Homework 8: Solutions

12.3.10 Suppose that $r \geq (1+\epsilon)r_0$, as in Theorem 11.8. Show that if $1 \leq k = O(1)$ then $G_{\mathcal{X},r}$ is k-connected w.h.p.

Solution: Let T be a spanning tree of Γ as promised by Lemma 11.11 and let T^* denote the vertices of $\mathcal{X}$ in the cells of T. If we remove $O(1)$ vertices from G then what remains of T^* will be connected. This is because each vertex of T corresponds to a clique of size $\Omega(\log n)$. Removing vertices from bad cells cannot disconnect the graph, as (11.16) tells us that we connect the vertices of bad cells, directly to T^*.

12.3.11 Show that if $2 \leq k = O(1)$ and $r \gg n^{-\frac{k}{2(k-1)}}$ then w.h.p. $G_{\mathcal{X},r}$ contains a k-clique. On the other hand, show that if $r = o(n^{-\frac{k}{2(k-1)}})$ then $G_{\mathcal{X},r}$ contains no k-clique.

Solution: Let Z denote the number of k-cliques. Suppose first that $r = \frac{1}{\omega n^{\frac{k}{2(k-1)}}}$ where $\omega \to \infty$. Then

$$\mathbf{E}(Z) \leq n\binom{n}{k-1}(\pi r^2)^{k-1} \leq n^k\left(\frac{\pi}{\omega^2 n^{k/(k-1)}}\right)^{k-1} = \frac{\pi^{k-1}}{\omega^{2(k-1)}} = o(1).$$

Suppose now that $r = \frac{\omega}{n^{\frac{k}{2(k-1)}}}$ where $\omega \to \infty, w = o(\log n)$. We there are exactly $k-1$ points within distance $r/2$ of x and the remaining $n-k-o(n)$ points are at distance $2r$ or more from x. Let $Z = |\{x : \mathcal{E}_x \text{ occurs}\}|$.

$$\begin{aligned}\mathbf{E}(Z) \geq n\binom{n-o(n)}{k-1}\left(\frac{\pi r^2}{4}\right)^{k-1} & \left(1-4\pi r^2\right)^{n-k-o(n)} \\ &\geq \frac{n^k}{2k!}\left(\frac{\pi\omega^2}{4n^{k/(k-1)}}\right)^{k-1} = \frac{\pi^{k-1}\omega^{2(k-1)}}{2k!4^{k-1}} \to \infty.\end{aligned}$$

We use the Chebyshev inequality. We have $\mathbf{E}(Z^2) \leq \mathbf{E}(Z)^2$. This is because if $\mathcal{E}_x, \mathcal{E}_y$ both occur then $|x-y| \geq 2r$ and the expectation of Z^2 simplifies.

12.3.12 Suppose that $r \gg \sqrt{\frac{\log n}{n}} = o(1)$. Show that w.h.p. the diameter of $G_{\mathcal{X},r} = \Theta\left(\frac{1}{r}\right)$.

Solution: We partition $D = [0,1]^2$ into m^2 cells of size $1/m \times 1/m$ where $m = \lceil 10/r \rceil$. Because $r \gg \sqrt{\frac{\log n}{n}}$ we know that w.h.p. each cell is non-empty. Furthermore, if $X, Y \in \mathcal{X}$ lie in adjacent cells then they are connected by an edge in $G = G_{\mathcal{X},r}$. It follows that we can connect pairs of vertices in G by a path of length at most $2m$. We simply follow a sequence of adjacent cells. Thus the diameter of G is $O(1/r)$. The diameter is clearly $\Omega(1/r)$ because any path from a vertex in the top left-most cell to a vertex in the top right-most cell must use at least $\frac{1-2/m}{r}$ edges.