Homework 5: Solutions

6.7.9 Following the hint we partition $[n]$ into 3 sets A, B, C of size $n/3$. The bipiartite graph H induced by A, B is distributed as $G_{n/3,n/3,p}$ and since $\frac{n}{3}p \gg \log \frac{n}{3}$ this graph has a perfect matching w.h.p. Fix a perfect matching M of H and define another random bipartite graph K with **vertices** M, C and an edge (e, x) for each $e = \{u, v\} \in M, x \in C$ such that the edges $\{x, u\}, \{x, v\}$ both exist. The random graph K is distributed as $G_{n/3,n/3,p^2}$ and since $\frac{n}{3}p^2 \gg \log \frac{n}{3}$ this graph has a perfect matching w.h.p. This perfect matching corresponds to $n/3$ vertex disjoint triangles.

6.7.10 Arguing as in the proof of Theorem 6.1 we see that

$$\mathbf{E}Y \leq 2\sum_{k=2}^{n/2} \binom{n}{k}\binom{n}{k-1}\binom{k(k-1)}{2k-2} p^{2k-2}(1-p)^{k(n/2+\epsilon-k)}$$

The only change here is that we can only guarantee that S has at least $k(n/2+e-k)$ neighbors not in T. Continuing,

$$\begin{aligned}\mathbf{E}Y &\leq 2\sum_{k=2}^{n/2} \left(\frac{ne}{k}\right)^k \left(\frac{ne}{k-1}\right)^{k-1} \left(\frac{Kke\log n}{2n}\right)^{2k-2} n^{-Kk(1/2+\epsilon-k/n)} \\ &\leq \frac{n^2}{\log^2 n}\sum_{k=2}^{n/2}\left(\frac{ne}{k}\cdot\frac{ne}{k-1}\cdot\left(\frac{Kke\log n}{2n}\right)^2\cdot n^{-K\epsilon}\right)^k \\ &= o(1),\end{aligned}$$

if $K \geq 2/\epsilon$.

6.7.17 Running DFS on the graph G_R induced by the red edges, we see that if there is no red path of length $n/1000$ then we find sets D, U, A with $|D| = |U| \geq \frac{999n}{2000}$ such that there is no red edge between D and U. Similarly, $[n]$ can be partitioned into D', U', A' such that $|D'| = |U'| \geq \frac{999n}{2000}$ and there is no blue edge between D' and U'.

Let $X = U \cap U', Y = U \cap D', X' = D \cap U', Y' = D \cap D'$ and let $x = |X|, y = |Y|, x' = |X'|, y' = |Y'|$. Then

$$x + y = |U \cap (U' \cup D')| = |U \setminus A'| \geq \frac{999n}{2000} - \frac{n}{1000} = \frac{997n}{2000}. \qquad (1)$$

Similarly,

$$x' + y', x + x', y + y' \geq \frac{997n}{2000}. \qquad (2)$$

It follows that either (i) $x, y' \geq \frac{997n}{4000}$ or (ii) $x', y \geq \frac{997n}{4000}$. (Failure of (i) and (ii) implies that (1) or (2) fail.) Suppose then that $x', y \geq \frac{997n}{2000}$. Now $X' \subseteq D$ and $Y \subseteq U$ and so there are no $X' : Y$ red edges. Furthermore,

$X' \subseteq U'$ and $Y \subseteq D'$ and so there are no $X' : Y$ blue edges either. In other words $X' : Y = \emptyset$. But,

$$\begin{aligned}\mathbf{P}\left(\exists \text{ disjoint } S, T : |S|, |T| \geq \frac{997n}{4000} \text{ and } S : T = \emptyset\right) \\ \leq 2^{2n}\left(1 - \frac{1000}{n}\right)^{(997n/4000)^2} = o(1).\end{aligned}$$