Random Graphs
Homework 1

1. Suppose that $p = c/n$ where c is constant.
   (i) Show that **whp** $G_{n,p}$ contains no set of vertices S with $s = |S| < (20c^2)^{-1}n$ such that S contains $2s$ edges.
   (ii) Show that **whp** the maximum degree $\Delta \leq \frac{\ln n}{\ln \ln n}$.
   (ii) Show that **whp** the vertices of degree at least $\frac{2 \ln n}{3 \ln \ln n}$ form an independent set.

2. Let $A_1, A_2, \ldots, A_N$ be events in some probability space. Let $\mathcal{E}_k$ be the event that exactly k of these events occur. Show that

$$\mathbf{Pr}(\mathcal{E}_k) - \sum_{t=k}^{M} (-1)^{t-k} \binom{t}{k} S_t \begin{cases} \leq 0 & M-k \textit{ even} \\ \geq 0 & M-k \textit{ odd} \end{cases}$$

   where

$$S_t = \sum_{X \subseteq [N],\, |X|=t} \mathbf{Pr}\left(\bigcap_{i \in X} A_i \right).$$

3. Let $p = n^{-1}(\ln n + c)$. What is the limiting probability that $G_{n,p}$ has exactly k isolated vertices?

4. Suppose that $p = (\ln(n^2/c)/n)^{1/2}$. What is the limiting probability (as $n \to \infty$) that $G_{n,p}$ has diameter 2?

5. Suppose that $0 < c < 1$ is constant and $p = c/n$.
   (i) What is the limiting probability that $G_{n,p}$ has girth g?
   (The girth of a graph is the length of its shortest cycle).
   (ii) What is the limiting probability that $G_{n,p}$ is a forest?

6. Let C be a component of size $k = O(1)$ in G_m where $m = o(n)$. What is the limiting probability that C is still a component in $G_{m+\alpha n}$?

7. Let $B_{m,n,p}$ be the random sub-graph of the complete bipartite graph $K_{m,n}$ obtained by keeping each independently with probability p.
   (i) When does $B_{\alpha n, \beta n, c/n}$ have a giant component **whp**?
   (ii) When is $B_{n,n,p}$ connected **whp**?
   (iii) When does $B_{n,n,p}$ have a 2-factor **whp**?

8. Suppose that we give each edge of $K_{\alpha n, \beta n}$ an independent uniform $[0, 1]$ edge weight. What is the limiting expected value of the length of a minimum spanning tree?

9. B_{k-out} is the random bipartite graph with vertex bipartition $[n] + [n]$. Each vertex chooses k random neighbours from the other side of the bipartition. For what values of k is it true that B_{k-out} has a perfect matching **whp**?

10. G_{k-out} is the random graph with vertex set $[n]$. Each vertex chooses k random neighbours. Show that G_{k-out} is k-connected **whp**, for $k \geq 2$.

11. Let T be a fixed tree on k vertices. Show that if K is sufficently large then **whp** $G_{kn,Kn\log n}$ contains n vertex disjoint copies of T covering every vertex of $[kn]$.
12. Let H be a fixed unicyclic graph on k vertices. Show that if K is sufficently large then **whp** $G_{kn,Kn^{3/2}\sqrt{\log n}}$ contains n vertex disjoint copies of H covering every vertex of $[kn]$.