenvectors of scale-free networks, cond-mat/0103337.

[11] H. Jeong, B. Tomber, R. Albert, Z. Oltvai and A. L. Barábasi, The large-scale organization of metabolic networks, *Nature*, **407** (2000), 378-382.

[12] J. Kleinberg, S. R. Kumar, P. Raphavan, S. Rajagopalan and A. Tomkins, The web as a graph: Measurements, models and methods, *Proceedings of the International Conference on Combinatorics and Computing*, 1999.

[13] Linyuan Lu, The Diameter of Random Massive Graphs, *Proceedings of the Twelfth ACM-SIAM Symposium on Discrete Algorithms*, (2001) 912-921.

[14] C. McDiarmid, Concentration. Probabilistic methods for algorithmic discrete mathematics, *Algorithms Combin.*, **16**, Springer, Berlin, (1998) 195–248.

[15] M. Mihail and C. Papadimitriou, On the eigenvalue power law, preprint.

[16] O. Perron, Theorie der algebraischen Gleichungen, II (zweite Auflage), de Gruyter, Berlin (1933).