NOTE3

Note Title 1/29/2004

Diameter of preferential attachment graph.

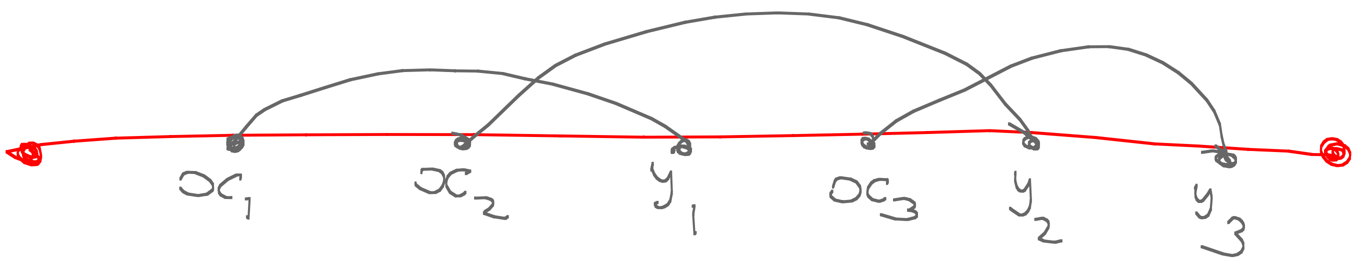
Model

Let $0 < x < y < 1$ be ordered outcomes of choosing two numbers independently and uniformly from $[0, 1]$.

Let (x_i, y_i), $i = 1, 2, \cdots, mn$ be mn independently chosen such pairs.

Suppose now that after relabelling,

$y_1 < y_2 < \cdots < y_{mn}$

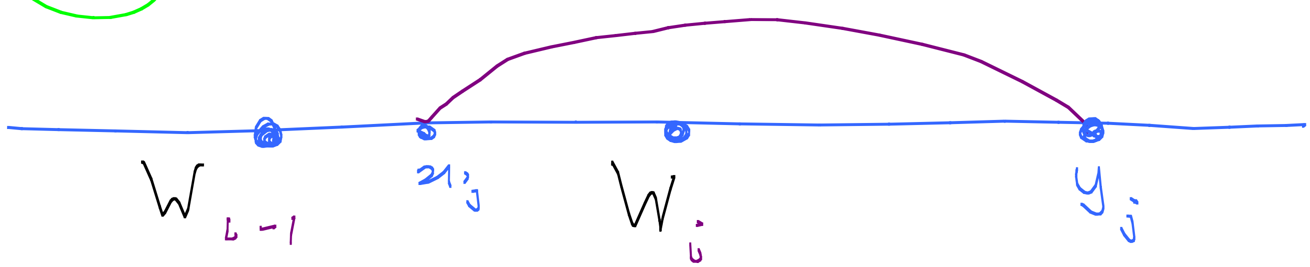


[If $z_1 < z_2 < \cdots < z_{2mn}$ is the ordering of the whole $2mn$ numbers and $x_k = z_{a(k)}$, $y_k = z_{b(k)}$ then $(a(k), b(k))$ is a random pairing of $\{1, 2, \ldots, 2mn\}$.]

Graph: Edge (i,j), $i<j$

if $\underset{\substack{\parallel \\ W_{i-1}}}{y_{(i-1)m}} < x_j < \underset{\substack{\parallel \\ W_i}}{y_{im}}$

G



Define

$$w_i = W_i - W_{i-1}, \quad 1 \le i \le n$$

Nb: If $y_1, y_2, \dots, y_{mn}$ are fixed then x_i is chosen uniformly in $[0, y_i]$.

Theorem Bollobás, Riordan

$$\mathrm{diam}(G) \sim \frac{\log n}{\log \log n} \text{ whp}$$

We simplify matters by conditioning on $W_1, W_2, \ldots, W_n$

Typical W_i : $2^{a-1} < (\log n)^7 \leq 2^a$

$2^b < 2n/3 \leq 2^{b+1}$

(0) $$E(\omega_{i_1}\omega_{i_2}\cdots\omega_{i_k}) \lesssim \prod_i \frac{1}{2\sqrt{n\, i_k}}$$

for $k = O(\log n)$.

(i) $\left|W_i - \sqrt{\frac{i}{n}}\right| \leq \frac{1}{10}\sqrt{\frac{i}{n}}$ $\quad 2^{a-1} \leq i \leq n$

(ii) $I_t = [2^t + 1, 2^{t+1}]$

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I_t contains at least 2^{t-1} vertices with $\omega_i \geq \frac{1}{10\sqrt{in}}$, $a \leq t \leq b$

(iii) $\omega_i \geq \frac{(\log n)^2}{n}$ $\quad i < n^{1/5}$

(iv) $\omega_i \leq n^{-4/5}$ $\quad i > \frac{n}{(\log n)^5}$

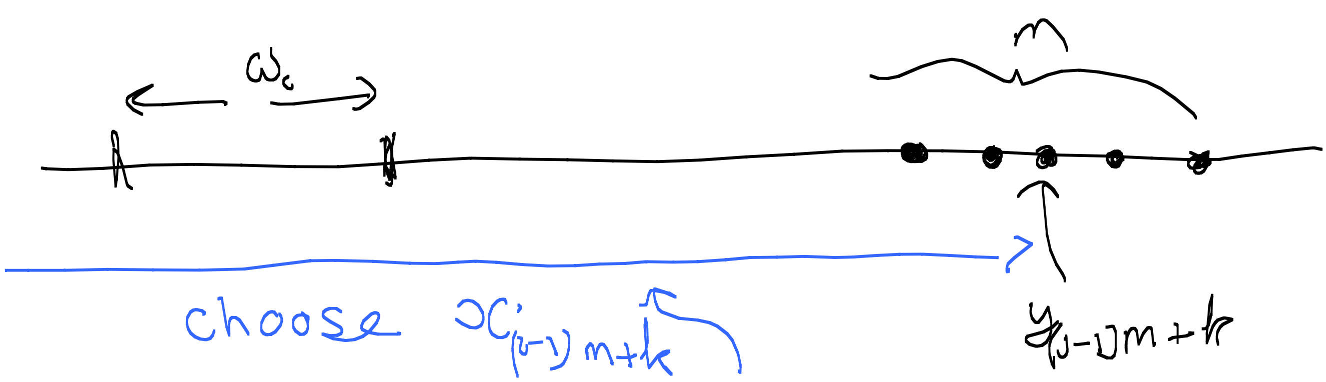
(v) $\omega_1 \geq \frac{4}{\sqrt{n}\log n}$

$W_1^2, W_2^2, \ldots, W_n^2$ are the order statistics of n independent $U[0,1]$ random variables.

Lower Bound on diameter

Claim: $i < j$

$$\Pr\left((i,j) \text{ exists} \mid \text{other edges incident } j \right) \le E\left[\left(\frac{\omega_i}{W_{j-1}} \right) \right] \le \frac{\omega}{\sqrt{i\, j}}$$



Let $u = n-1$, $v = n$ and

$P = (u_0 = u, u_1, u_2, \ldots, u_k = v)$.

$$\Pr(P \text{ exists in } G) \leq \frac{1}{\sqrt{uv}} \prod_{i=1}^{k-1} \frac{w}{u_i}$$

$$\Pr(\text{Path of length } k \text{ from } u \text{ to } v) \leq \frac{w^{k-1}}{n} \sum_{u_1, u_2, \ldots u_{k-1}} \prod_i \frac{1}{u_i}$$

$$\leq \frac{w^{k-1}}{n} \prod_{i} \sum_{j=1}^{n} \frac{1}{j}$$

$$\lessapprox \frac{(w \log n)^k}{n}$$

$$= O(1)$$

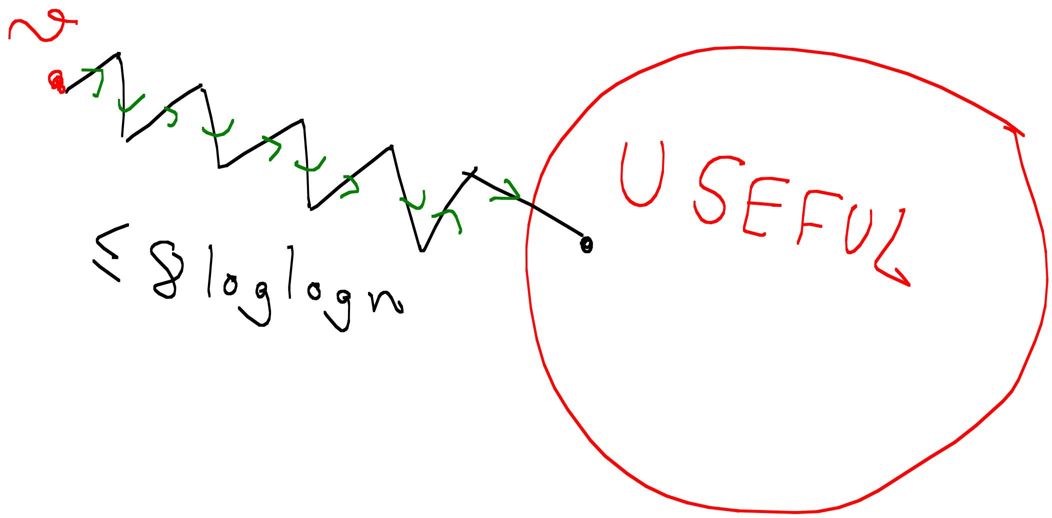
if $\quad k \leq (1-\epsilon) \frac{\log n}{\log \log n}$

Upper bound (HARDER!)

USEFUL: $i \leq \frac{n}{(\log n)^5}$ & $\omega_i \geq \frac{(\log n)^3}{n}$

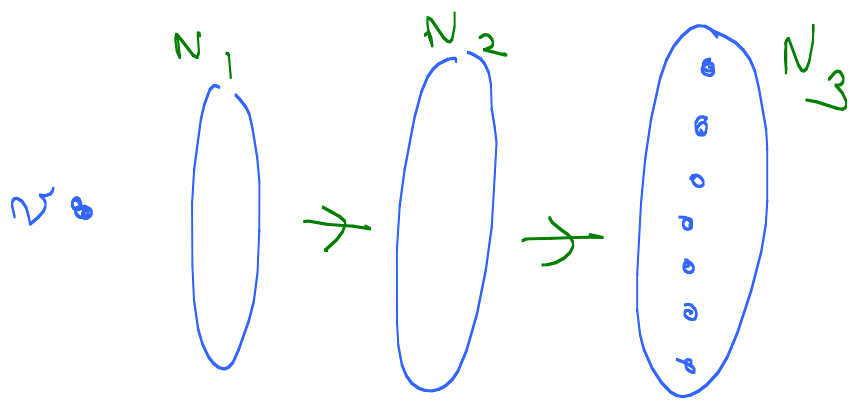
GOOD: $\omega_i \geq \frac{1}{10\sqrt{in}}$

Lemma



Proof

Grow from v — only use 2 arcs



$$\underset{\text{neither useful}}{\Pr(x \to y)} \le \underbrace{\frac{(\log n)^2}{n}}_{w_y} \cdot \underbrace{\left(\frac{n^{1/5}}{n}\right)^{1/2}}_{w_x} = n^{-3/5+o(1)}$$

vertices in k^{th} level $N_k \le 2^k$

$\Pr(\exists\, v_1, v_2, w_1, w_2$ in k th level, both choosing same in $(k+1)$ th level$)$

$\le 2^{4k} n^{-6/5+o(1)}$

$\le n^{-7/6}$ if $k = O(\log\log n)$

$\Pr(|N_3| \le 2) = O(n^{-7/6})$

and so with probability $1 - O(n^{-1/6})$

$\forall v \quad |N_k| \ge c 2^k \quad$ for $k = O(\log\log n)$

$c < 1$ constant.

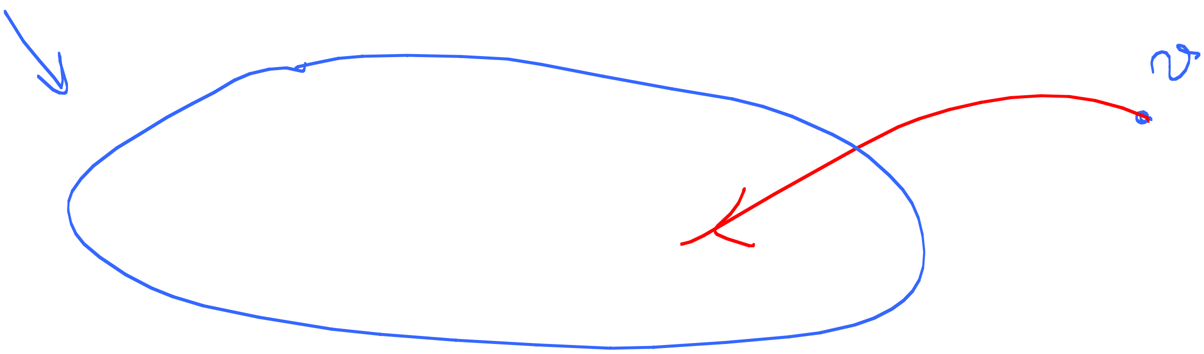
$v_0 \xrightarrow{\text{choose}} w$

$$P_1(w \text{ is useful}) \ge \frac{1}{(\log n)^3}$$

$I_t = [2^t, 2^{t+1}]$ $\qquad a \le t \le b'$

$\qquad\qquad 2^{b'+1} \le \min\left\{v, \frac{n}{(\log n)^5}\right\}$

$\bigcup I_t = J$



$$P_1(v \text{ points to } J) \ge \frac{W_{2^{b'}} - W_{2^a}}{W_v}$$

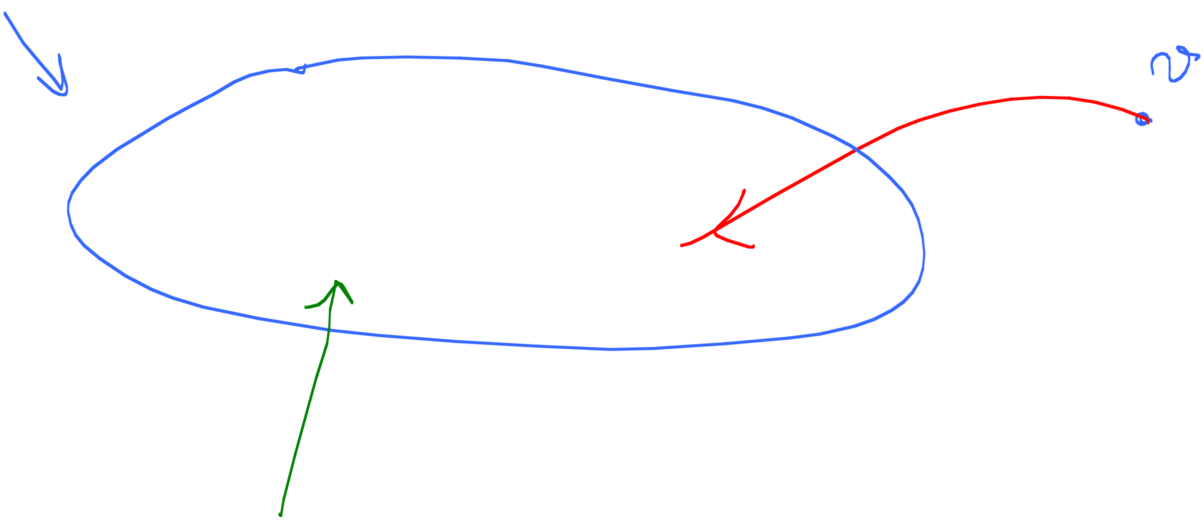
(1) $v \le \frac{n}{(\log n)^5}$ $\qquad\qquad n^{-\frac{1}{2}+o(1)}$

$W_{2^{b'}} \ge \frac{1}{8} W_v$ $\qquad 1 \ge \lambda \ge W_{n^{1/5}} \ge n^{-2/5+o(1)}$

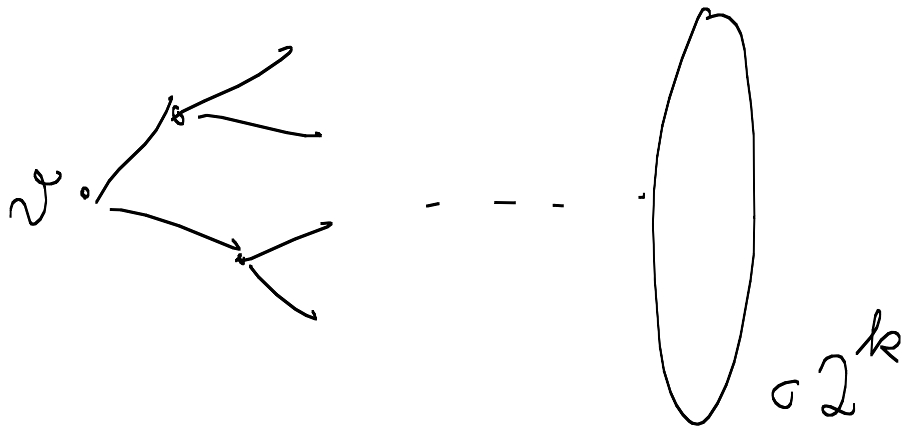
(ii) $v > \frac{n}{(\log n)^5} \Rightarrow W_{2^t} \geq W_{\frac{n}{(\log n)^5}} \geq \frac{100}{(\log n)^3}$

So, probability of $\swarrow$ $\geq \frac{100}{(\log n)^3}$

$\bigcup I_t = \bar{J}$

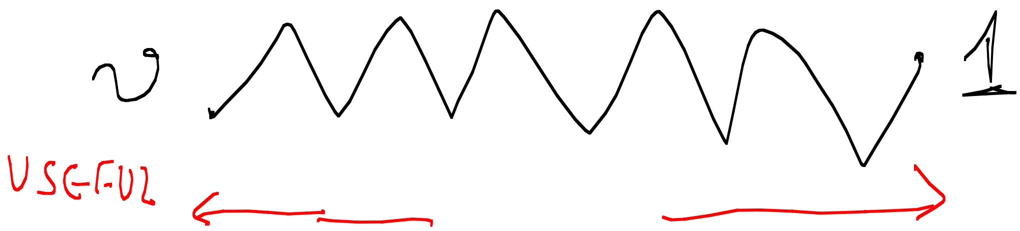


Useful vertices contain at least $\frac{1}{100}$ of weight.



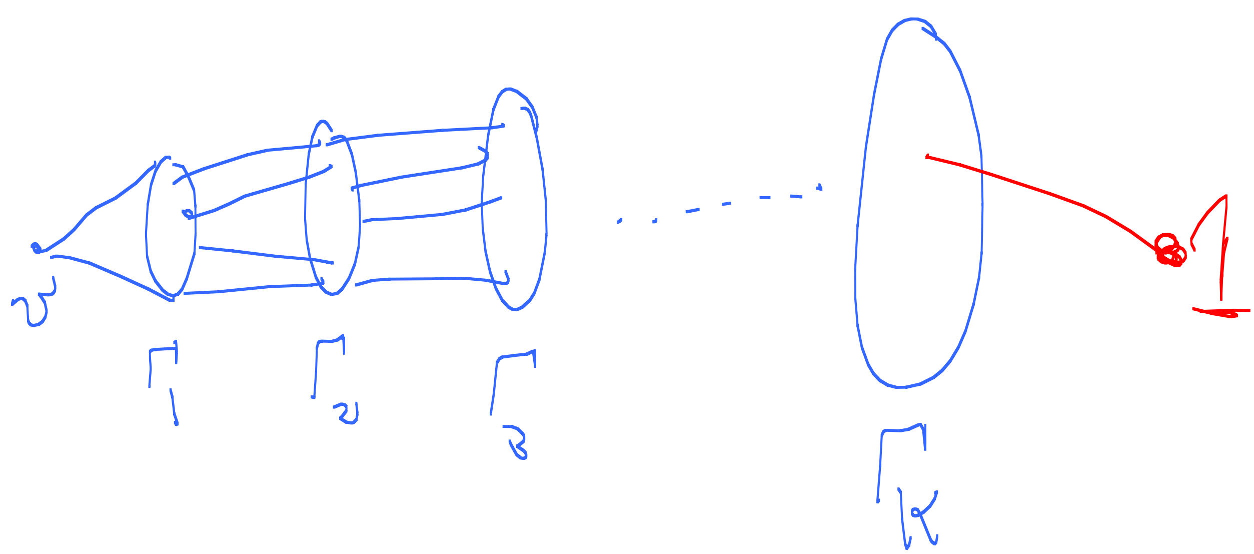
$$\Pr\left(N_{k+1} \cap \mathrm{USEFUL} = \emptyset\right) \leq \left(1 - \frac{1}{(\log n)^3}\right)^{c2^k} \leq n^{-2}$$

Lemma

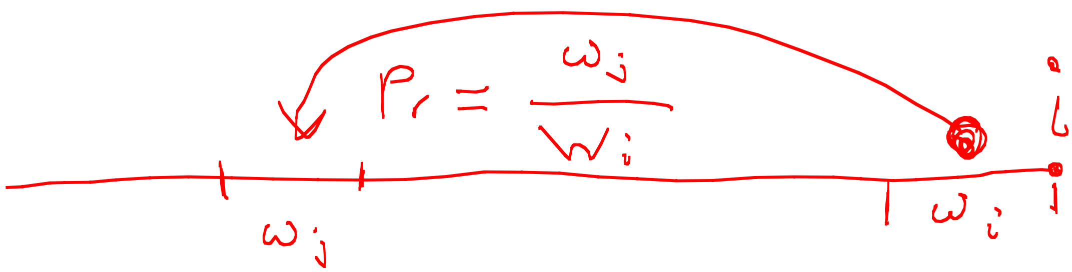


$$\leq \left(\frac{1}{2}+\epsilon\right)\frac{\log n}{\log\log n}$$

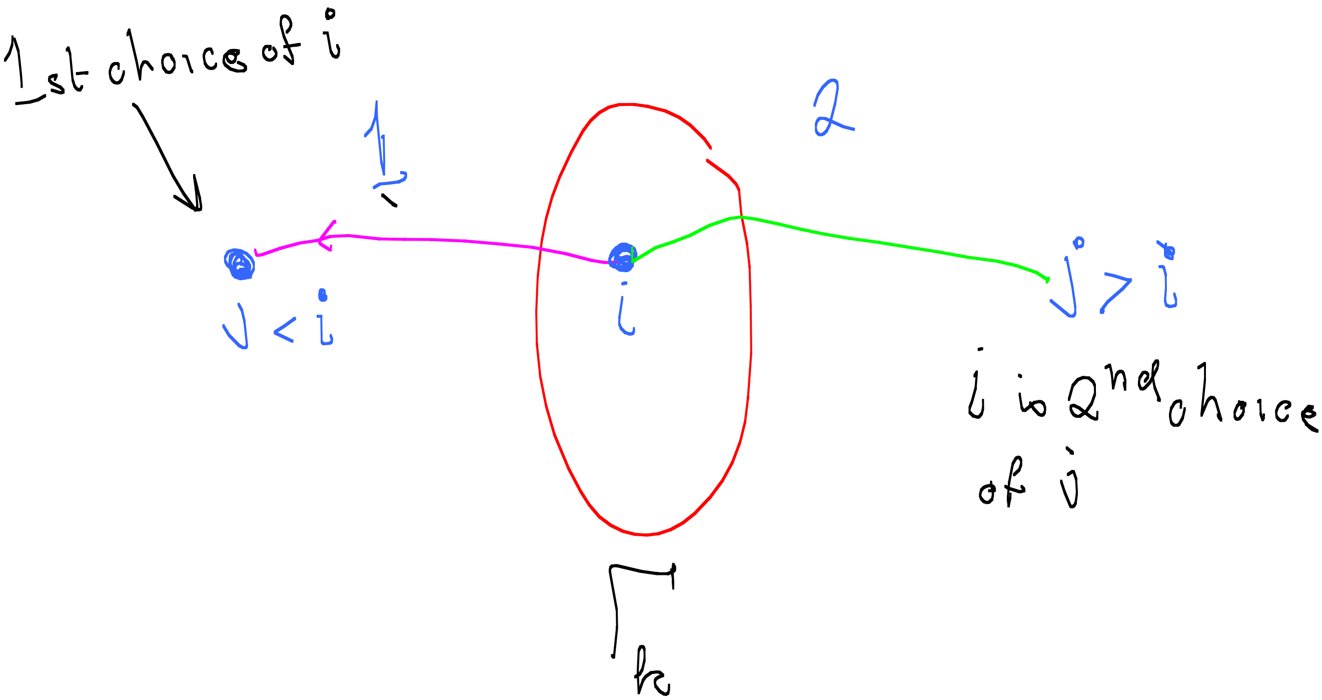
Proof



Random edges

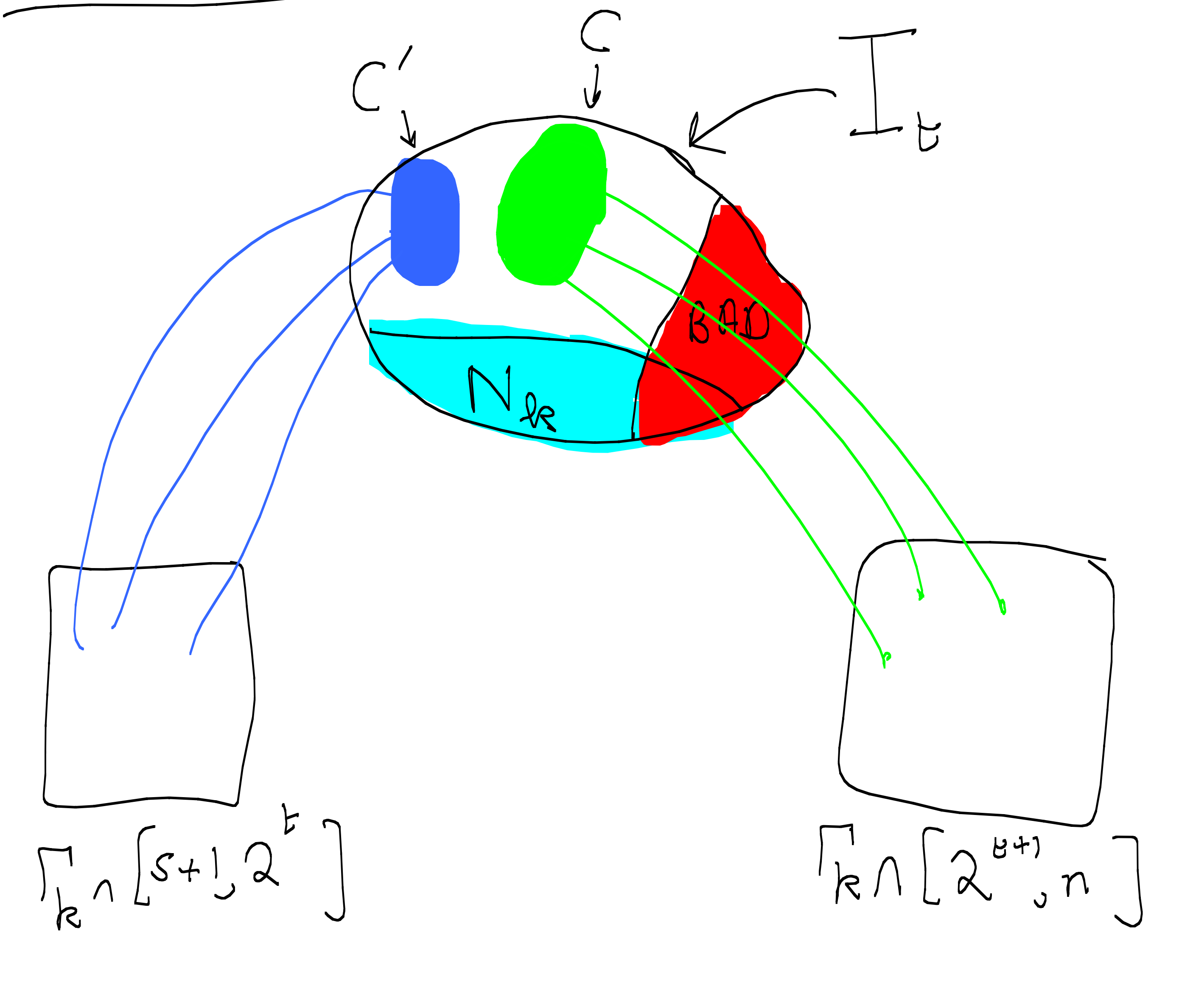


In growing Γ_{k+1} from Γ_k only use

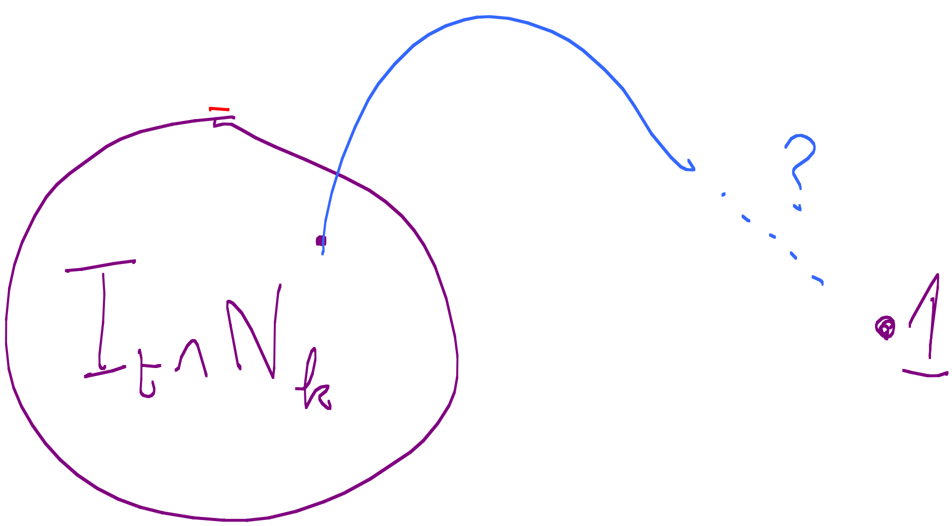


$I_t = [2^t + 1, 2^{t+1}]$ $\qquad t \geq a$

$N_k = \Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_k$



Case 1: $\exists t : X = |I_t \cap N_k| \geq 2^{t-10}$.



$$\Pr(\text{NO} \frown 1) \leq \left(1 - \frac{w_1}{W_{max:t}}\right)^{2^{t-10}}$$

$$\le \left(1 - \frac{4/(\log n \sqrt{n})}{1.1(2^{t+1}/n)^{1/2}}\right)^{2^{t-10}}$$

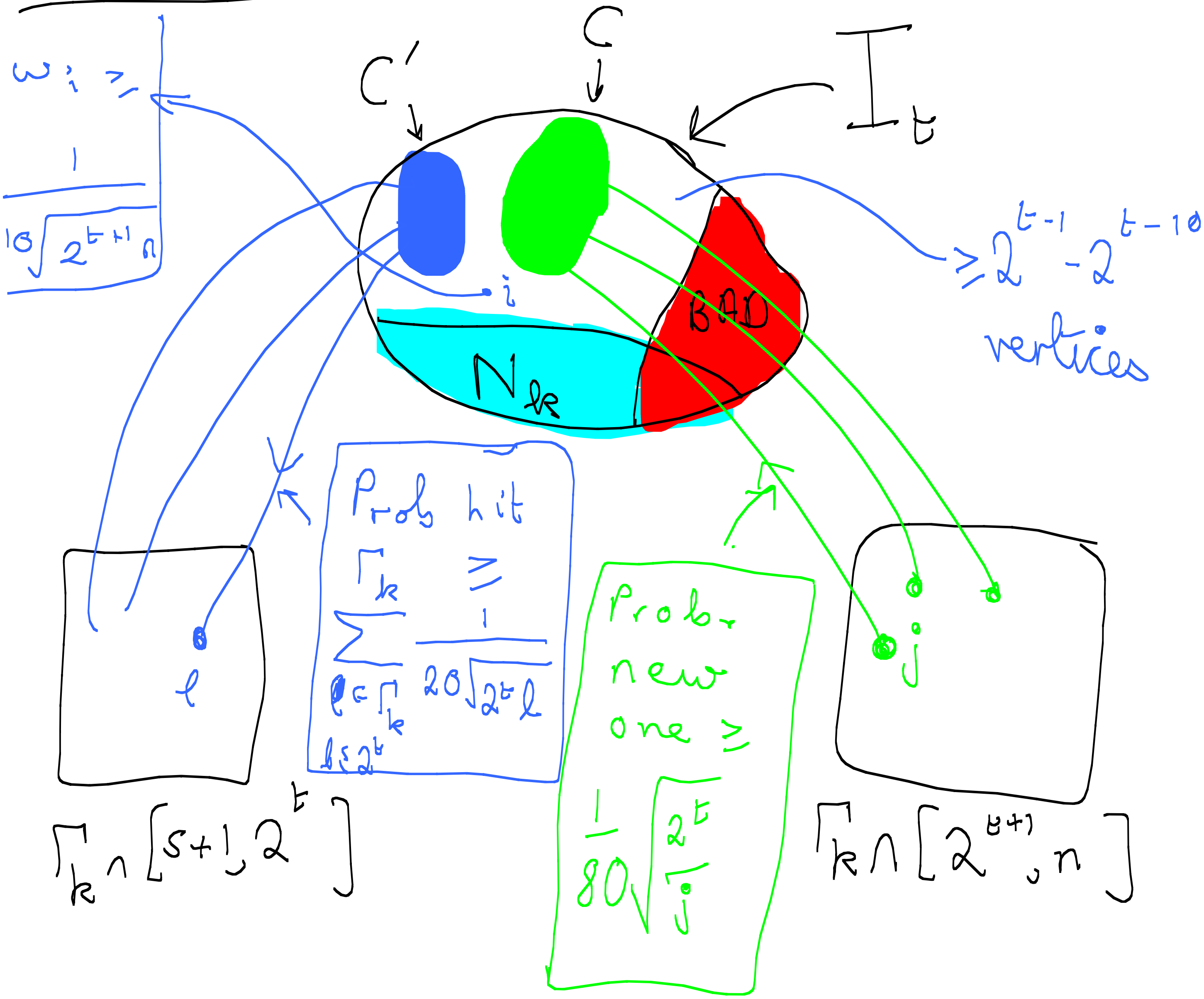
$$\le \exp\left\{-\frac{c\,2^{t/2}}{\log n}\right\}$$

$$\le \exp\left\{-c(\log n)^{5/2}\right\}$$

Case 2: $X < 2^{t-10}$:

$$I_t = [2^t + 1, 2^{t+1}] \qquad t \geq a$$

$$N_k = \Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_k$$



$$E\left(|\Gamma_{k+1} \cap I_t|\right) \geq \underbrace{\frac{2^{t/2}}{80} \sum_{i \in \Gamma_k \setminus I_t} \frac{1}{\sqrt{i}}}_{\mu(t)}$$

We track the growth of

$$f_k = \sum_{i \in \Gamma_k} \frac{1}{\sqrt{i\,n}}$$

$$\mu(t) \geq \frac{2^{t/2}\sqrt{n}}{160} f_k$$

$$\mu(t_1) + \mu(t_2) \geq \frac{2^{t/2}\sqrt{n}}{80} f_k$$

for all but ≤ 1 value of t.

Whp

$$|\Gamma_{k+1} \cap I_t| \geq \frac{1}{4}\mu(t)$$

if $\mu(t) \geq 20 \log n$

and then

$$\sum_{i \in \Gamma_{k+1} \cap I_t} \frac{1}{\sqrt{in}} \geq \frac{\mu(t)}{4\sqrt{2^{t+1} n}}$$

$$\geq \frac{1}{1000} f_k$$

#t for which

$$\frac{2^{t/2}\sqrt{n}}{160} f_k \geq 20 \log n$$

is at least

$$\underbrace{\min\left\{\log_2\left[\frac{n}{3}\cdot\frac{\sqrt{n}\, f_k}{3200\log n}\right],\ b-a\right\}}_{\otimes}$$

So we get a high prob. recurrence:

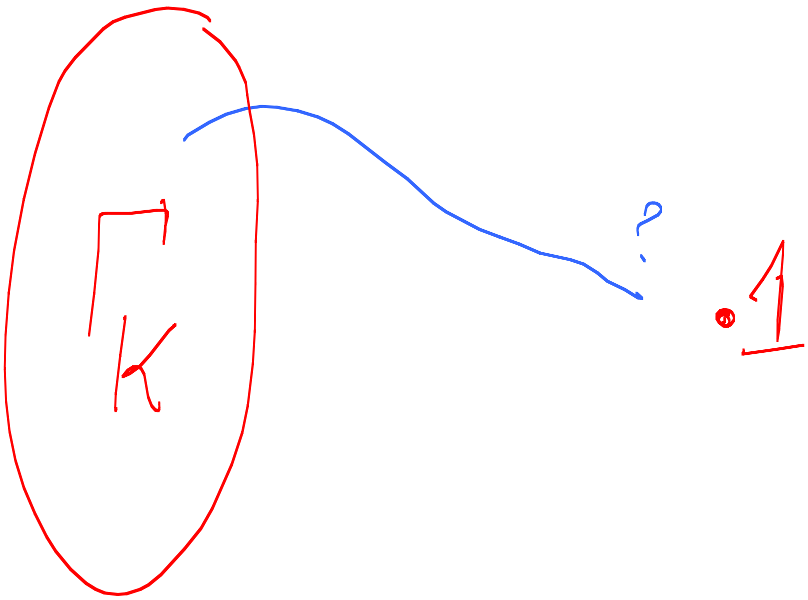
$$f_{k+1} \geq \frac{\left(x - 1 \right)}{1000} f_k$$

$$f_0 = \frac{1}{\sqrt{n v}} \geq \frac{(\log n)^2}{n}$$

$$\Downarrow$$

After $\leq \left(\frac{1}{2} + \epsilon\right) \dfrac{\log n}{\log\log n}$ rounds

$$f_K \geq \frac{(\log n)^3}{\sqrt{n}}$$



$$\Pr(\text{no such edge}) \leq$$

$$\prod_{i \in \Gamma_k} \left(1 - \frac{w_1}{W_i}\right) \leq$$

$$\prod_{i \in \Gamma_k} \left(1 - \frac{4}{\sqrt{n}\log n} \cdot \sqrt{\frac{n}{2i}}\right)$$

$$\leq \exp\left\{ -\frac{4}{\sqrt{n}\log n} \cdot \frac{n}{\sqrt{2}} \cdot \sum_{i \in \mathcal{N}_k} \frac{1}{\sqrt{i\,n}} \right\}$$

$$= \exp\left\{ -\frac{4\sqrt{n}}{\sqrt{2}\log n} f_K \right\}$$

$$\leq n^{-1-\epsilon}$$