Homework 2: Solutions

1.4.8 The probability that $G_{n,p} = K_n$ is $p^N \to 0$ and so its diameter will be at least two w.h.p. On the other hand, let $\mathcal{A}_{x,y}$ be the event that there does not exist $z \neq x, y$ such that $\{x,z\},\{y,z\} \in E(G_{n,p})$. Then,

$$\mathbf{P}(\exists x, y : \mathcal{A}_{x,y}) \leq n^2(1-p)^{n-2} \leq n^2 e^{-(n-2)p} \to 0.$$

2.4.2 Using the first moment method, we see that

$$\begin{aligned}\mathbf{P}(\exists\text{unicyclic component}) &\leq \sum_{k=3}^{n} \binom{n}{k} C(k,k) p^k (1-p)^{k(n-k)+\binom{k}{2}-k} \\ &\leq \sum_{k=3}^{n} \frac{n^k}{k!} k^{k-2} \binom{k}{2} p^k \exp\left\{-p\left(k(n-k)+\binom{k}{2}-k\right)\right\} \\ &\leq \sum_{k=3}^{n} (nep)^k \exp\left\{-p\left(k(n-k)+\binom{k}{2}-k\right)\right\}. \end{aligned} \tag{1}$$

Let u_k be the summand in (1). Then we have

$$u_k \leq \begin{cases} (nep)^k e^{-knp/2} & k \leq n/2. \\ (nep)^k e^{-k^2 p/3} & k > n/2. \end{cases}$$

So,

$$\begin{aligned}\mathbf{P}(\exists\text{unicyclic component}) &\leq \sum_{k=3}^{n/2} (e\omega e^{-\omega/2})^k + \sum_{k=n/2}^{n} (e\omega e^{-\omega/12})^k \\ &= o(1) + o(1).\end{aligned}$$

2.4.5 Let X denote the number of pairs (e, H) where H is a unicyclic graph with n vertices $[n]$ and n edges and e is an edge of $C(H)$, where $C(H)$ is the unique cycle of H. Then

- $X = n^{n-2}(N-n+1)$ where $N = \binom{n}{2}$, counting $(e, e+T)$ where T is a spanning tree.
- $X = \sum_{k=1}^{n} kC_k$ where C_k is the number of unicyclic H whose cycle has k edges.

So,

$$\frac{n^{n-2}(N-n+1)}{C(n,n)} = \frac{\sum_{k=1}^{n} kC_k}{C(n,n)} = \mathbf{E}(|C(H)|).$$