Homework 6: Solutions

6.7.12 Write \( G_{n,p} = G_{n,p_1} \cup G_{n,p_2} \) where \( p_1 = \frac{100}{n} \) and \( (1-p) = (1-p_1)(1-p_2) \) and \( p_2 \sim p \).

Now Theorem 6.8 implies that w.h.p. \( G_{n,p_1} \) contain a path of length greater than \( n/2 \). And then Theorem 6.1 implies that w.h.p. we can complete this path to a caterpillar.

6.7.16 Let \( T' \) be obtained from \( T \) by removing the leaves. Then write \( G_{n,p} = G_{n,p_1} \cup G_{n,p_2} \) where \( p_1 = p_2 ~ \frac{K \log n}{2n} \) and \( 1-p = (1-p_1)(1-p_2) \). Build a copy of \( T' \) in \( G_{n,p_1} \) as follows: fix one vertex \( v \) of \( T' \) as a root and then do a breadth first search to construct \( V_i, i = 1, 2, \ldots \) where \( V_i \) is the set of vertices at distance \( i \) from \( v \). We then embed \( T' \) into \( G_{n,p_1} \) in the order \( V_0, V_1, V_2, \ldots \). Suppose that we have embedded \( V_0, V_1, \ldots, V_i \) as \( W_0, W_1, \ldots, W_i \). Then to create \( V_{i+1} \) we must for each \( v \in V_i \) find up to \( c_i \log n \) neighbors from a set of size at least \( c_2 n \). This will always be possible with probability \( 1 - o(1) \) and so we succeed in embedding \( T' \).

After this, we can use Theorem 6.1 to find a matching that will allow us to add the leaves to create \( T \).

6.7.17 Running DFS on the graph \( G_R \) induced by the red edges, we see that if there is no red path of length \( n/1000 \) then we find sets \( D, U, A \) with \( |D| = |U| \geq \frac{999n}{2000} \) such that there is no red edge between \( D \) and \( U \). Similarly, \( [n] \) can be partitioned into \( D', U', A' \) such that \( |D'| = |U'| \geq \frac{999n}{2000} \) and there is no blue edge between \( D' \) and \( U' \).

Let \( X = U \cap U', Y = U \cap D', X' = D \cap U', Y' = D \cap D' \) and let \( x = |X|, y = |Y|, x' = |X'|, y' = |Y'| \). Then

\[
x + y = |U \cap (U' \cup D')| = |U \setminus A'| \geq \frac{999n}{2000} - \frac{n}{1000} = \frac{997n}{2000}.
\]

Similarly,

\[
x' + y' = x + x', y + y' \geq \frac{997n}{2000}.
\]

It follows that either (i) \( x, y' \geq \frac{997n}{4000} \) or (ii) \( x', y \geq \frac{997n}{4000} \). (Failure of (i) and (ii) implies that (1) or (2) fail.) Suppose then that \( x', y \geq \frac{997n}{4000} \). Now \( X' \subseteq D \) and \( Y \subseteq U \) and so there are no \( X' : Y \) red edges. Furthermore, \( X' \subseteq U' \) and \( Y \subseteq D' \) and so there are no \( X' : Y \) blue edges either. In other words \( X' : Y = \emptyset \). But,

\[
P \left( \exists \text{ disjoint } S, T : |S|, |T| \geq \frac{997n}{4000} \text{ and } S : T = 0 \right)
\leq 2^{2n} \left( 1 - \frac{1000}{n} \right)^{\left(\frac{997n}{4000}\right)^2} = o(1).
\]