Hamilton Cycles in Random Graphs

Theorem

Let $m = \frac{1}{2} n (\log n + \log\log n + c_n)$. Then

$$\lim_{n \to \infty} \Pr(G_{n,m} \text{ is Hamiltonian}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to \infty \end{cases}$$

$$= \lim_{n \to \infty} \Pr(\delta(G_{n,m}) \geq 2).$$

The proof of this is complicated and so we start by proving a weaker theorem.

Let $p = \frac{25\log n}{n}$. Then

$G_{n,p}$ is Hamiltonian whp.

Write

$$G_{n,p} = G_{n,p_1} \cup G_{n,p_2}$$

where

$$p_1 = \frac{20\log n}{n}$$

and

$$1-p = (1-p_1)(1-p_2), \quad p_2 \approx \frac{5\log n}{n}$$

We first show that whp $G_1 = G_{n,p_1}$ has a Hamilton path.

Let $\lambda(G)$ denote the length of a longest path in G.

Let $\mathcal{E}_v$ be the event

$$\lambda(G_1 \setminus v) = \lambda(G_1)$$

Then

G_1 does not have a Hamilton path

$\Rightarrow \exists v : \mathcal{E}_v$ occurs.

G, not Hamiltonian

longest path P

v

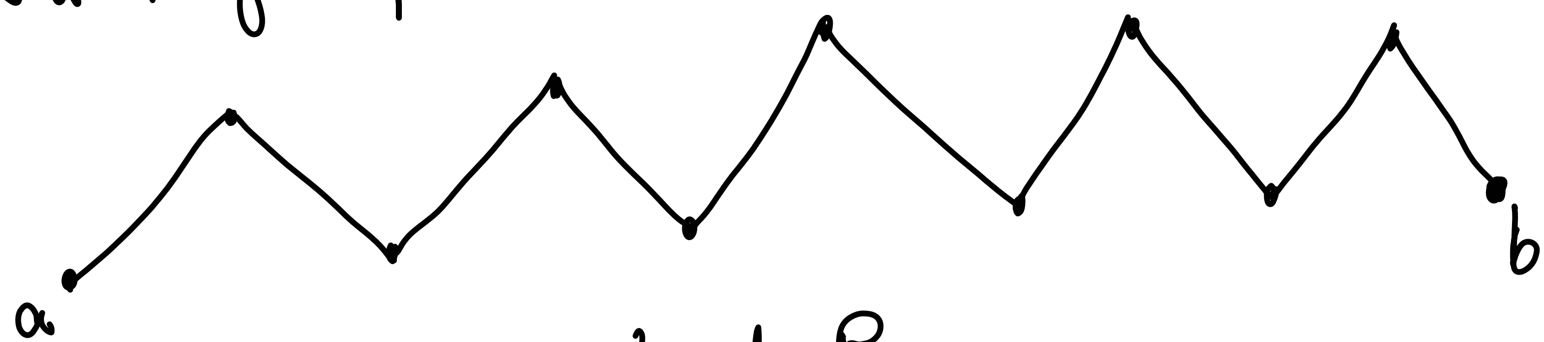
$\mathcal{E}_v$ occurs.

We show now that

$$\Pr\Big(\bigcup_v \mathcal{E}_v\Big) \leq n \Pr(\mathcal{E}_n) = o(1).$$

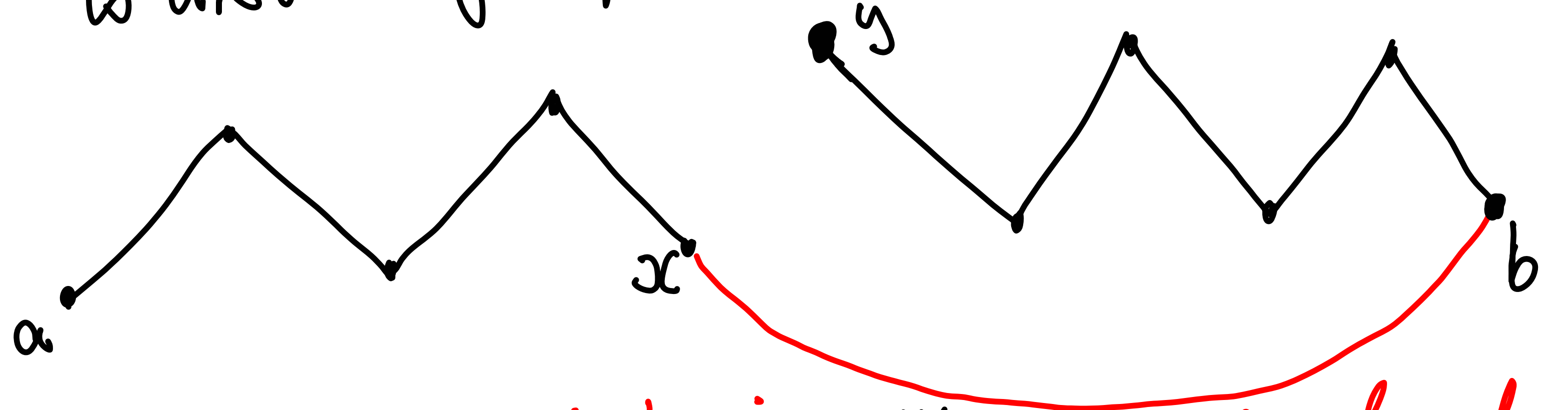
Posá Lemma

P is a longest path



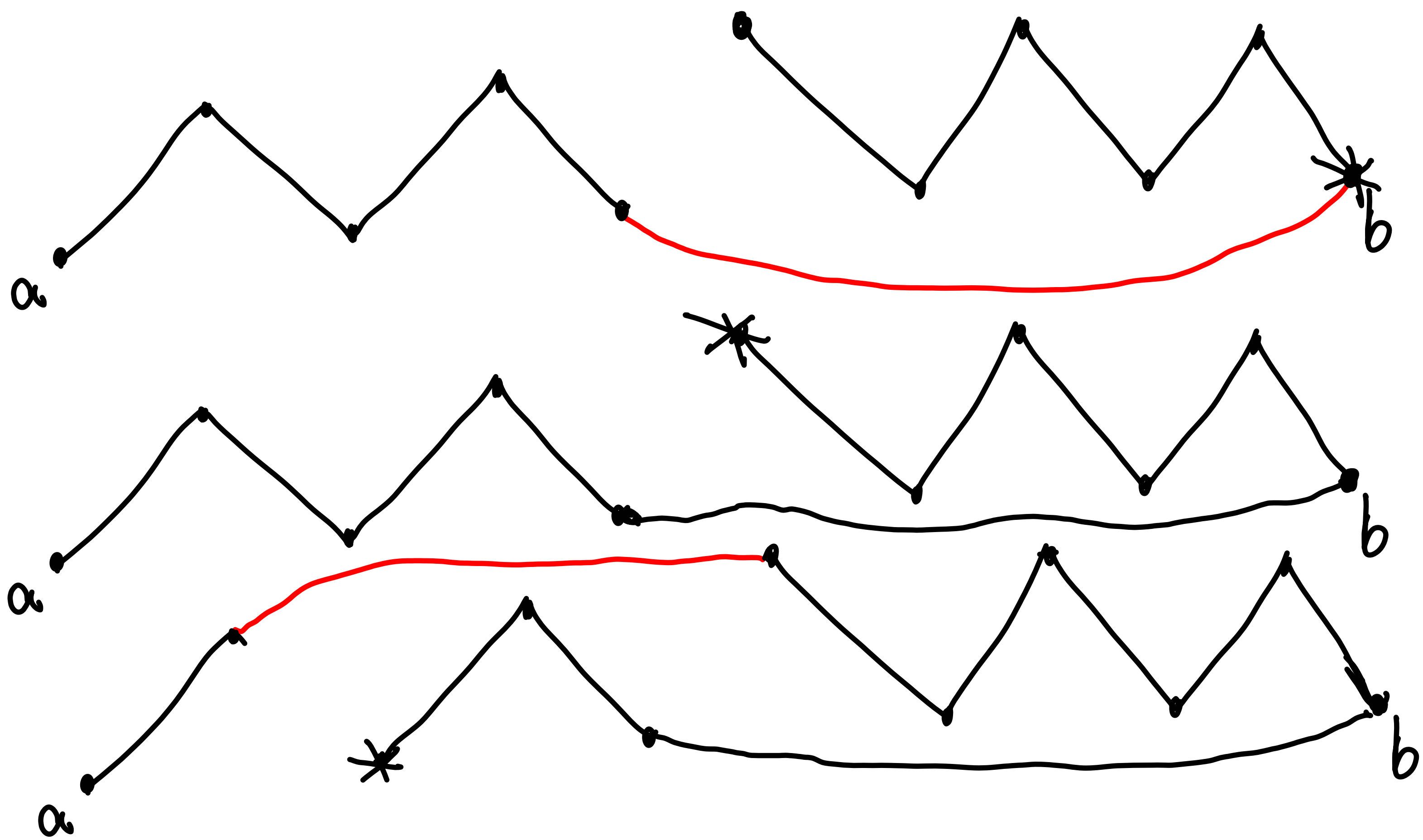
No edge from b go outside P.

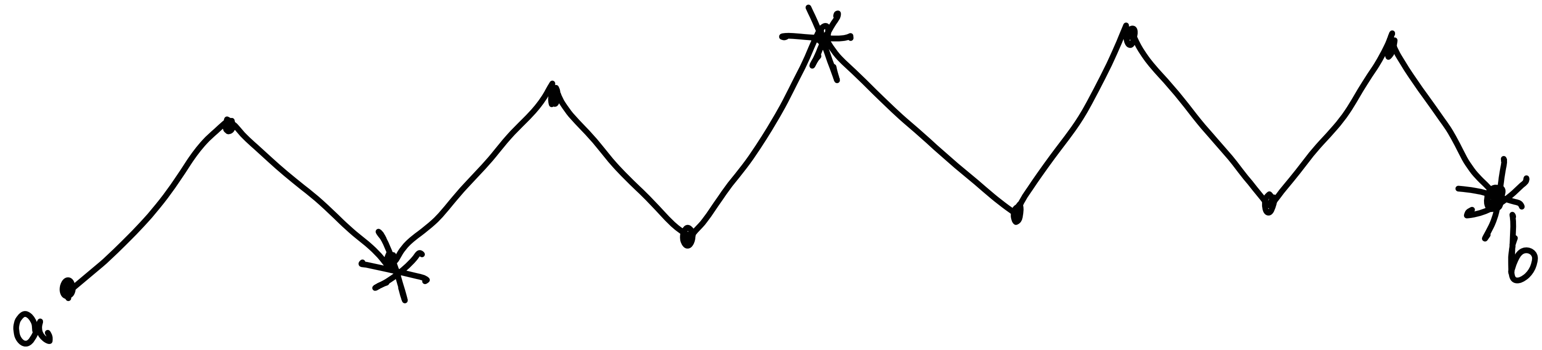
P' is also longest path:



P' is obtained by a rotation with a as fixed endpoint.

Now let END denote the set of v such that $\exists$ longest path P_v from a to v such that P_v is obtained from P by a sequence of rotations with a fixed.





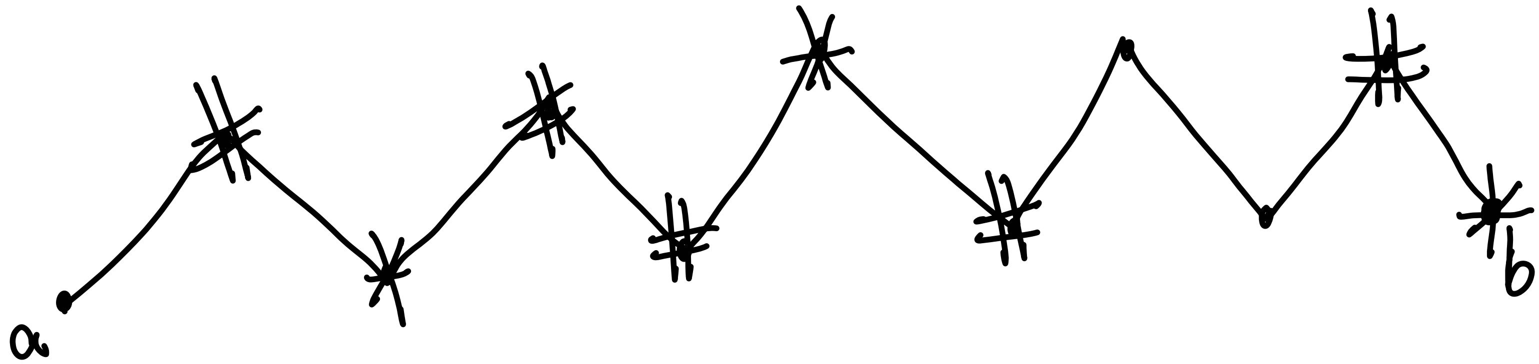
END = {*}

Lemma

If $v \in P \setminus END$ and v is adjacent to $w \in END$ then there exist $x \in END$ such that the edge $(x, v) \in P$ or $(v, x) \in P$.

Corollary

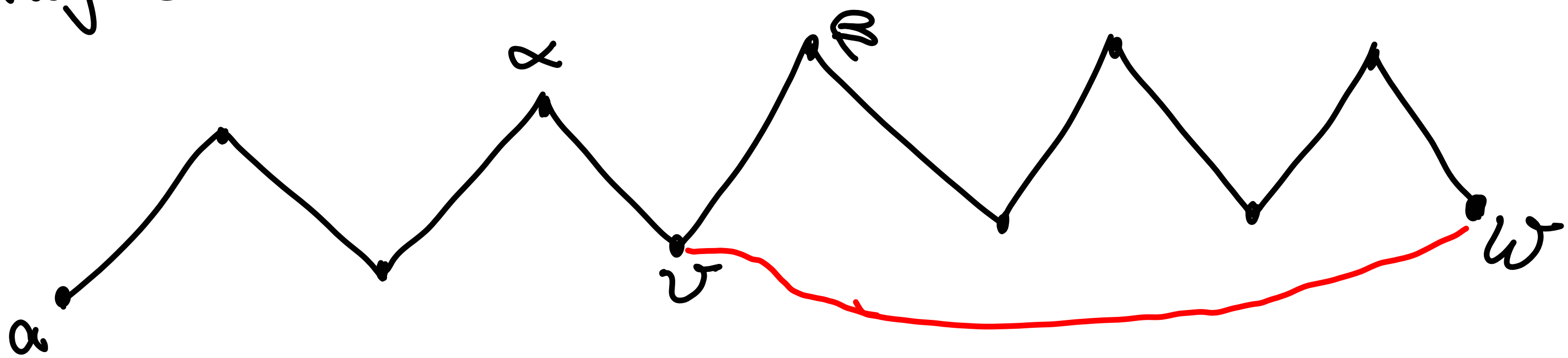
$$|N(END)| < 2|END|.$$

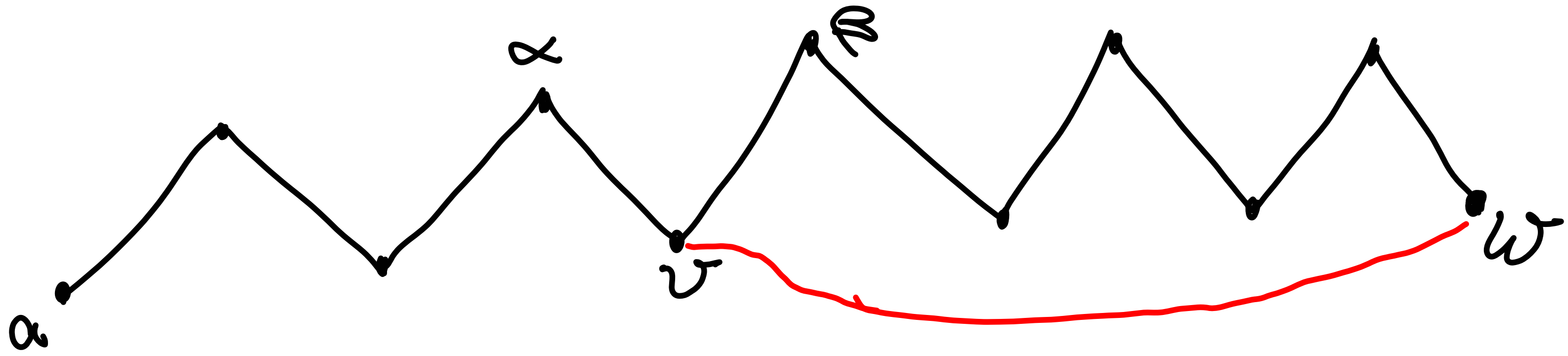


$N(END) = \{\#\}$

Proof of Lemma

Suppose that x, y are the neighbours of v on P and that $v, x, y \notin END$ and that v is adjacent to $w \in END$. Consider P_w





Now $\{\alpha, \beta\} = \{x, y\}$ because if a rotation deleted (x, v) say then x or v becomes an endpoint.

But then $\beta \in END$.

$\square$

Lemma

Whp $S \subseteq [n-1], |S| \leq \frac{1}{4}n \implies$

$|N(S)| \geq 2|S|$

in $G_1 \setminus \{n\}$

Proof

$P_1(\exists S : |S| \leq \frac{1}{4}n$ and $|N(S)| < 2|S|) \leq$

$$\sum_{k=1}^{\frac{1}{4}n-1} \binom{n-1}{k}\binom{n-1}{2k} \underbrace{(1-p_1)}_{\leq e^{-kp_1/4} \leq n^{-5k}}{}^{k(n-1-3k)} \leq$$

$$\sum_{k=1}^{n/4} \left[\frac{ne}{k} \cdot \frac{n^2e^2}{k^2} \cdot n^{-5}\right]^k = O(n^{-2}).$$

It follows that if P is a longest path in $G_1 \setminus \{n\}$ and END is defined w.r.t. P then

$$\Pr(|END| \leq n/4) = O(n^{-2}).$$

Now the edges incident with n are unconditioned by $G_1 \setminus \{n\}$ and (see p3)

$$\mathcal{E}_n \Rightarrow \nexists \text{ edge from } n \text{ to END}.$$

So

$$\Pr(\mathcal{E}_n) \leq O(n^{-2}) + (1-p_1)^{n/4} = O(n^{-2}).$$

So $\Pr(G_1$ does not have a Hamilton path$) = O(n^{-1})$.

Now use the G_{n,p_2} edges.

Let P be a Hamilton path in G_1 and let END be defined w.r.t. P.

By arguing as for $G_1 \setminus \{n\}$ we see that $|END| \geq n/4$ whp.

Let a be the fixed endpoint of P.

Then

$G_{n,p}$ not Hamiltonian $\Rightarrow \nexists$ a G_{n,p_2} edge from a to END.

Thus

$$\Pr(G_{n,p} \text{ is not Hamiltonian}) = o(1) + (1-p_2)^{n/4} = o(1).$$

Let us now go to $G = G_{n,m}$, $m = \frac{1}{2}n(\log n + \log\log n + c)$

and $G_{n,p}$, $p = \frac{m}{N}$.

Let a vertex of G be large if its degree is at least $\lambda = \frac{\log n}{100}$, and small otherwise.

<u>Lemma</u>

Whp $v, w \in SMALL \Rightarrow dist(v,w) \geq 5$

<u>Proof</u>

$$\Pr_p(\neg) \leq \binom{n}{2}\left(\sum_{l=0}^{3}\binom{n}{l}p^{l+1}\right)\left(\sum_{k=0}^{\lambda}\binom{n}{k}p^k(1-p)^{n-k}\right)^2$$

$$\lesssim \frac{1}{2}n(\log n)^4\left(\sum_{k=0}^{\lambda}\frac{(\log n)^k}{k!}\cdot\frac{e^{-c}}{n\log n}\right)^2$$

$$\lesssim \frac{1}{2} n (\log n)^4 \left(\sum_{k=0}^{\lambda} \underbrace{\frac{(\log n)^k}{k!} \cdot \frac{e^{-c}}{n \log n}}_{u_k} \right)^2 \qquad \frac{u_{k+1}}{u_k} > 100$$

$$\leq n (\log n)^4 \left(\frac{(\log n)^{\lambda}}{\lambda!} \, \frac{e^{-c}}{n \log n} \right)^2 \qquad \lambda! \geq \left(\frac{\lambda}{e}\right)^{\lambda}$$

$$= O\left(\frac{(\log n)^3}{n} \cdot (100e)^{\frac{2\log n}{100}} \right)$$

$$= O(n^{-3/4}).$$

So $\Pr_m(\neg) = O(m^{1/2} n^{-3/4}) = o(1).$ $\square$

Lemma

Whp $|SMALL| \le n^{1/4}$.

Proof

$$\Pr_p(|SMALL| > n^{1/4})$$

$$\le n \sum_{k=0}^{\log n/100} \underbrace{\binom{n-1}{k} p^k (1-p)^{n-1-k}}_{u_k}$$

$$u_{k+1} / u_k = \frac{n-1-k}{k+1} \cdot p \cdot \frac{1}{1-p} \ge 50.$$

$$\le 2n \left(\frac{nep \log n}{100\, n} \right)^{\frac{\log n}{100}} \cdot \frac{1}{n}$$

$$\le n^{1/5}.$$

Now apply Markov and monotonicity to go to $G_{n,m}$.

$\square$

Lemma

Whp $\nexists$ a cycle C_4 containing a small vertex.

Proof

$$\Pr_p(\neg) \leq \frac{1}{2} n^4 p^4 \sum_{k=0}^{\log n/100} \binom{n}{k} p^k (1-p)^{n-1-k}$$

$$\leq (\log n)^4 n^{-3/4}.$$

So $\Pr_m(\neg) \leq \tilde{O}(m^{\frac{1}{2}} n^{-3/4}) = o(1)$.

$\square$

Lemma

Whp, $\forall\, |S| \leq \frac{n}{(\log n)^3}$, $e(S) \leq 2|S|$

Proof

$\Pr_p\left(\exists S : |S| \leq \frac{n}{(\log n)^3} \text{ and } e(S) > 2|S|\right) \leq$

$$\sum_{s=4}^{n/(\log n)^3} \binom{n}{s} \binom{\binom{s}{2}}{2s} p^{2s}$$

$$\leq \sum_s \left(\frac{ne}{s} \cdot \left(\frac{se\log n}{2n}\right)^2 \right)^s$$

$$= \sum_s \left(\frac{s}{n} \cdot \frac{e^3 (\log n)^2}{2} \right)^s$$

$= O(n^{-3})$.

So $\Pr_m(\neg) = O(m^{1/2} n^{-3}) = o(1)$.

$\square$

Lemma

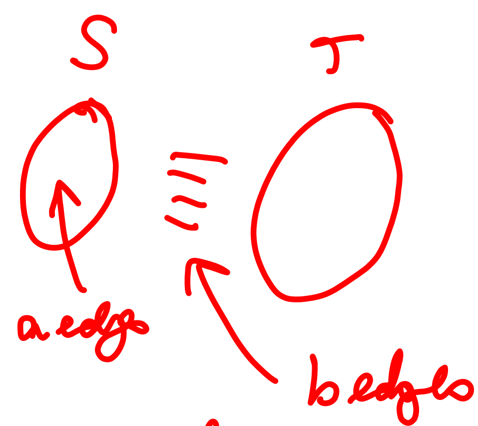
$S \subseteq LARGE$, $|S| \le \frac{n}{\log n} \Rightarrow |N(S)| \ge \frac{\log n}{1000}|S|$.

{large vtces}

Proof

(a) $1 \le |S| \le \frac{n}{(\log n)^3}$ $\qquad s = |S|$

$T = N(S)$ $\qquad t = |T|$



$$Pr_p(\exists S) \le \sum_{s=\sqrt{\lambda}}^{n/(\log n)^3} \sum_{t=0}^{\frac{\log n}{1000}s} \binom{n}{s}\binom{n}{t}\binom{\binom{s+t}{2}}{\frac{\log n}{200}s} p^{\frac{\log n}{200}s} \times (1-p)^{s(n-s-t)}$$

$2a+b \ge \frac{\log n}{100}s$

$a \le 2s$

$\Downarrow$

$a+b \ge \frac{\log n}{200}s$

$$\leq \sum_{s,t} \left(\frac{ne}{s}\right)^s \left(\frac{ne}{t}\right)^t \left(\frac{(s+t)^2 e\,(200)}{2s\log n} \cdot \frac{(1+o(1))\log n}{n}\right)^{s\log n/200}$$

$$\times\, n^{-s(1-1/(\log n)^2)}$$

The summand increases with $t \ll n$ and so we can put $t = \frac{\log n}{1000}s$ and then very crudely

$$\leq n \sum_{s \geq \sqrt{\lambda}} \left(\frac{ne}{s} \cdot \left(\frac{1000\, ne}{s\log n}\right)^{\frac{\log n}{1000}} \left(\frac{e(\log n)^2 s}{1000\, n}\right)^{\log n/200} \left(\frac{1+o(1)}{n}\right)\right)^s$$

$$< n \sum_{s \geq \sqrt{\lambda}} \left(\frac{e^6 (\log n)^9 s^4}{10^{12}\, n^4} \right)^{\frac{\log n}{1000} s}$$

$$= O(n^{-\Omega(\sqrt{\lambda}\log n)}) \text{ and so } \Pr_m(\neg) = o(1).$$

(b) $\frac{n}{(\log n)^3} \le |S| \le \frac{n}{\log n}$

$$\Pr_p[\neg] \le \sum_{s,t} \binom{n}{s}\binom{n}{t}\binom{st}{t} p^t (1-p)^{s(n-s-t)}$$

S:T edge

T=N(S)

$$\le \sum_{s,t} \underbrace{\left(\frac{ne}{s}\right)^s \left(\frac{ne}{t}\right)^t (sep)^t \, n^{-s\left(1-\frac{s+t}{n}\right)}}_{u_{s,t}}$$

$$\frac{u_{s,t+1}}{u_{s,t}} = \frac{ne}{t+1}\cdot\left(\frac{t}{t+1}\right)^t\cdot(sep)\cdot n^{s/n}$$

$$\ge 10.$$

So

$$\Pr(\neg) \leq 2\sum_{s} \left(\frac{ne}{s}\right)^{s} \left(10^{3} e^{2+o(1)}\right)^{\frac{s\log n}{1000}} n^{-s\left(1-\frac{s(1+\log n/1000)}{n}\right)}$$

$$= 2\sum_{s \geq \frac{n}{(\log n)^3}} \left(\frac{e}{s}\cdot\left(10^{3} e^{2+o(1)}\right)^{\frac{\log n}{1000}}\cdot n^{\frac{1}{1000}+o(1)}\right)^{s}$$

$$= O\left(n^{-\Omega(n^{1/2})}\right)$$

and so

$$\Pr_m(\neg) = o(1).$$

$\square$

Suppose now that $X \subseteq E(G)$ and

(i) $|X| = \log n$

(ii) X is a matching

(iii) X is not incident with a small vertex

(iv) X avoids the edges of some longest path of G.

We say that X is deletable.

Let $G_X = G \setminus X$

Lemma

Suppose that $\delta(G) \geq 2$ and

(i) $v, w \in SMALL \Rightarrow dist(v,w) \geq 5$ and $v \notin$ any C_4.

(ii) $S \subseteq LARGE$, $|S| \leq \frac{n}{\log n} \Rightarrow |N(S)| \geq \frac{\log n}{1000}|S|$.

(iii) X is deletable.

Then $S \subseteq [n]$, $|S| \leq 10^{-4} n \Rightarrow |N_X(S)| \geq 2|S|$

$\nwarrow$ nbrs in G_X.

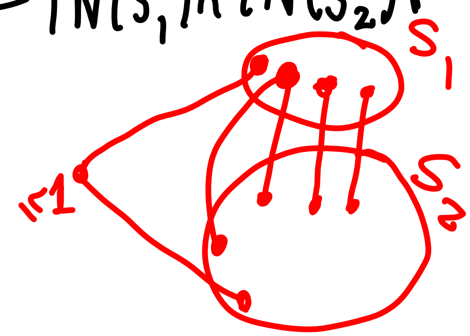
Proof

Let $S_1 = S \cap SMALL$ and $S_2 = S \setminus S_1$

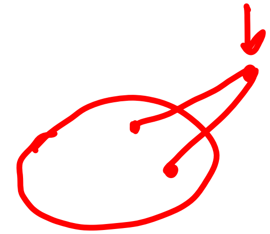
$$|N(S)| \geq |N(S_1)| + |N(S_2)| - |N(S_1) \cap S_2| - |N(S_2) \cap S_1| - |N(S_1) \cap N(S_2)|$$

$$\geq |N(S_1)| + |N(S_2)| - 2|N(S_1) \cap S_2| - |S_2|$$

$$\geq |N(S_1)| + |N(S_2)| - 3|S_2|.$$



DOES NOT HAPPEN



S_1

Now $|N(S_1)| \geq 2|S_1|$

and $|N(S_2)| \geq 9|S_2|$.

(i) $|S_2| \leq \frac{n}{\log n} \Rightarrow |N(S_2)| \geq \frac{\log n}{1000} |S_2|$

(ii) $|S_2| > \frac{n}{\log n}$. Take $S_2' \subseteq S_2$, $|S_2| = \frac{n}{\log n}$

$$|N(S_2)| \geq |N(S_2')| - |S_2|$$

$$\geq \frac{n}{1000} - |S_2|$$

$$\geq 9|S_2|.$$

So $|N(S)| \geq 2|S_1| + 6|S_2|$

and

$$|N_X(S)| \geq |N(S)| - |S_2|$$

X is a matching and it avoids SMAZZ

$$\geq 2|S_1| + 5|S_2|$$

$$\geq 2|S|.$$

Summary

(I) $\lim_{n\to\infty} \Pr(\delta(G_{n,m}) \geq 2) = e^{-e^{-c}}$.

(II) $G_{n,m}$ is connected whp

(III) $|SMALL| \leq n^{1/4}$, $v,w \in SMALL \Rightarrow dist(v,w) \geq 5$,
$\not\exists\, C_4 : C_4 \cap SMALL \neq \emptyset$, whp.

(IV) If $\delta(G) \geq 2$ and X is deletable then whp

$$|N_X(S)| < 2|S| \Rightarrow |S| \geq 10^{-4} n.$$

$\mathcal{G} = \{$all graphs on $[n]$ with m edges$\}$

$\mathcal{G}_0 = \{G \in \mathcal{G} : \delta(G) \geq 2$ and (II) — (IV) hold$\}$

$\mathcal{G}_1 = \{G \in \mathcal{G}_0 : G$ is not Hamiltonian$\}$

Couloring Argument

Suppose $G \in \mathcal{G}_1$ and X be deletable

Let P be a longest path in G_X.

Then

$$|END_X| \geq 10^{-4} n$$

(add subscript X to END)

Now for each $b \in END$, start with P_b and do all possible rotations, starting from P_b, but with b as a fixed endpoint. Let $END_X(b)$ be the set of endpoints produced.

We do a bit of re-naming

$$END_X \leftarrow END_X \cup \{a\}$$

$$END_X(a) \leftarrow END_X / \{a\}$$

Now we can say that for $b \in END$, we have

$$|END_X(b)| \geq \frac{n}{1000}$$

Now for $G \in \mathcal{G}$ and $X \subseteq E(G)$, $|X| = \omega = \log n$ choose some fixed longest path P_X of G_X.

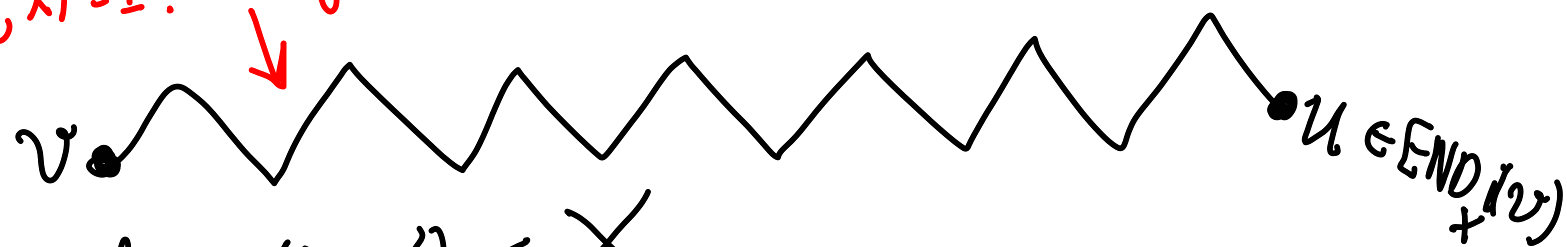
Furthermore choose so that if $G, G' \in \mathcal{G}$ and $G_X = G'_Y$ then $P_X = P'_Y$ i.e. path depends on G_X and not G, X.

$$\theta(G, X) = \begin{cases} 1 : & \text{(a) } G \in \mathcal{G}, \\ & \text{(b) } X \cap E(P_X) = \emptyset \\ & \text{(c) } X \text{ is deletable} \\ 0 : & \text{otherwise} \end{cases}$$

Note that $a(G,X)=1$ implies

If $u \in END_X(v)$ then $(u,v) \notin X$

$a(G,X)=1$. Longest in G_X and G



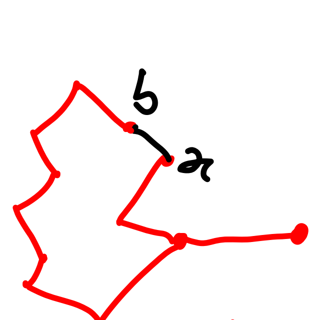
$\nexists$ edge $(u,v) \in X$

Either

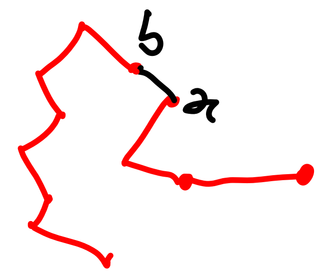
(i) $\ell(P_0) = n-1$

$\Rightarrow$ G is Hamiltonian

(ii)



G is connected



longer path

Now a double counting estimate for $\sum_G \sum_X a(G,X)$.

(i) Fix $G \in \mathcal{G}_1$.

$$\sum_X a(G,X) \geq \binom{m}{\omega}\left(1 - \frac{n + n^{1/4}\log n + \omega}{m - \omega}\right)^{\omega}$$

$$\geq \binom{m}{\omega} \Big/ 10.$$

$$\left(1 - \frac{\approx 2}{\omega}\right)^{\omega}$$

Random choose ω edges $e_1, e_2, \ldots$

$$\Pr(a(G,X)=1) \geq \prod_{i=0}^{\omega-1} \Pr(e_i \text{ avoids } P_X, \text{SMALL}, e_1, \ldots, e_{i-1} \mid e_1, \ldots, e_{i-1})$$

So

$$|\mathcal{G}_1| \leq \frac{10}{\binom{m}{\omega}} \sum_{G \in \mathcal{G}} \sum_X a(G,X).$$

(ii) Now fix a graph H with $m-w$ edges.

Let
$$S_H = \sum_{G,X:\, G_X = H} a(G,X).$$

If $S_H > 0$ then H has the expansion properties and its END sets are large. Thus we expect

$$S_H \leq \binom{N-m+w}{w}\left(1 - \frac{\binom{n/1000}{2}}{N}\right)^{w} \leq \binom{N-m+w}{w} e^{-(10^{-6}-o(1))w}.$$

There $\binom{N}{m-w}$ ways to add w edges to H. bounds the probability that a randomly chosen set of w edges avoids joining a to $END_H(a)$ for $a \in END_H$.

Thus

$\beta \approx 10^{-6}$

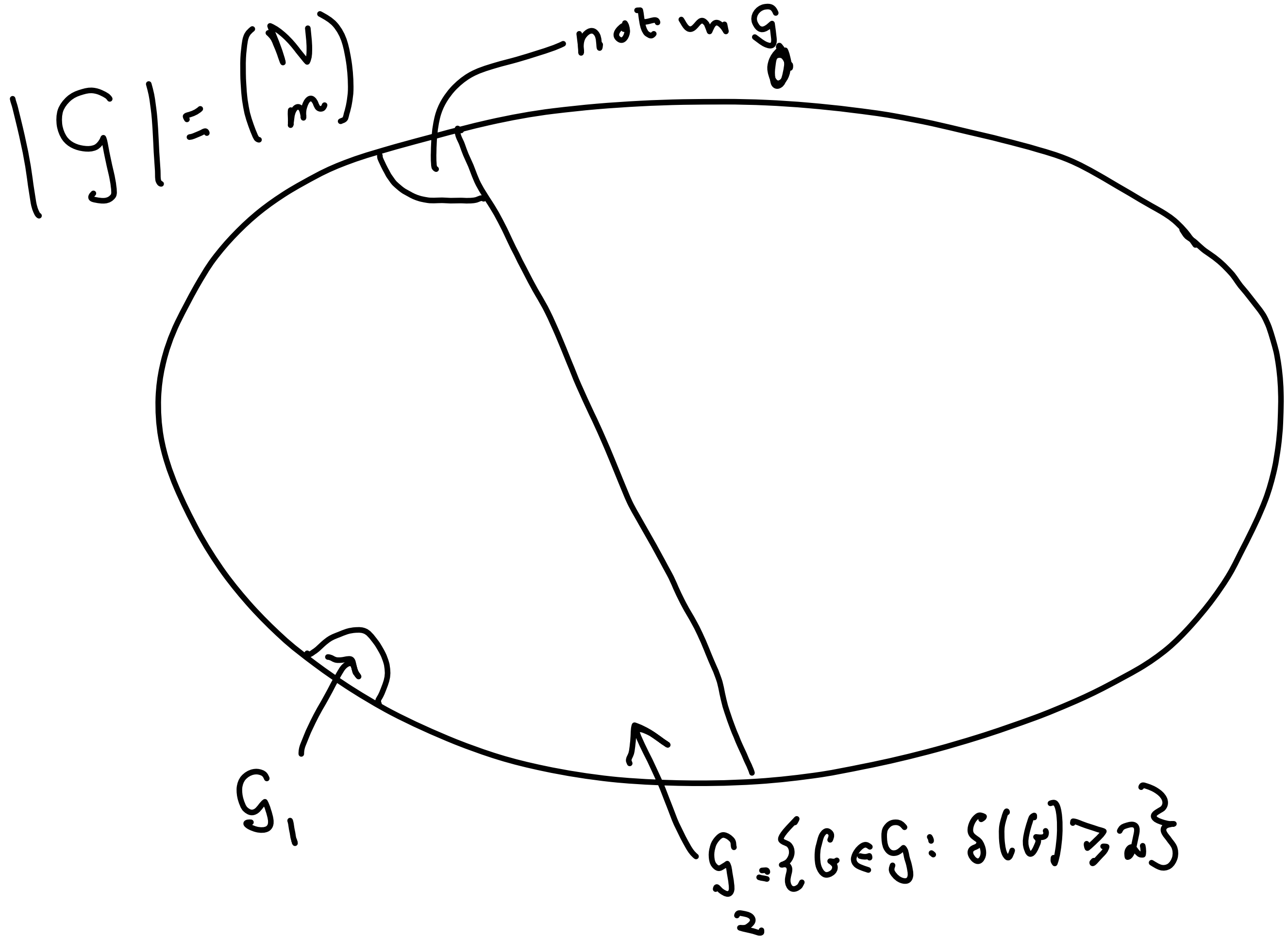
$$\sum_{G \in \mathcal{G}} \sum_{X} a(G, X) \leq \binom{N}{m-w} \binom{N-m+w}{w} e^{-\beta w}$$

$$= \binom{N}{m} \binom{m}{w} e^{-\beta w}$$

and so

$$|\mathcal{G}_1| \leq \frac{10}{\binom{m}{w}} \sum_{G \in \mathcal{G}} \sum_{X} a(G, X).$$

$$\leq 10 e^{-\beta w} \binom{N}{m}.$$



$$\Pr(G \text{ is not Ham } \& \ \delta(G) \geq 2) = \Pr(G \in (\mathcal{G}_2 \setminus \mathcal{G}_0) \cup \mathcal{G}_1) = o(1).$$

$$\Pr(G \text{ is Ham } \& \ \delta(G) \geq 2) = e^{-e^{-c}} - o(1).$$

$\square$