Homework 3: Solutions

2.4.7 Let $L_\pm = (1 \pm \epsilon_n)\frac{\log n}{c - \log c}$ and let X_- denote the number of components that are paths of length L_- and let X_+ denote the number of components that are paths of length at least L_+.

Now we know that w.h.p. there are no path components of length more than $A \log n$ for some constant $A > 0$. This is because (a) there are no components of size greater than $A \log n$ and (b) the giant component has too many edges to be a path. Thus,

$$\begin{aligned}
\mathbf{P}(X_+ > 0) &\leq o(1) + \sum_{k=L_+ +1}^{A\log n} \binom{n}{k} k! \left(\frac{c}{n}\right)^{k-1} \left(1 - \frac{c}{n}\right)^{k(n-k)} \\
&\leq o(1) + \frac{(1+o(1))n}{c} \sum_{k=L_+ +1}^{A\log n} (ce^{-c})^k \\
&\leq o(1) + \frac{2An\log n}{c} \cdot (ce^{-c})^{L_+} \\
&\leq o(1) + \frac{2An\log n}{c} \cdot \frac{(ce^{-c})^{\epsilon_n L_+}}{n} \\
&= o(1).
\end{aligned}$$

Now, with $X = X_-$ and $L = L_-$,

$$\begin{aligned}
\mathbf{E}(X) &= \binom{n}{L+1}(L+1)! \left(\frac{c}{n}\right)^L \left(1 - \frac{c}{n}\right)^{L(n-L) - \binom{L+1}{2} + L} \\
&\geq (1 - o(1))n(ce^{-c})^L \\
&= (1 - o(1))(ce^{-c})^{-\epsilon_n L} \\
&\to \infty.
\end{aligned}$$

Furthermore, if $P_1, P_2, \ldots, P_M$ is an enumeration of the paths of length L in K_n and X_i is thge indicator for P_i being a path component then

$$\begin{aligned}
\mathbf{E}(X^2) &= \mathbf{E}(X) + \sum_{i \neq j} \mathbf{P}(X_i = X_j = 1) \\
&\leq \mathbf{E}(X) + \mathbf{E}(X)^2.
\end{aligned}$$

This is because if $i \neq j$ then

$$\mathbf{P}(X_i = X_j = 1) = \begin{cases} \mathbf{P}(X_i = 1)\mathbf{P}(X_j = 1) & P_i \cap P_j = \emptyset. \\ 0 & P_i \cap P_j \neq \emptyset. \end{cases}$$

Thus

$$\mathbf{P} X > 0 \geq \frac{\mathbf{E}(X)^2}{\mathbf{E}(X^2)} \geq \frac{1}{\frac{1}{\mathbf{E}(X)} + 1} \to 1.$$

2.4.14 The expected number of sets of size at most s that contain at least $ks/2$ edges is at most

$$\sum_{t=2k+1}^{s} \binom{n}{t}\binom{\binom{t}{2}}{kt/2} p^{kt/2} \leq \sum_{t=2k+1}^{s} \left(\frac{ne}{t}\right)^t \left(\frac{t^2 e}{kt}\right)^{kt/2} p^{kt/2}$$
$$= \sum_{t=2k+1}^{s} \left(\left(\frac{t}{n}\right)^{k/2-1} \left(\frac{e^{1+2/k}c}{k}\right)^{k/2}\right)^t = o(1)$$

if say, $s \leq s_0 = \theta n$ where $\theta = \frac{1}{2}(e^{1+2/k}c/k)^{-k/(k-2)}n$.

This means that w.h.p. every set of size at most s_0 contains a vertex with fewer than than k neighbors in the set. Thus w.h.p. either the k-core is empty or it has size greater than s_0.

2.4.19 Suppose first that $c < 1$. Let X_k denote the number of tree components of size k and let X denote the number of edges on non-tree components. Thwn

$$\mathbf{E}|E_{n,p}| = \binom{n}{2}\frac{c}{n} = \sum_{k=1}^{n}(k-1)\mathbf{E}X_k + \mathbf{E}X$$
$$= \sum_{k=1}^{n}\mathbf{E}kX_k + \mathbf{E}X - \sum_{k=1}^{n}\mathbf{E}X_k = n + o(n) - \sum_{k=1}^{n}\mathbf{E}X_k.$$

Thus, from calculations in the chapter,

$$\frac{cn}{2} = n + o(n) - \frac{n}{c}\sum_{k=1}^{n}\frac{k^{k-2}}{k!}(ce^{-c})^k.$$

Dividing through by n and letting $n \to \infty$ gives the answer for $c < 1$.

For $c > 1$ we have

$$\frac{1}{c}\sum_{k=1}^{n}\frac{k^{k-2}}{k!}(ce^{-c})^k = \frac{1}{c}\sum_{k=1}^{n}\frac{k^{k-2}}{k!}(xe^{-x})^k = \frac{1}{c}\left(x - \frac{x^2}{2}\right).$$