Homework 4: Solutions

3.3.1 Let X denote the number of vertices of degree k in $G_{n,m}$. Then,

$$\mathbf{E}(X) = n\frac{\binom{n-1}{k}\binom{N-n+1}{m-k}}{\binom{N}{m}} =$$

$$n\binom{n-1}{k}\cdot\frac{(N-n+1)(N-n)\cdots(N-n-m+k+2)\,m(m-1)\cdots(m-k+1)}{N(N-1)\cdots(N-m+1)} \approx$$

$$\frac{n^k}{k!}\frac{m^k}{N^k}\prod_{i=0}^{m-k+1}\frac{N-n+1-i}{N-i} = \frac{n^k}{k!}\frac{m^k}{N^k}\prod_{i=0}^{m-k+1}\left(1-\frac{n-1}{N-i}\right) \approx$$

$$\frac{n^k}{k!}\frac{m^k}{N^k}\exp\left\{-\sum_{i=0}^{m-k+1}\frac{n}{N-i}\right\} = \frac{n^k}{k!}\frac{m^k}{N^k}\exp\left\{-\frac{n}{N}\sum_{i=0}^{m-k+1}\left(1+O\left(\frac{i}{N}\right)\right)\right\} \approx$$

$$\frac{n^k}{k!}\frac{m^k}{N^k}\exp\left\{-\frac{nm}{N}\right\} \approx \frac{d^k e^{-d}}{k!}.$$

After this (a) we use the Chebyshev inequality and (b) see why we prefer $G_{n,p}$.

4.3.3 Fix $v \in [n]$. Let $\mathcal{A}_i, i = 0, \ldots, i_0 = \left\lfloor \frac{2\log n}{3\log d} \right\rfloor$ be the event that $|S_i(v)| \in [(d/2)^i, (2d)^i]$ for all $v \in [n]$. Clearly $\mathcal{A}_0$ is true. Now if $p = m/N$ so that $d \approx np$ then in $G_{n,p}$,

$$\mathbf{P}(\neg\mathcal{A}_{i+1} \mid S_i(v), \mathcal{A}_j, j \le i) = \mathbf{P}\left(Bin\left((n - \sum_{j=0} i|S_j(v)|, 1-(1-p)^{|S_i|}\right)\right).$$

Observe that $d^{i_0} \approx n^{2/3}$. Given the conditioning, $\left|\sum_{j=0} i|S_j(v)|\right| = o(n)$ and $1-(1-p)^{|S_i|} \approx |S_i|p$. Thus,

$$\mathbf{P}(\neg\mathcal{A}_{i+1} \mid S_i(v), \mathcal{A}_j, j \le i) = \mathbf{P}\left(Bin\left((n-o(n),(1-o(1))|S_i|p\right)\right).$$

The expected value of the binomial in the above is $\approx d|S_i|$ and applying the Chernoff bounds we get that

$$\mathbf{P}(\neg\mathcal{A}_{i+1} \mid S_i(v), \mathcal{A}_j, j \le i) \le e^{-\Omega(d|S_i|)} \text{ and } d|S_i|) \gg \log n.$$

This implies that

$$\mathbf{P}(\neg\mathcal{A}_{i+1} \mid \mathcal{A}_j, j \le i) \le n^{-\omega} \text{ where } \omega \to \infty.$$

But, then

$$\mathbf{P}(\neg\mathcal{A}_i) \le \mathbf{P}(\neg\mathcal{A}_1) + \mathbf{P}(\neg\mathcal{A}_2 \mid \neg\mathcal{A}_1) + \mathbf{P}(\neg\mathcal{A}_3 \mid \mathcal{A}_1, \mathcal{A}_2)\cdots \le in^{-\omega}.$$

and we can use the union bound to deal with all choices of v, i.

4.3.7 If $G_{n,n,p}$ is not connected then there is a component of size at most n. So, assuming that $\omega = o(\log n)$,

$$
\begin{aligned}
&\mathbf{P}(G_{n,n,p} \text{ is not connected}) \\
&\leq \sum_{k=1}^{n-1} \sum_{\ell=1}^{n-k} \binom{n}{k}\binom{n}{\ell} k^{\ell-1}\ell^{k-1}p^{k+\ell-1}(1-p)^{k(n-\ell)+\ell(n-k)} \\
&\leq \sum_{k=1}^{n-1} \sum_{\ell=1}^{n-k} \left(\frac{ne}{k}\right)^k \left(\frac{ne}{\ell}\right)^\ell k^{\ell-1}\ell^{k-1}p^{k+\ell-1}e^{-(k(n-\ell)+\ell(n-k))p} \\
&= \frac{1}{p}\sum_{k=1}^{n-1} \sum_{\ell=1}^{n-k} \left(\frac{k}{\ell}\right)^{\ell-k} \left(p\exp\left\{1-\omega+\frac{2k\ell p}{k+\ell}\right\}\right)^{k+\ell} \\
&\leq \frac{1}{p}\sum_{k=1}^{n-1} \sum_{\ell=1}^{n-k} \left(\frac{k}{\ell}\right)^{\ell-k} \left(p\exp\left\{1-\omega+\frac{(k+\ell)p}{2}\right\}\right)^{k+\ell} \\
&\leq \frac{1}{p}\sum_{s=2}^{n} s\left(p\exp\left\{1-\omega+\frac{sp}{2}\right\}\right)^{s} \\
&= \frac{1}{p}\sum_{s=2}^{n} u_s.
\end{aligned}
$$

Now

$$\frac{1}{p}\sum_{s=2}^{n/2} u_s \leq \frac{1}{p}\sum_{s=2}^{n/2}\left(pe^{1-\omega}n^{1/4+o(1)}\right)^s = O(n^{-1/2+o(1)}) = o(1).$$

$$\frac{1}{p}\sum_{s=n/2}^{n} u_s \leq \frac{1}{p}\sum_{s=n/2}^{n}\left(e^{1-\omega}n^{-1/2+o(1)}\right)^s = o(1).$$