Random vertex deletion in a scale free random graph

Colin Cooper, Alan Frieze, Juan Vera

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- C Probability $\alpha - \alpha_1$: Add m random edges with endpoints chosen independently as in (1).

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- If a loop is created in C, try again.

Let $D_k(t)$ be the number of vertices of degree k in G_t and let $\overline{D}_k(t) = \mathbf{E}(D_k(t))$.
Let

$$\beta = \frac{2\alpha}{3\alpha - 1 - \alpha_1} \text{ and } \rho = \frac{\alpha_1}{\alpha}.$$

Theorem

Assume that $\alpha + \alpha_1 > 1$. Then there exists a constant $C = C(m, \alpha, \alpha_1)$ such that for $k \geq 1$ and $1/2 < \alpha \leq 1$,

$$\left| \frac{\overline{D}_k(t)}{t} - Ck^{-1-\beta} \right| = O\left(t^{-\rho/8}\right) + O\left(k^{-2-\beta}\right).$$

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$$\overline{D}_k(t+1) = \overline{D}_k(t) +$$

$$(2\alpha - \alpha_1)m\mathbf{E}\left(-\frac{kD_k(t)}{2e_t} + \frac{(k-1)D_{k-1}(t)}{2e_t} \,\middle|\, e_t > 0\right)\Pr(e_t > 0)$$

$$+ (1-\alpha)(k+1)\mathbf{E}\left(\frac{D_{k+1}(t) - D_k(t)}{v_t} \,\middle|\, e_t > 0\right)\Pr(e_t > 0)$$

$$+ \sum_{i=2}^{m}\mathbf{E}\left(\frac{(k-i)^i D_{k-i}(t)}{2^i e_t^i} \mid e_t > 0\right)\Pr(e_t > 0)$$

$$+ 1_{k=m}(\alpha_1 + (2\alpha - \alpha_1)\Pr(e_t = 0)).$$

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$$\Pr(\exists x : \ degree(x, G_t) \geq 200m^2 t^{1-\rho/2}(\ln t)^2) \leq t^{-10}.$$

As a consequence we can write

$$\overline{D}_k(t+1) = \overline{D}_k(t) + (A_2(k+1) + B_2)\frac{\overline{D}_{k+1}(t)}{t} +$$
$$(A_1 k + B_1 + 1)\frac{\overline{D}_k(t)}{t} + (A_0(k-1) + B_0)\frac{\overline{D}_{k-1}(t)}{t} + \alpha_1 1_{k=m} + O(t^{-\rho/8}).$$

$$A_2 = \frac{1-\alpha}{\alpha+\alpha_1-1} \qquad B_2 = 0$$

$$A_1 = -\frac{(2\alpha-\alpha_1)(\alpha_1+1-\alpha)}{2\alpha(\alpha+\alpha_1-1)} - \frac{1-\alpha}{\alpha+\alpha_1-1} \qquad B_1 = -\frac{\alpha_1}{\alpha+\alpha_1-1}$$

$$A_0 = \frac{(2\alpha-\alpha_1)(\alpha_1+1-\alpha)}{2\alpha(\alpha+\alpha_1-1)} \qquad B_0 = 0$$

Assuming $\overline{D}_k(t) \sim d_k t$ we consider the recurrence: $d_{-1} = 0$ and for $k \geq -1$,

$$\begin{aligned}(A_2(k+2) + B_2)d_{k+2} + (A_1(k+1) + B_1)d_{k+1} + (A_0 k + B_0)d_k \\ = -\alpha_1 1_{k=m-1}. \end{aligned} \tag{2}$$

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To justify this attention we prove: Let d_k be a solution to (2) such that $d_k \leq \frac{C}{k}$ for $k > 0$ for some C, then there exists a constant $M > 0$ such that for $t \geq 2,\ k \geq -1$,

$$|\overline{D}_k(t) - td_k| \leq Mt^{1-\rho/8}.$$

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We first tackle homogeneous equation:

$$(A_2(k+2)+B_2)e_{k+2}+(A_1(k+1)+B_1)e_{k+1}+(A_0k+B_0)e_k=0,$$

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for $k \geq 1$.

We use Laplace's method and make the substitution

$$e_k = \int_{t=a}^{t=b} t^{k-1}v(t)dt$$

for $a, b, v(t)$ to be determined.

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Substituting gives

$$[t^k \phi_1(t) v(t)]_a^b - \int_a^b t^k \phi_1(t) v'(t) dt + \int_a^b t^{k-1} \phi_0(t) v(t) dt = 0.$$

So, $v(t)$ will give a solution to the homogeneous equation if

$$[t^k v(t)\phi_1(t)]_a^b = 0 \qquad \text{and} \qquad \frac{v'(t)}{v(t)} = \frac{\phi_0(t)}{t\phi_1(t)}.$$

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The differential equation is homogeneous and can be integrated to give,

$$v(t) = C_0(t-1)^\beta \left(t - \frac{(\alpha_1+1-\alpha)(2\alpha-\alpha_1)}{2\alpha(1-\alpha)}\right)^{-\beta}$$

where $C_0 = C_0(\alpha, \alpha_1) \neq 0$ and $\beta = \frac{2\alpha}{3\alpha-1-\alpha_1}$

Since

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Substituting we get the following solution to the homogeneous equation: valid for $k \geq 1$,

$$u_1(k) = \int_0^1 t^{k-1} \left(\frac{1-t}{1-\gamma t}\right)^\beta dt$$

where

$$\gamma = \frac{2\alpha(1-\alpha)}{(\alpha_1 + 1 - \alpha)(2\alpha - \alpha_1)} < 1.$$

$$
\begin{aligned}
u_1(k) &= \int_0^1 t^{k-1}(1-t)^\beta \frac{1}{(1-\gamma t)^\beta}\,dt \\
&= \int_0^1 t^{k-1}(1-t)^\beta \sum_{j=0}^{\infty} \binom{\beta+j-1}{j} (\gamma t)^j\,dt \\
&= \sum_{j=0}^{\infty} \binom{\beta+j-1}{j} \gamma^j \int_0^1 t^{k+j-1}(1-t)^\beta\,dt \\
&= \sum_{j=0}^{\infty} \binom{\beta+j-1}{j} \gamma^j \frac{\Gamma(k+j)\Gamma(\beta+1)}{\Gamma(k+j+\beta+1)} \\
&= \sum_{j=0}^{\infty} \gamma^j \frac{\Gamma(\beta+j)}{\Gamma(j+1)\Gamma(\beta)} \frac{\Gamma(k+j)\Gamma(\beta+1)}{\Gamma(k+j+\beta+1)}
\end{aligned}
$$

assuming k is large, using Stirling for $\Gamma(k+j), \Gamma(k+j+\beta+1)$, we get

$$
\begin{aligned}
&= (1+O(k^{-1}))\beta \sum_{j=0}^{\infty} \gamma^j \frac{\Gamma(j+\beta)}{\Gamma(j+1)} (k+\beta+j)^{-\beta-1} \\
&= (1+O(k^{-1}))C_1 k^{-1-\beta}
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$$|\overline{D}_k(t) - t d_k| \leq M t^{1-\rho/8}.$$

Let

$$C = -\frac{2A_2 w_2 + (A_1 + B_1)w_1}{2A_2 u_1(2) + (A_1 + B_1)u_1(1) \leftarrow \neq 0}$$

and

$$d_0 = -\frac{A_2(Cu_1(1) + w_1)}{B_1},$$

then there is a solution to (2) such that

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Since $u_1(k) = o(k^{-1})$, we deduce that $\overline{D}_k(t)/t$ is asymptotically equal to $Cu_1(k)$.

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Our high probability bound on $v_t - \nu t$ implies that for t large, $v_t < k_0(t) = \frac{t+B+1}{-A_1}$. Assume, t is large enough, $> T$ say, and adjust M later to deal with small t.

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Define $\Delta_k(t) = \overline{D}_k(t) - d_k t$.

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For $k \geq 0$,

$$\Delta_k(t+1) = A_2(k+1)\frac{\Delta_{k+1}(t)}{t} + (t + A_1 k + B_1 + 1)\frac{\Delta_k(t)}{t} + A_0(k-1)\frac{\Delta_{k-1}(t)}{t} + O(t^{-\rho/8}). \qquad (3)$$

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If $t + A_1 k + B_1 + 1 \leq 0$ then $k \geq k_0(t+1) > v_{t+1}$ and (i) $\overline{D}_{t+1} = 0$ and $(t+1)d_k \leq Ct/k = O(1)$ and so $\mathcal{A}(k, t+1)$ is true.

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If $t + A_1 k + B_1 + 1 > 0$ then $\mathcal{A}(k, t+1)$ follows easily from (3).

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$$\Pr(|e_t - \eta t| \geq t^{1-\rho/8}) = O(t^{-\rho/4}(\log t)^5).$$

$$v_t = X_1 + X_2 + \cdots + X_t$$

where $X_1 = X_2 = 1$. For $i > 2$ we have $X_i = +1$ with probability α_1, $X_i = 0$ with probability $\alpha - \alpha_1$ and $X_i = -1$ with probability $1 - \alpha$, when $v_{i-1} > 0$, otherwise $X_i = 2$.

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The distribution of v_t dominates that of $Y_1 + Y_2 + \cdots + Y_t$ where the Y_i are independent and $Y_i = +1$ with probability α_1, $Y_i = 0$ with probability $\alpha - \alpha_1$ and $Y_i = -1$ with probability $1 - \alpha$.

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It follows from Hoeffding's theorem that

$$\Pr(|Y_1 + Y_2 + \cdots + Y_t - \nu t| \geq u) \leq 2 \exp\left\{-\frac{u^2}{2t}\right\}$$

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This verifies the concentration for v_t.

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\begin{aligned}
\mathbf{E}(e_{t+1} \mid e_t, v_t > 0) &= e_t + \alpha m - (1-\alpha) \sum_{k \geq 0} k \frac{D_k(t)}{v_t} \\
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So we see that,

$$\mathbf{E}(e_{t+1}) = \mathbf{E}(e_t) + \alpha m - \frac{2(1-\alpha)}{(\alpha+\alpha_1-1)t}\mathbf{E}(e_t) + O(t^{-1/2}\log t).$$

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So we can write

$$|\mathbf{E}(f_t)| \;\le\; O(t^{1/2}\log t)$$

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$$\mathbf{Var}(e_{t+1}) \leq \mathbf{Var}(e_t) + \mathbf{E}(Y_t^2) + 2(\mathbf{E}(e_t Y_t) - \mathbf{E}(e_t)\mathbf{E}(Y_t)).$$

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We show $\Pr\left(\sum k^2 D_k(t) \geq t^{2-\rho}(\log t)^5\right) = O(t^{-10})$ and so

$$\mathbf{E}(Y_t^2) = O(t^{1-\rho}(\log t)^5).$$

$$\mathbf{E}(e_t Y_t \mid G_t) = e_t(\alpha m - 2(1-\alpha) v_t^{-1} e_t)$$

and so

$$\begin{aligned}
\mathbf{E}(e_t Y_t) &= \alpha m \mathbf{E}(e_t) - 2(1-\alpha)\mathbf{E}(v_t^{-1} e_t^2) \\
&\leq \alpha m \mathbf{E}(e_t) - 2(1-\alpha)\frac{\mathbf{E}(e_t^2)}{\nu t + t^{1/2}\log t} + o(1) \\
\mathbf{E}(e_t)\mathbf{E}(Y_t) &= \alpha m \mathbf{E}(e_t) - 2(1-\alpha)\mathbf{E}(e_t)\mathbf{E}(v_t^{-1} e_t) \\
&\geq \alpha m \mathbf{E}(e_t) - 2(1-\alpha)\frac{\mathbf{E}(e_t)^2}{\nu t - t^{1/2}\log t} + o(1)
\end{aligned}$$

$$\mathbf{E}(e_t Y_t \mid G_t) = e_t(\alpha m - 2(1-\alpha)v_t^{-1}e_t)$$

and so

$$\begin{aligned} \mathbf{E}(e_t Y_t) &= \alpha m \mathbf{E}(e_t) - 2(1-\alpha)\mathbf{E}(v_t^{-1}e_t^2) \\ &\leq \alpha m \mathbf{E}(e_t) - 2(1-\alpha)\frac{\mathbf{E}(e_t^2)}{\nu t + t^{1/2}\log t} + o(1) \\ \mathbf{E}(e_t)\mathbf{E}(Y_t) &= \alpha m \mathbf{E}(e_t) - 2(1-\alpha)\mathbf{E}(e_t)\mathbf{E}(v_t^{-1}e_t) \\ &\geq \alpha m \mathbf{E}(e_t) - 2(1-\alpha)\frac{\mathbf{E}(e_t)^2}{\nu t - t^{1/2}\log t} + o(1) \end{aligned}$$

It follows that

$$\mathbf{Var}(e_{t+1}) \leq \mathbf{Var}(e_t) + O(t^{1-\rho}(\log t)^5) + O(t^{1/2}\log t)$$

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Applying the Chebychev inequality we obtain

$$\Pr(|e_t - \eta t| \geq t^{1-\rho/8}) \leq \frac{\mathbf{Var}(e_t)}{t^{2-\rho/4}} = O(t^{-\rho/4}(\log t)^5).$$

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$$\Pr\left(\sum k^2 D_k(t) \geq t^{2-\rho}(\log t)^5\right) = O(t^{-10})$$

General idea: We compare the process with deletions to a process without deletions and bound the above probability in the latter.

Final loose end:

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General idea: We compare the process with deletions to a process without deletions and bound the above probability in the latter.

Process without deletions: Let $\rho = \alpha_1/\alpha$. At each step, add m edges between old vetices with probability $1-\rho$ and a new vertex and m edges with probability ρ.

Fix $s \leq t$ and let $X_\tau = deg(s, \tau)$ for $\tau = s, s+1, \ldots, t$ and let $\lambda = \frac{(s/t)^{(1+\rho)/2}}{20m^2 \ln t}$.

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Let Y be a 0,1 random variable with $\Pr(Y = 1) = 1 - \rho$. Then conditional on $X_\tau = x$, we have

$$X_{\tau+1} = X_\tau + YB\left(2m, \frac{x}{2m\tau}\right) + (1-Y)B\left(m, \frac{x}{2m\tau}\right)$$

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So,

$$
\begin{aligned}
\mathbf{E}(e^{\lambda X_{\tau+1}} \mid X_\tau = x) &\le e^{\lambda x}\left((1-\rho)\left(1 - \frac{x}{2m\tau} + \frac{x}{2m\tau}e^{\lambda}\right)^{2m} + \right. \\
&\qquad \left. \rho\left(1 - \frac{x}{2m\tau} + \frac{x}{2m\tau}e^{\lambda}\right)^{m}\right) \\
&\le e^{\lambda x}\left((1-\rho)\exp\left\{\frac{x}{\tau}(e^{\tau}-1)\right\} + \right. \\
&\qquad \left. \rho\exp\left\{\frac{x}{2\tau}(e^{\tau}-1)\right\}\right) \\
&= \exp\left\{\lambda x\left(1 + \frac{(2-\rho)(1+2m^2\lambda)}{2\tau}\right)\right\},
\end{aligned}
$$

after using $x \le m\tau$.

Thus

$$\mathbf{E}(e^{\lambda X_{\tau+1}}) \le \mathbf{E}\exp\left\{X_\tau \lambda \left(1 + \frac{(2-\rho)(1+2m^2\lambda)}{2\tau}\right)\right\}.$$

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If we put $\lambda_t = \lambda$ and $\lambda_{\tau-1} = \lambda_\tau \left(1 + \frac{(2-\rho)(1+2m^2\lambda_\tau)}{2\tau}\right)$ then provided $\lambda_s \leq 1$ we will have

$$\mathbf{E}(e^{\lambda X_t}) \leq e^{m\lambda_s}.$$

Now provided $\lambda_\tau \leq \Lambda = \frac{1}{2m^2 \ln t}$ we can write

$$\lambda_{\tau-1} \leq \lambda_\tau \left(1 + \frac{(2-\rho)(1+2m^2\Lambda)}{2\tau}\right)$$

and then

$$\begin{aligned} \lambda_s &\leq \lambda \prod_{\tau=s}^{t} \left(1 + \frac{(2-\rho)(1+2m^2\Lambda)}{2\tau}\right) \\ &\leq 10\lambda(t/s)^{1-\rho/2} \end{aligned}$$

which is $\leq \Lambda$ by the definition of λ.

Putting $u = 200m^2(t/s)^{1-\rho/2}(\ln t)^2$ we get

$$
\begin{aligned}
\Pr(X_t \geq u) &\leq e^{m\lambda_s - \lambda u} \\
&\leq \exp\{\lambda(10m(t/s)^{1-\rho/2} - u)\} \\
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Now note that

$$\sum_{s=1}^{t}(200m^2(t/s)^{1-\rho/2}(\ln t)^2)^2 \leq t^{2-\rho}(\log t)^5$$

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Now note that

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We now only have to show that when there are deletions, degrees are less likely to grow large.