Homework 5: Solutions

6.7.4 Applying Hall’s theorem and defining $L = \lceil 2 \log n \rceil$,

$$P(\text{no p.m.}) \leq \sum_{k=L}^{n} \binom{n}{k} \binom{n}{k-1} \left( \frac{(k-1)}{L} \right)^k$$

$$\leq \sum_{k=L}^{n} \left( \frac{ne}{k} \right)^k \left( \frac{ne}{k-1} \right)^{k-1} \left( \frac{k-1}{n} \right)^{Lk}$$

$$\leq \sum_{k=L}^{n} \left( \frac{k}{n} \right)^{L-2} \cdot e^2 \cdot \left( \frac{k-1}{k} \right)^L$$

$$= \sum_{k=L}^{n} u_k.$$

Now if $k = (1 - \epsilon)n$ then we have $(k/n)^{L-2} \leq e^{-\epsilon(L-2)} \leq e^{-3+o(1)}$ if $\epsilon \geq 3/L$. We deduce from this that

$$\sum_{k=L}^{n_0} u_k = o(1) \text{ where } n_0 = n - \frac{4n}{L}.$$

So we need to be more careful for $k > n_0$. If there is a set of size $k$ on one side of the partition with at most $k - 1$ neighbors, then there is a set $X$ of size $\ell = n - k + 1$ on the other side of the partition with at most $\ell - 1$ neighbors. We estimate this by

$$\left( \binom{n}{\ell-1} \right) P \left( \text{Bin} \left( n, 1 - \left( 1 - \frac{\ell}{n} \right)^L \right) \leq \ell - 1 \right).$$

(1)

We choose $X$ in $(\binom{n}{\ell-1})$ ways and then $1 - \left( 1 - \frac{\ell}{n} \right)^L$ lower bounds the probability that a vertex chooses a neighbor in $X$.

Summing over $\ell \leq \ell_0 = n - n_0 + 1$, this is at most

$$2 \sum_{\ell=1}^{\ell_0} \binom{n-1}{\ell-1} 2^{\ell-2} \left( \frac{\ell}{n} \right)^{\ell-1} \left( 1 - \frac{\ell}{n} \right)^{L(n-\ell+1)}$$

(2)

$$\leq 2 \sum_{\ell=1}^{\ell_0} \left( \frac{ne}{\ell-1} \right)^{2\ell-2} \left( \frac{\ell}{n} \right)^{\ell-1} e^{-L(1+o(1))}$$

$$\leq 2 \sum_{\ell=1}^{\ell_0} \left( \frac{ne^2 \ell L}{\ell-1} \right)^{\ell-1} n^{-(2-o(1))\ell}$$

$$= o(1).$$

Explanation for (2): the binomial probability in (1) is dominated by the $\ell - 1$ term, leading to the factor 2.
6.7.9 Following the hint we partition \([n]\) into 3 sets \(A, B, C\) of size \(n/3\). The bipartite graph \(H\) induced by \(A, B\) is distributed as \(G_{n/3,n/3,p}\) and since \(\frac{n}{3}p \gg \log \frac{n}{3}\) this graph has a perfect matching w.h.p. Fix a perfect matching \(M\) of \(H\) and define another random bipartite graph \(K\) with vertices \(M, C\) and an edge \((e, x)\) for each \(e = \{u, v\} \in M, x \in C\) such that the edges \(\{x, u\}, \{x, v\}\) both exist. The random graph \(K\) is distributed as \(G_{n/3,n/3,p^2}\) and since \(\frac{n}{3}p^2 \gg \log \frac{n}{3}\) this graph has a perfect matching w.h.p. This perfect matching corresponds to \(n/3\) vertex disjoint triangles.

6.7.10 Arguing as in the proof of Theorem 6.1 we see that

\[
EY \leq 2 \sum_{k=2}^{n/2} \binom{n}{k} \binom{n}{k} \frac{k(k-1)}{2k-2} p^{2k-2} (1-p)^{k(3n/4-k)}
\]

The only change here is that we can only guarantee that \(S\) has at least \(k(3n/4-k)\) neighbors not in \(T\). Continuing,

\[
EY \leq 2 \sum_{k=2}^{n/2} \left( \frac{ne}{k} \right)^k \left( \frac{ne}{k-1} \right)^{k-1} \left( \frac{Kk\log n}{2n} \right)^{2k-2} n^{-Kk(3/4-k/n)}
\]

\[
\leq \frac{n^2}{\log^2 n} \sum_{k=2}^{n/2} \left( \frac{ne}{k} \cdot \frac{ne}{k-1} \cdot \left( \frac{Kk\log n}{2n} \right)^2 \cdot n^{-K/2} \right)^k
\]

\[= o(1),\]

if \(K \geq 4\).