Homework 7: Solutions

7.6.10 Show that if $d > 2k\log k$ for a positive integer $k \geq 2$ then w.h.p. $G(n, d/n)$ is not k-colorable.

Hint:Consider the expected number of proper k-coloring's:

$$\sum_{n_1+\cdots+n_k=n} \binom{n!}{n_1!\cdots n_k!}(1-p)^{\sum_i n_i(n_i-1)/2}$$

Solution: Let $d = 2k\log k + 2\epsilon$. The expected number of k-colorings can be bounded by

$$\begin{aligned}
&\sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k}(1-p)^{\sum_{i=1}^k \binom{n_i}{2}} \\
&\leq \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} \exp\left\{-\frac{d}{n}\sum_{i=1}^k \binom{n_i}{2}\right\} \\
&\leq \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} \exp\left\{-\frac{d}{n}\cdot k\binom{n/k}{2}\right\} \\
&\leq \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} \exp\left\{-(\log k+\epsilon)n\right\} \\
&\leq k^n \exp\left\{-(\log k+\epsilon)n\right\} \\
&= o(1).
\end{aligned}$$

7.6.11 Let $p = K\log n/n$ for some large constant $K > 0$. Show that w.h.p. the diameter of $\mathbb{G}_{n,p}$ is $\Theta(\log n/\log\log n)$.
(Breadth First Search.)

Solution: Fix $v \in [n]$ and let $N_i = N_i(v)$ be the set of vertices at distance i from v. Let $N_{\leq k} = \bigcup_{i\leq k} N_i$. Using the Chenoff bounds we see that for large K, we have that w,h,p,

$$\frac{1}{2}K\log n \leq \deg(x) \leq 2K\log n \qquad \text{for all } x \in [n]. \tag{1}$$

It follows that w.h.p. if $k_0 = \frac{\log n}{2\log\log n}$ then

$$|N_{\leq k_0}| \leq \sum_{k\leq k_0} (2K\log n)^k = n^{1/2+o(1)}$$

and so the diameter of $\mathbb{G}_{n,p}$ is at least $\frac{\log n}{2\log\log n}$.

Now consider a Breadth First Search (BFS) that constructs $N_1, N_2, \ldots, N_{k_1}$ where

$$k_1 = \frac{3\log n}{5\log\log n}.$$

It follows that for $k \leq k_1$ we have

$$|N_{i\leq k}| \leq n^{3/4} \text{ and } |N_k|p \leq n^{-1/5}. \qquad (2)$$

Observe now that the edges from N_i to $[n] \setminus N_{\leq i}$ are unconditioned by the BFS up to layer k and so for $x \in [n] \setminus N_{\leq k}$,

$$\mathbf{P}(x \in N_{k+1} \mid N_{\leq k}) = 1-(1-p)^{|N_k|} \geq |N_k|p(1-|N_k|p) \geq \rho_k = |N_k|p(1-n^{-1/5}).$$

The events $x \in N_{k+1}$ are independent and so $|N_{k+1}|$ stochastically dominates the binomial $B(n - n^{3/4}, \rho_k)$. Assume inductively that $|N_k| \geq (2\log n)^k$ for some $k \geq 1$. This is true w.h.p. for $k = 1$ by (1). Let $\mathcal{A}_k$ be the event that (2) holds. It follows that

$$\mathbf{E}(|N_{k+1}| \mid \mathcal{A}_k) \geq K|N_k| \log n(1 - O(n^{-1/5})).$$

It then follows from the Chernoff bounds that

$$\mathbf{P}(|N_{k+1}| \leq (2\log n)^{k+1}) \leq \exp\left\{-\frac{1}{3}K|N_k|\log n\right\} = o(n^{-anyconstant}).$$

There is a small point to be made about conditioning here. We can condition on (1) holding and then argue that this only multiplies small probabilities by $1 + o(1)$ if we use $\mathbf{P}(A \mid B) \leq \mathbf{P}(A)/\mathbf{P}(B)$.

It follows next that if

$$k_2 = \frac{2\log n}{3\log\log n}$$

then w.h.p. we have

$$|N_{k_2}| \geq n^{2/3}.$$

Analogously, if we do BFS from w to create $N'_k, i = 1, 2, \ldots, k_2$ then $|N'_{k_2}| \geq n^{2/3}$. If $N_{\leq k_2} \cap N'_{\leq k_2} \neq \emptyset$ then $dist(v, w) \leq 2k_2$ and we are done. Otherwise, we observe that the edges $N_{k_2} : N'_{k_2}$ between N_{k_2} and N'_{k_2} are unconditioned (except for (1)) and so

$$\mathbf{P}(N_{k_2} : N'_{k_2} = \emptyset) \leq (1-p)^{n^{2/3}\times n^{2/3}} = o(n^{-2}).$$

If $N_{k_2} : N'_{k_2} \neq \emptyset$ then $dist(v, w) \leq 2k_2 + 1$ and we are done. Note that given (1), all other unlikely events have probability $O(n^{-anyconstant})$ of occurring and so we can inflate these latter probabilities by n^2 to account for all choices of v, w.

11.6.10 Show that w.h.p. $\mathbb{G}_{n,3}$ is not planar.
(Remove short cycles and consider Euler's formula when there are no small faces.)

Solution: Let L be a large constant and let X denote the number of cycles of $G = \mathbb{G}_{n,3}$ of length less than L. Then

$$\mathbf{E}(X) \leq \sum_{k=3}^{L-1} \binom{n}{i} i! \left(\frac{3}{n}\right)^i \leq 3^L.$$

The Markov inequality then implies that $X \le \log n$, w.h.p. Let H be a subgraph of G obtained by removing one edge from each cycle of length at most L. Then w.h.p. $|E(H)| \ge 3n/2 - \log n$. Because H has girth at least L, if it were planar and if $\phi(H)$ is the number of faces of some plane embedding of H,

$$2 = n - |E(H)| + \phi(H) = n - |E(H)| + \frac{2|E(H)|}{L} = \\ n\left(1 - \left(\frac{3}{2} - o(1)\right)\left(1 - \frac{2}{L}\right)\right) < 0,$$

contradiction. Here we have used $2|E(H)| \ge L\phi(H)$.