HW9: outline solutions

1. A *tournament* T is an orientation of the complete graph K_n. In a random tournament, edge $\{u,v\}$ is oriented from u to v with probability 1/2 and from v to u with probability 1/2. Show that w.h.p. a random tournament is strongly connected.

   Solution: If T is not strongly connected then there exists a set S of size at most $n/2$ such that all edges in $S:\bar{S}$ are oriented the same way i.e all are S to $\bar{S}$ or vice-versa. The probability of this is at most
$$2\sum_{k=1}^{n/2}\binom{n}{k}\frac{1}{2^{k(n-k)}} \leq 2\sum_{k=1}^{n/2}\left(\frac{ne}{k2^{n-k}}\right)^k = o(1).$$

2. Let T be a random tournament. Show that w.h.p. the size of the largest acyclic sub-tournament is asymptotic to $2\log_2 n$. (A tournament is acyclic if it contains no directed cycles).

   Solution: Let X_k denote the number of sets of size k that induce an acyclic tournament. If S is acyclic then S can be ordered $x_1, x_2, \ldots, x_k$ so that if $i<j$ then the edge is oriented from x_i to x_j. Thus,
$$\mathbb{E}(X_k) \leq \binom{n}{k}k!\frac{1}{2^{k(k-1)/2}} \leq \left(\frac{ne}{2^{(k-1)/2}}\right)^k.$$

   So, $\mathbb{E}(X_k) \to 0$ if $k \geq (2+\varepsilon)\log_2 n$. If $k \leq (2-\varepsilon)\log_2 n$ then the second moment method suffices.

3. Suppose that the edges of $G_{n,p}$ where $0<p\leq 1$ is a constant, are given exponentially distributed weights with rate 1. Show that if X_{ij} is the shortest distance from i to j then

   (a) For any fixed i,j,
$$\mathbb{P}\left(\left|\frac{X_{ij}}{\log n/n}-\frac{1}{p}\right| \geq \epsilon\right) \to 0.$$

   (b)
$$\mathbb{P}\left(\left|\frac{\max_j X_{ij}}{\log n/n}-\frac{2}{p}\right| \geq \epsilon\right) \to 0.$$

Solution: one argues that the number of edges between any set S of size k and its complement $\bar{S}$ is $(1+o(1))k(n-k)p$. This follows from the Chernoff bounds. It follows that the expression for $\mathbb{E}(Y_n)$ in Chapter 19.2 of the book becomes $\mathbb{E}(Y_n) \approx \sum_{k=1}^{n-1}\frac{1}{k(n-k)p}$. The rest of the proof of this section is only changed by the factor $1/p$.