Perfect Matchings in Random Graphs

Let $K_{n,n,p}$ be the random bipartite graph with vertex bipartition $A=B=[n]$ in which each of the n^2 possible edges appears independently with probability p.

Theorem

Let $p = \dfrac{\log n + c_n}{n}$.

$$\lim_{n\to\infty} \Pr(K_{n,n,p} \text{ has a perfect matching}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-2e^{-c}} & c_n \to c \\ 1 & c_n \to \infty \end{cases}$$

$$= \lim_{n\to\infty} \Pr(\delta(K_{n,n,p}) \geq 1).$$

Let X_0 = # isolated vertices.

$$E(X_0) = 2n(1-p)^n$$
$$\approx 2e^{-c}.$$

By previously used techniques we

$$\Pr(X_0 = 0) \approx e^{-2e^{-c}}.$$

We will now use Hall's condition.

$G = K_{n,n,p}$ contains a perfect matching iff

$$\forall S \subseteq A, \quad |N(S)| \geq |S|. \qquad (*)$$

It is convenient to replace $(*)$ by

$$\forall S \subseteq A, \ |S| \leq \tfrac{1}{2}n, \ |N(S)| \geq |S|$$
$$\forall T \subseteq B, \ |T| \leq \tfrac{1}{2}n, \ |N(T)| \geq |T|. \qquad (**)$$

$$\Pr(\exists v : v \text{ isolated})$$

$$\leq \Pr(\nexists \text{ a perfect matching}) \leq$$

$$\Pr(\exists v : v \text{ isolated}) \ +$$

$$\Pr(\exists k, S \subseteq A, T \subseteq B, \ |S| = k \geq 2, |T| = k-1$$
$$N(S) \subseteq T \text{ and } e(S:T) \geq 2k-2\}$$

S:T edges

? Why $e(S:T) \geq 2k-2$?

Take a pair S, T with $|S|+|T|$ as small as possible.

(i) If $|S| > |T|+1$, remove $|S|-|T|-1$ vertices from S

(ii) Suppose $\exists\, w \in T$ such that w has < 2 nbs in S. Remove w and its (unique) nbr in S.

Repeat until (i) & (ii) do not hold. $|S|$ will stay at least 2 if $S \geq 1$.

$$E(\#\text{ sets } S, T) \leq$$

$$2\sum_{k=2}^{n/2} \binom{n}{k}\binom{n}{k-1}\binom{k(k-1)}{2k-2} p^{k}(1-p)^{k(n-k)}$$

$$\leq 2\sum_{k=2}^{n/2} \left(\frac{ne}{k}\right)^{k}\left(\frac{ne}{k-1}\right)^{k-1}\left(\frac{ke(\log n+c)}{2n}\right)^{2k-2} e^{-npk(1-\frac{k}{n})}$$

$$\leq 8\sum_{k=2}^{n/2} \underbrace{n\left(\frac{e^{O(1)}(\log n)^{2}n^{k/n}}{n}\right)^{k}}_{u_k}$$

Case 1: $2 \leq k \leq n^{3/4}$

$$u_k = n\left(\frac{e^{O(1)}(\log n)^2 n^{k/n}}{n}\right)^k$$

$$= e^{O(k)} n^{1+o(1)-k}.$$

So $$\sum_{k=2}^{n^{3/4}} u_k = O(1/n).$$

Case 2: $n^{3/4} < k \leq n/2$.

$$u_k = n\left(\frac{e^{O(1)}(\log n)^2 n^{k/n}}{n}\right)^k$$

$$\leq n^{1-k/3}$$

So $$\sum_{k=n^{3/4}}^{n/2} u_k = O(n^{-n^{3/4}/4}).$$

So,

$$\Pr(\nexists \text{ a perfect matching}) =$$

$$\Pr(\exists \text{ isolated vertex}) + o(1).$$

We now consider $G_{n,p}$.

We could try to replace Hall's Theorem by Tutte's theorem, but it is simpler to use Hall's theorem.

<u>Theorem</u>

Let $p = \frac{\log n + c_n}{n}$.

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \Pr(G_{n,p} \text{ has a perfect matching}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to \infty \end{cases}$$

$$= \lim_{n\to\infty} \Pr(\delta(G_{n,p}) \geq 1).$$

First of all if

X_0 = # of isolated vertices.

$$E(X_0) = n(1-p)^{n-1}$$

$$\approx e^{-c}$$

By previously used techniques we

$$\Pr(X_0 = 0) \approx e^{-e^{-c}}.$$

Suppose $n = 2m$ and $A = \{1, 2, \dots, m\}$

$B = \{m+1, \dots, n\}$

We will choose $A^* \subseteq A$, $B \subseteq B$ $|A^*| = |B^*| = s$ where s is small, such that whp $G_{n,p}$ contains a perfect matching between

$$\hat{A} = (A \setminus A^*) \cup B^*$$

and $$\hat{B} = (B \setminus B^*) \cup A^*.$$

Let

$$V_0 = \{ v : |N(v)| \leq \frac{\log n}{100} \}$$

$$A_0 = \{ v \in A \cap V_0 : |N(v) \cap A| > |N(v) \cap B| \}$$

$$B_0 = \{ w \in B \cap V_0 : |N(w) \cap B| > |N(w) \cap A| \}$$

$$A_1 = \{ v \in A \setminus A_0 : |N(v) \cap B| < \frac{\log n}{200} \}$$

$$B_1 = \{ w \in B \setminus B_0 : |N(w) \cap A| < \frac{\log n}{200} \}$$

Suppose

$$|A_0 \cup A_1| = |B_0 \cup B_1| + r$$

where $r \geq 0$.

Choose $R \subseteq B \setminus (B_0 \cup B_1)$ with $|R| = r$.

$$A^* = A_0 \cup A_1$$

$$B^* = B_0 \cup B_1 \cup R$$

We show that, conditional on $\delta \geq 1$, there is whp a perfect matching between $\hat{A}$ and $\hat{B}$.

Lemma

Whp $|V_0| \leq n^{1/10}$.

Proof

$$E(|V_0|) \leq n \sum_{k=0}^{\frac{1}{100}\log n} \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$\leq 2n \binom{n-1}{\frac{1}{100}\log n} p^{\frac{1}{100}\log n} \frac{e^{-c}}{n}$$

$$\leq 2\left(100e^{1+o(1)}\right)^{\frac{\log n}{100}} e^{-c}.$$

Now use Markov inequality. $\square$

Similarly, Whp

$$|A_1|, |B_1| \leq n^{2/3}.$$

Lemma

Whp $|A_1 \cup B_1| \leq n^{6/10}$.

Proof

$$\mathbb{E}(|V_0|) \leq n \sum_{k=0}^{\frac{1}{200}\log n} \binom{\frac{1}{2}n-1}{k} p^k (1-p)^{\frac{1}{2}n-1-k}$$

$$\leq 2n \binom{\frac{1}{2}n-1}{\frac{1}{200}\log n} p^{\frac{1}{200}\log n} \frac{e^{-c}}{n^{1/2}}$$

$$\leq 2n^{1/2} \left(200e^{1+o(1)}\right)^{\frac{\log n}{200}} e^{-c}.$$

Now use Markov inequality. $\square$

Lemma

Why, $v \in V_0,\ w \in A_1 \cup B_1 \Rightarrow N(v) \cap N(w) = \emptyset$

Proof

$$\Pr\left(\exists\, v, w : N(v) \cap N(w) \neq \emptyset\right)$$

$$\leq 3\binom{n}{3} p^2 \left(\sum_{k=0}^{\frac{1}{100}\log n} \binom{n-3}{k} p^k (1-p)^{n-3-k} \right) \leftarrow \frac{1}{n^{1-\epsilon}}$$

$$\times \left(\sum_{k=0}^{\frac{1}{200}\log n} \binom{\frac{1}{2}n-3}{k} p^k (1-p)^{\frac{1}{2}n-3-k} \right) \leftarrow \frac{1}{n^{\frac{1}{2}-\epsilon}}$$

$$\leq 3\binom{n}{3} \left(\frac{\log n + c}{n}\right)^2 n^{-4/3}$$

$$= o(1).$$

$\square$

Lemma

Whp $\nexists\, v: |N(v) \cap (A_1 \cup B_1 \cup V_0)| \geq 3$

Proof

$$\Pr(\exists v) \leq n \binom{n}{3} p^3$$

$$n \binom{n}{3} p^3 \left(\sum_{k=0}^{\frac{1}{100}\log n} \binom{\frac{1}{2}n-5}{k} p^k (1-p)^{\frac{1}{2}n-5-k} \right)^3$$

$$\leq n (\log n)^3 \cdot n^{-6/5}$$

$$= o(1).$$

$\square$

Lemma

Whp $S \subseteq A \setminus (A_0 \cup A_1)$ implies

$$|N_B(S)| \geq \frac{\log n}{500} |S| \text{ for } |S| \leq \frac{n}{(\log n)^3}$$

Proof

We first show that whp

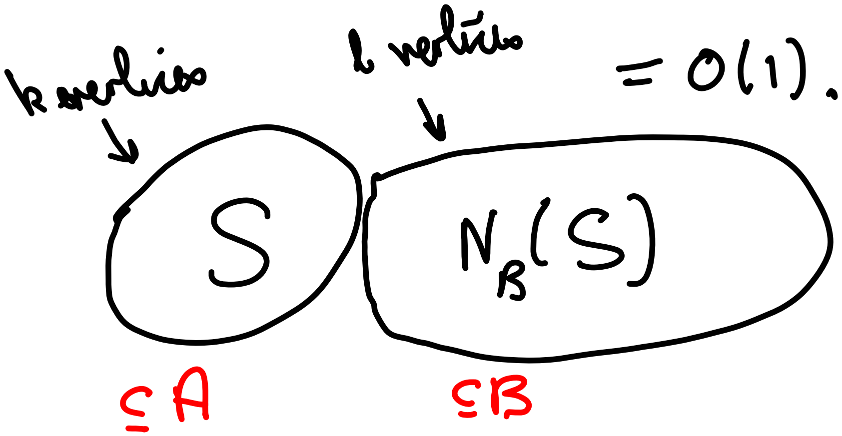
$$|S| \leq \frac{n}{10(\log n)^2} \quad \text{implies} \quad e(S) < 2|S|.$$

#edges inside S

$$\Pr(\exists S:\ e(S) \geq 2|S|) \leq \sum_{k=4}^{\frac{n}{10(\log n)^2} \leftarrow n_0}$$

$$\sum_{k=4} \binom{n}{k} \binom{\binom{k}{2}}{2k} p^{2k} \leq \sum_{k=4}^{n_0} \left(\frac{ne}{k}\right)^{k} \left(\frac{ke}{2} \frac{(\log n + c)}{n}\right)^{2k}$$

$$= \sum_{k=4}^{n_0} \left(\frac{k}{n} \cdot \frac{e^3}{4} \cdot (\log n + c)^2 \right)^{k} = O(1).$$



$$k + l \leq \frac{n}{10(\log n)^2}$$

$$\Rightarrow \frac{k \log n}{200} < 2(k+l)$$

$$\Rightarrow l > \frac{k \log n}{500}$$

So $k \leq \frac{n}{(\log n)^3}$

Lemma

Whp $\nexists$ $v \in A$, $u_1, u_2, u_3 \in B$ such that $u_i \in N(A_1 \cup V_0) \cap N(v)$ for $i = 1, 2, 3$.

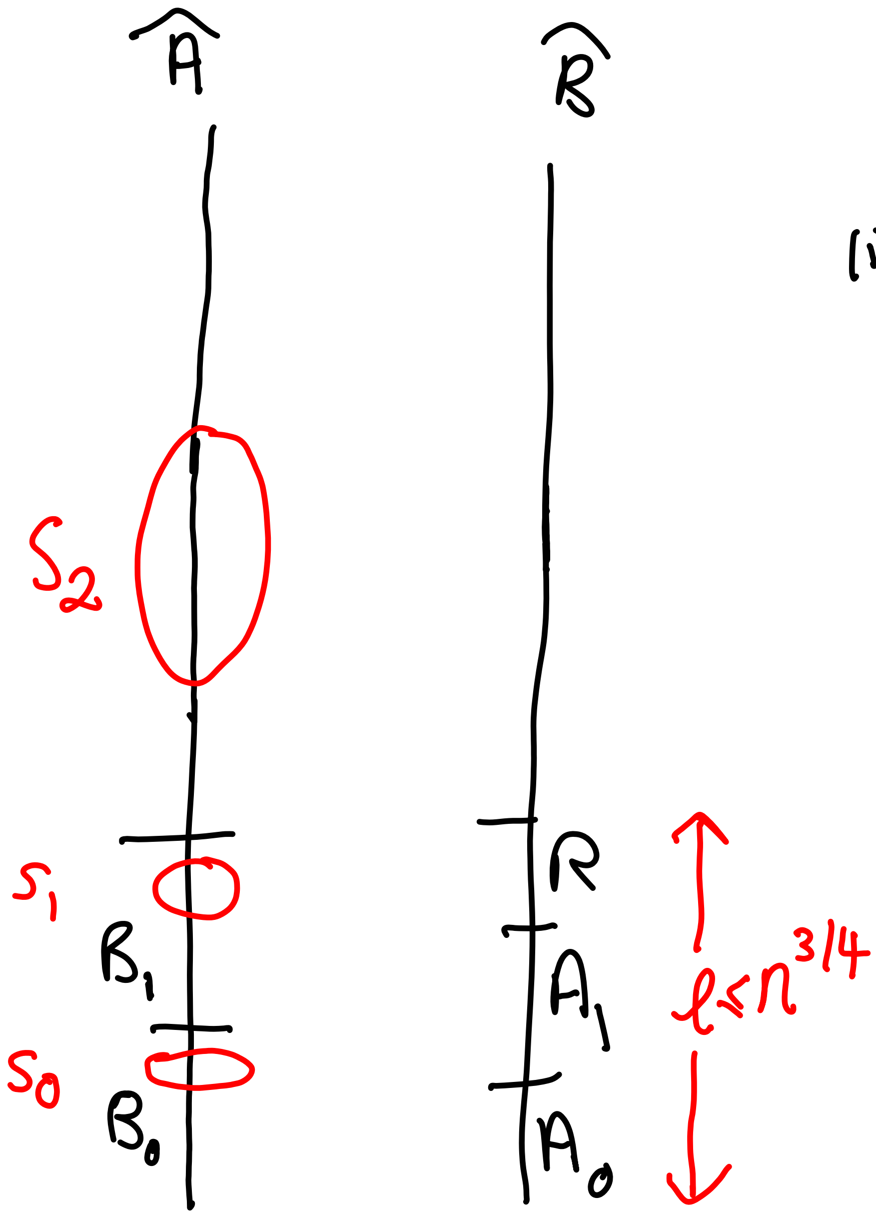
Proof

$$\Pr(\neg) \leq n \binom{n}{3} p^3 \left(np \sum_{k=0}^{\log n/100} \binom{\frac{1}{2}n}{k} p^k (1-p)^{n-k} \right)^3$$

$$\leq n (\log n)^2 n^{-6/5}$$

$$= o(1).$$

$\square$



Try Hall Condition

(1) $|S| \le \dfrac{n}{(\log n)^3}$

$$S = \underbrace{S_0}_{S \cap B_0} \cup \underbrace{S_1}_{S \cap B_1} \cup S_2$$

$$|N_B(S)| \ge |S_0| + \frac{\log n}{600}|S_1| + \frac{\log n}{500}|S_2| - 2|S_2|$$

$$\ge |S|.$$

(ii) $\frac{n}{2(\log n)^3} < |S| \leq n/4$

<u>Lemma</u>

Whp $\nexists\ S \subseteq A$, , such that $|N_B(S)| \leq |S| + 2\ell$

[This completes proof that Hall's condition holds whp.

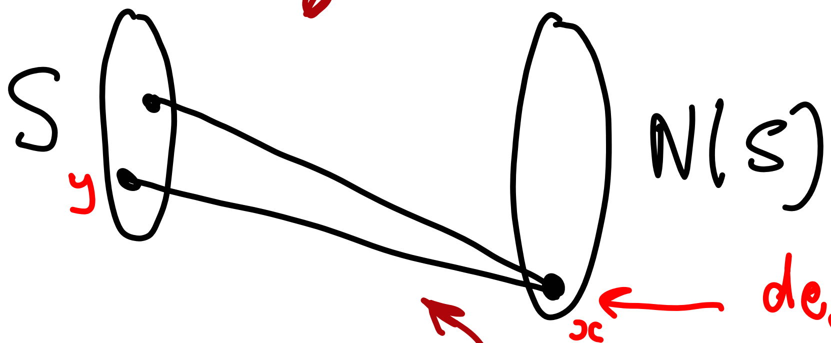
$N_{\hat{B}}(S) \geq N_B(S \setminus (A_0 \cup A_1)) - \ell \geq |S| - \ell - \ell + 2\ell$.]

<u>Proof</u>

As before we actually consider $S \subseteq A$,

$|S| \leq n/4$ and double our estimate.

#edges $\geq 2k+2\ell$



$|N(S)| = k-1$

degree ≥ 2, else we can remove y from S. Can't go below n/ by previous analysis

Probability $\leq$

$$2\sum_{k=\frac{n}{2(\log n)^3}}^{\frac{1}{4}n} \binom{\frac{1}{2}n}{k}\binom{\frac{1}{2}n}{k+2\ell}\binom{k(k+2\ell)}{2(k+2\ell)} p^{2k+4\ell}(1-p)^{k(\frac{1}{2}n-k-2\ell)}$$

$$\leq 2\sum_{k} \frac{n^k e^k}{2^k k^k}\cdot\frac{n^{k+2\ell}e^{k+2\ell}}{2^{k+2\ell}(k+2\ell)^{k+2\ell}}\left(\frac{ke(\log n+c)}{2n}\right)^{2k+4\ell} n^{-k/5}$$

$$\leq 2 \sum_{k} \frac{n^k e^k}{2^k k^k} \cdot \frac{n^{k+2l} e^{k+2l}}{2^{k+2l} (k+2l)^{k+2l}} \left(\frac{ke(\log n + c)}{2n} \right)^{2k+4l} n^{-k/5}$$

$$\leq \sum_{k} \left(\frac{e^{O(1)} (\log n)^2}{n^{1/5}} \right)^k$$

$$= o(1). \qquad \square$$

Check: $k^{2l} = \left(k^{2l/k}\right)^k = \left(e^{O(1)}\right)^k.$