1. Suppose that $p = c/n$ where $c > 1$ is constant. Show that w.h.p. there does not exist a set of $s \leq \frac{1}{2}\log_c n$ vertices that induce a connected subgraph with more than cycle.

2. Suppose that $p = c/n$ where $c > 4\log 2$ is constant. Show that w.h.p. $e(A, B) > 0$ for every pair of sets A, B of size $n/2$ each that partition $[n]$. Here $e(A, B)$ is the number of edges joing a vertex in A to a vertex in B.

3. Suppose that $np \leq \log n$. Let a vertex be *small* if its degree is at most $np/10$. Show that w.h.p. no two small vertices are within distance 10 of each other.

4. Suppose that $m = \binom{n}{2}p$ and that $\mathcal{P}$ is a monotone increasing property. Prove that $\mathbf{Pr}(G_{n,m} \in \mathcal{P}) \leq 10\mathbf{Pr}(G_{n,p} \in \mathcal{P})$ without invoking the Central Limit Theorem i.e. just by comparing binomial coefficients.

5. Show that if $p \geq 2n^{-1/2}\log n$ then the diameter of $G_{n,p}$ is at most two. (It will be exactly two unless p is very close to one). What is the equivalent statement for $G_{n,m}$?

6. Suppose that $p = c/n$ where $c > 1$ is a large constant.

   - Show that with probability at least 1/2, there are at most $3c^{\log\log n}$ vertices X that lie on cycles of length at most $\log\log n$.
   - Show that w.h.p. there is no independent set of size greater than $\frac{2\log c}{c}n$.
   - Deduce that for large n, there exists a graph with girth at least $\log\log n$ and chromatic number at least $\frac{c-o(1)}{2\log c}$.

7. Consider the following algorithm for finding an independent set I in $G_{n,p}$ where $p = o(1)$ and $np \to \infty$. Initially $I = \emptyset$ and let $N(I)$ denote the set of neighbors of I. At an odd numbered step add any vertex to I that is not in $N(I)$. At an even step add a random vertex not in $N(I)$ to I. choose a vertex arbitrarily from those vertices not in $N(I)$. Prove that w.h.p. at the end of the process $|I| \leq (2-\epsilon)\log_b n$ where $b = 1/(1-p)$ and ϵ is a positive constant. This is regardless of the choices made at the odd steps.

8. Let $A_1, A_2, \ldots, A_N$ be events and for $i \in [N]$ let θ_i be the 0/1 indicator for the occurence of A_i. Let
$$f_k(\theta_1, \ldots, \theta_N) = \sum_{\substack{S \subseteq [N] \\ |S| = k}} \prod_{i \in S} \theta_i \prod_{j \notin S} (1 - \theta_j).$$
Let X denote the number of i such that A_i occurs. Show that

   (a)
$$f_k(\theta_1, \ldots, \theta_N) = \begin{cases} 1 & X = k \\ 0 & X \neq k \end{cases}$$

   (b) Let $S_j = \binom{X}{j}$ for $j \in [N]$. Deduce that
$$\mathbf{Pr}(X = k) = \sum_{j \geq k} (-1)^{j-k} \binom{j}{k} \mathbf{E}(S_j).$$

9. Prove a hitting time version for perfect matchings.

10. The k-core $C_k(G)$ of a graph G is the largest subgraph with minimum degree k and let $c_k(G)$ denote its size. Prove that if it is non-empty, then it is unique. What is a simple algorithm for constructing it.

    Consider the graph process $G_0, G_1, \ldots,$. Show that if $k \geq 3$ then w.h.p. there exists m and a constant $\alpha_k > 0$ such that $c_k(G_{m-1}) = 0$ and $c_k(G_m) \geq \alpha_k n$.

11. Let $p = c/n$ where $c > 1$ is a constant. Let $0 < x < 1$ be the solution to $x = 1 - e^{-cx}$. Show that w.h.p. $G_{n,p}$ has $\approx cx(2-x)n/2$ edges.
(Hint: remove an edge and put it back.)

12. Let c, x be as in the previous question. Show that w.h.p. $C_2(G_{n,p})$ has $\approx (x - cx + cx^2)n$ vertices and $\approx cx^2n/2$ edges.

13. Let c, x be as in the previous question. Show that wh.p. $C_2(G_{n,p})$ is non-planar.
(Hint: bound the number of edges in a planar graph of high girth.)