Web Course

Note Title 1/4/2004

(1) Web/Internet: examples of large real world networks.

Grows unpredictably - randomly

Model by random graph:

Classical random graph $G_{n,m}$ or $G_{n,p}$

Vertex set $[n]$:

$G_{n,m}$: Each graph with m edges equally likely

$G_{n,p}$: Each possible edge occurs independently with probability p.

$$\Pr(G_{n,p} = H) = p^{e(H)} (1-p)^{\binom{n}{2} - e(H)}$$

If $m \approx \binom{n}{2} p$ then $G_{n,p}$ and $G_{n,m}$ have similar properties.

Suppose $m = \frac{1}{2} cn$ or $p = c/n$.

c is constant = average degree

(expected average in p model.)

Degree distribution:

$$\Pr(d(1) = k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$\simeq \frac{c^k e^{-c}}{k!} \qquad (*)$$

Observed degree distribution of WWW/Internet more like (Faloutsos, Faloutsos, Faloutsos)

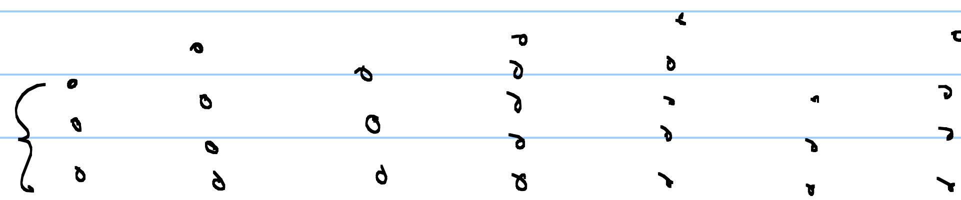
$$\Pr(d(1) = k) \simeq \frac{A}{k^\alpha} \qquad (\text{large } k)$$

Need new model

(A) Random graph with fixed degree sequence. Bender, Canfield, Bollobás, Molloy Reed

Choose uniformly at random from graphs with degree sequence $d_1, d_2, \dots, d_n$

Configuration model:

d_1 

Randomly pair up dots.

Collapse dots to n vertices.

(B) Fix $d_1 \geq d_2 \geq \cdots \geq d_n$

$$D = d_1 + d_2 + \cdots + d_n$$

Add an edge between i and j with probability

$$\frac{d_i d_j}{D}$$

(Assume $d_1^2 < D$)

Expected degree of i is

$$d_i \left(1 - \frac{d_i}{D}\right) \sim d_i$$

Models with growth more interesting

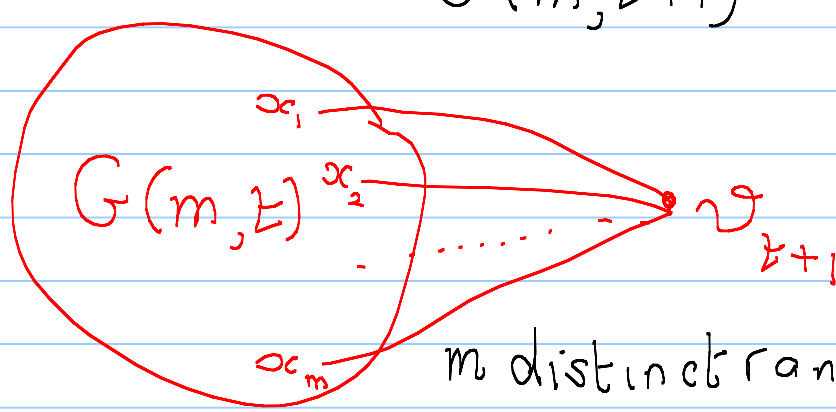
Barabasi, Albert

Bollobás, Riordan, Spencer, Tusnady

$G(m,t)$: vertex set $v_1, v_2, \cdots v_t$.

<u>Basic Model</u>

$G(m, t+1)$



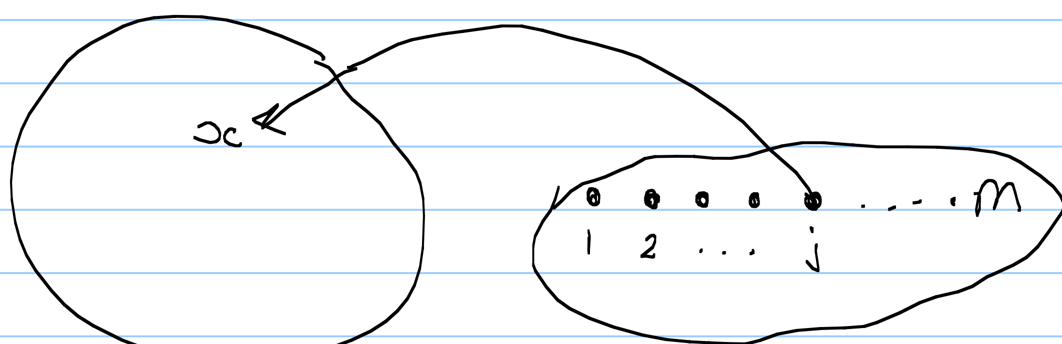
m distinct random edges

$$P_r(x_i = x) = \frac{d_t(x)}{2mt}, \quad 1 \le i \le m$$

$d_t(x)$ = degree of x in $G(m,t)$

Initially $K_{m+1} = G(m, m+1)$

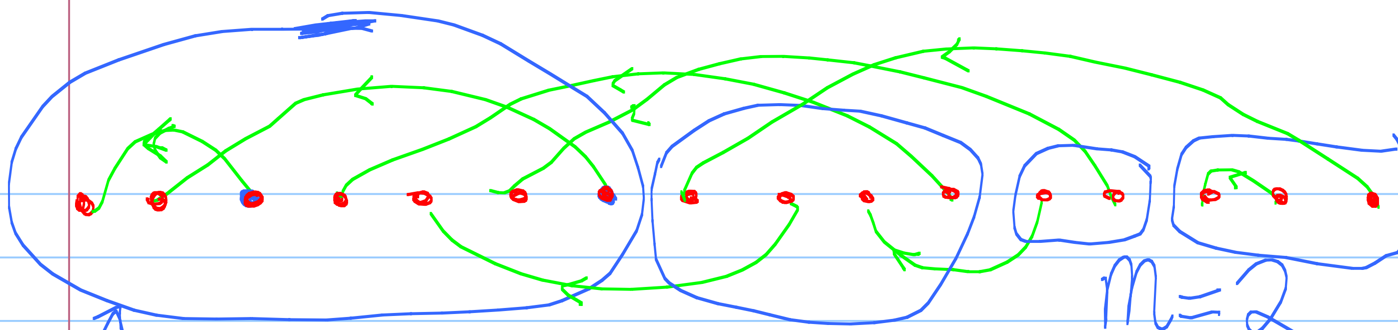
Alternative: B, R, S, T



Becomes one vertex

$Pr(x) = \dfrac{\text{degree}(x)}{2mt + j - 1}$

equivalent to



Vertex 1

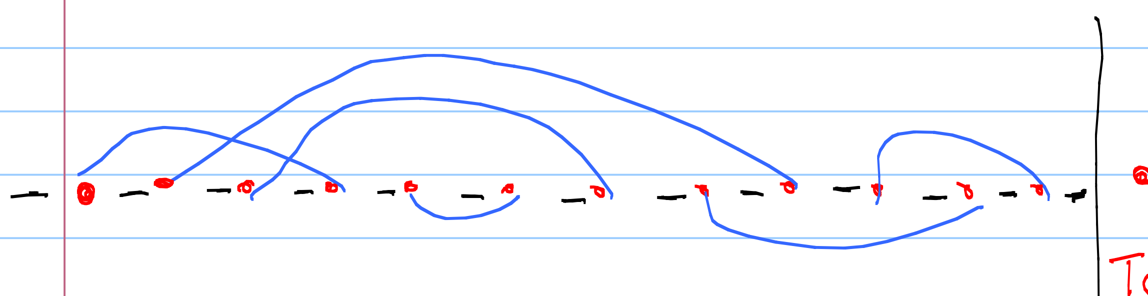
$m=2$

$t=4$

2mt points

pair up randomly

PAIRING MODEL



To add new edge pair • with random –

Degree sequence in $G(m,t)$.

Let $D_k(t)$ be the number of vertices of degree k in $G(m,t)$

$$\overline{D_k}(t) = E(D_k(t))$$

$$\overline{D}_k(t+1) = \overline{D}_k(t) + 1_{k=m} + m\left(\frac{(k-1)\overline{D}_{k-1}(t)}{2mt} - \frac{k\overline{D}_k(t)}{2mt}\right)$$

$\overline{D}_k(t) = 0$ for $k < m$.

Now assume inductively on $k+t$ that

$$\bar{D}_k(t) = d_k t$$

for some $d_m, d_{m+1}, \ldots$

Then d_k should satisfy

$$d_k(t+1) = d_k t + 1_{k=m} + (k-1)d_{k-1}/2 - k d_k/2$$

or $d_k = 1_{k=m} + \frac{k-1}{2} d_{k-1} - \frac{k}{2} d_k$

or

$$\frac{k+2}{2} d_k = 1_{k=m} + \frac{k-1}{2} d_{k-1}$$

So

$$d_k = d_m \prod_{l=m+1}^{k} \left(1 - \frac{3}{l+2}\right)$$

$$\sim C_1 \exp\left\{-\sum_{l=m+1}^{k} \frac{3}{l+2}\right\}$$

$$\sim \frac{C_2}{k^3}.$$

"Power Law".

Now we have to do the induction:

The hypothesis is that

$$\bar{D}_k(\tau) = d_k \tau$$

Given $(k, t+1)$ assume true

for $\kappa < k$ (all t) & $\kappa = k$, $\tau \le t$,

$$\overline{D}_k(t+1) = d_k t + 1_{k=m} + (k-1)\, d_{k-1}/2 - k\, d_k/2$$

$$= d_k(t+1)$$

Concentration

$$\Pr\left(\left|D_k(t) - d_k t\right| \geq u\right) \leq \exp\left\{-\frac{u^2}{8mt}\right\}$$

We will use the Azuma-Hoeffding martingale inequality.

Let $Y_1, Y_2, \ldots, Y_{mt}$ be the sequence of random choices made in the construction of $G(m,t)$.

$$Z_i = Z_i(Y_1, Y_2, \dots, Y_i)$$

$$= E(\overline{D}_k(t) \mid Y_1, Y_2, \dots, Y_i)$$

$$Z_0 = \overline{D}_k(t), \quad Z_{mt} = D_k(t)$$

$Z_0, Z_1, \dots Z_{mt}$ is Doob Martingale

Concentration result follows from

$$|Z_i - Z_{i-1}| \leq 4$$

Fix $X_1, Y_2, \dots, Y_i$ and $\hat{Y}_i \neq Y_i$

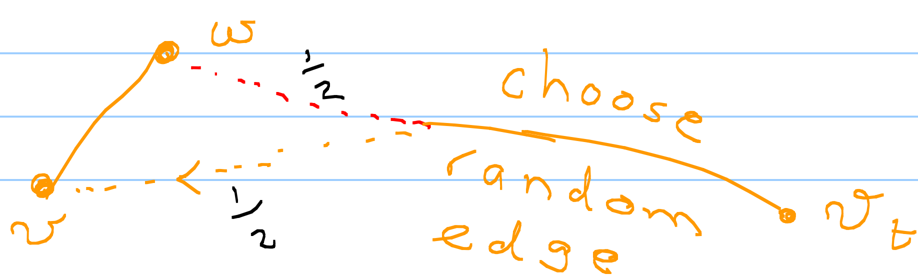
Define map

$$X_1, Y_2, \dots, Y_{i-1}, Y_i, Y_{i+1} \dots Y_{mt}$$

measure preserving $\Big\Updownarrow$ bijection

$$X_1, Y_2, \dots, Y_{i-1}, \hat{Y}_i, \hat{Y}_{i+1} \dots \hat{Y}_{mt}$$

Key feature of preferential attachment is that choice Y_i can be attributed to an edge already in graph i.e.



Let $Y_i = (x, v)$ $\quad x > v$

$\hat{Y}_i = (\hat{x}, \hat{v})$ $\quad \hat{x} > \hat{v}$

[$x = \hat{x}$ if $i \bmod m \neq 1$]

Now suppose $j > i$ and Y_j creates new edge by choosing (ξ, η) and then ξ

Then if $Y_j = (w, v)$ $[(w, x)]$ arises from choice of (x, v) then

$\hat{Y}_j = (w, \hat{v})$ $[(w, \hat{x})]$

Only $x, \hat{x}, v, \hat{v}$ can change degree.