Homework 6

6.7.4 Consider the random bipartite graph G with bi-partition A, B where $|A| = |B| = n$. Each vertex $a \in A$ independently chooses $\lceil 2\log n \rceil$ random neighbors in B. Show that w.h.p. G contains a perfect matching.

Solution: Arguing as for Theorem 6.1 of the book, with $\ell = \lceil 2\log n \rceil$,

$$\begin{aligned}
&\mathbf{P}(\exists S \subseteq A, T \subseteq B, N(S) \subseteq T, |T| = |S|-1) \\
&\leq \sum_{k=\ell+1}^{n} \binom{n}{k}\binom{n}{k-1}\left(\frac{\binom{k-1}{\ell}}{\binom{n}{\ell}}\right)^k \qquad (1)\\
&\leq \sum_{k=\ell+1}^{n} \binom{n}{k}\binom{n}{k-1}\left(\frac{k-1}{n}\right)^{k\ell} \\
&\leq \sum_{k=\ell+1}^{n} \left(\frac{n^2e^2k^\ell}{k^2n^\ell}\right)^k \\
&= \sum_{k=\ell+1}^{n} \left(\left(\frac{k}{n}\right)^{(\ell-2)} e^2\right)^k.
\end{aligned}$$

Putting $u_k = \left(\left(\frac{k}{n}\right)^{(\ell-2)} e^2\right)^k$ we see that if $k \leq n_0 = n - \frac{3n}{\ell}$ then $u_k \leq e^{-3+o(1)}$ and so

$$\sum_{k=\ell+1}^{n_0} u_k \leq \sum_{k=\ell+1}^{n_0} e^{-k+o(k)} = o(1).$$

For $n > n_0$ we write $k = n - m$ and replace the RHS of (2) by

$$\begin{aligned}
&\sum_{m=0}^{n-n_0} \binom{n}{m}\binom{n}{m+1}\left(\frac{\binom{n-m-1}{\ell}}{\binom{n}{\ell}}\right)^{n-m} \\
&\leq n\left(1-\frac{1}{n}\right)^{\ell n} + n\sum_{m=1}^{n-n_0}\left(\frac{ne}{m}\right)^{2m}\left(1-\frac{m+1}{n}\right)^{(n-m)\ell} \\
&\leq o(1) + n\sum_{m=1}^{n-n_0}\left(\frac{n^2e^2}{m^2}e^{-(m+1)\ell(1-m/n)}\right)^m \\
&\leq o(1) + n\sum_{m=1}^{n-n_0}\left(\frac{n^2e^2}{m^2}\cdot n^{-4+o(1)}\right)^m \\
&= o(1).
\end{aligned}$$

7.6.1 Let $p = d/n$ where d is a positive constant. Let S be the set of vertices of degree at least $\frac{2\log n}{3\log\log n}$. Show that S is an independent set w.h.p.

Solution: Let X denote the nuber of edges v, w with both endpoints having large degree and let $L = \frac{2\log n}{3\log\log n}$. Then,

$$\mathbf{E}(X) \leq n^2 p \left(\sum_{k=L-1}^{n-2} \binom{n-2}{k} p^k \right)^2 \tag{2}$$

$$\leq 2nd \left(\frac{nep}{L-1} \right)^{2(L-1)} \tag{3}$$

$$= 2nd \times n^{-4/3+o(1)} \tag{4}$$

$$= o(1).$$

We get (3) from (2) as follows: let u_k be the summand in (2). Then $u_{k+1}/u_k \leq np/k = o(1)$.

We get (4) because

$$\log \left(\frac{nep}{L-1} \right)^{2(L-1)} \approx \frac{4\log n}{3\log\log n} \times \log\log n.$$

7.6.9 Suppose that H is obtained from $G_{n,1/2}$ by planting a clique C of size $m = n^{1/2}\log n$ inside it. Describe a polynomial time algorithm that w.h.p. finds C. (Think that an adversary adds the clique without telling you where it is).

(How does adding the clique change the degree sequence?)

Solution: Theorem 3.5 implies that the minimum and maximum degrees $\delta(G_{n,1/2}), \Delta(G_{n,1/2})$ satisfy the following w.h.p.:

$$\Delta - \delta \leq O(n^{1/2}\log^{1/2} n).$$

It follows that the vertices of the planted clique will w.h.p. be the m vertices of largest degree.