Auctions

Bids made on a single item.

N bidders.

Each has own <u>private</u> valuation of item for sale.

(a) <u>English Auction.</u>

Bids increase. Last person to bid pays last bid.

(b) <u>Dutch Auction</u>

Start at high price. First bidder wins. Pays bid.

(c) <u>Sealed Bid First Price</u>

Bids placed in envelopes.

Highest price wins at that price.

(d) <u>Sealed Bid Second Price</u>

Bids placed in envelopes.

Highest price wins at second price.

$$a \equiv d$$

Optimal to bid evaluation.

[More discussion later]

$$b \equiv c$$

Basically same auction.

Analysis of Symmetric Nash Equilibrium in (c) and (d).

Bidder i values object at value x_i.

$$\Pr[x_i \leq x] = F(x), \quad x \in [0, w]$$

Bids $\beta_i(x_i)$.

Symmetric if $\beta_1 = \beta_2 = \cdots = \beta_N$

Second Price Auction

$$\text{Payoff to } i = \begin{cases} x_i - \max\limits_{j \neq i} b_j & b_i > \\ 0 & \text{otherwise} \end{cases}$$

Prop

$\beta^{II}(x) = x$ is dominant strategy

Proof

$i = 1.$ $\quad P_1 = \max\limits_{j \neq 1} b_j$

Bid $z_1 < x_1$	$x_1 > z_1 \geq P_1$	$P_1 > x_1 > z_1$	$x_1 > P_1 > z_1$
Profit z_1	$x_1 - P_1$	0	0
x_1	$x_1 - P_1$	0	$x_1 - P_1$

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Profit z_1	$x_1 - P_1$	0	$x_1 - P_1$
x_1	$x_1 - P_1$	0	0

$$y_1 = \max\{x_2, x_3, \ldots, x_n\}.$$

$$P[y_1 \le y] = F(y)^{N-1} = G(y).$$

Expected Payment

$$m^{II}(x) = G(x)\, E[y_1 \mid y_1 < x].$$

First Price Auction

$$\text{Payoff to } i = \begin{cases} x_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases}$$

Assume $j \neq 1$ follow $\beta \nearrow$

$$y_1 = \max_{i \neq 1} x_j$$

Bidder 1 bids b.

(i) $b \leq \beta(\omega)$.

(ii) $\beta(0) = 0$.

$$E(\text{payoff}) = G(\beta^{-1}(b)) \times (x - b)$$

$\Pr[\beta(y_1) < b]$

Maximising w.r.t. b. $\quad G' = g$

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(x-b) - G(\beta^{-1}(b)) = 0$$

$y = \beta^{-1}(b)$; $\quad b = \beta(y)$; $\quad \frac{db}{dy} = \beta'(y) = \beta'(\beta^{-1}(b)) = 1 \Big/ \frac{dy}{db}$.

Heuristic: $b = \beta(x)$ in symmetric equilibrium

$$G(x)\beta'(x) + g(x)\beta(x) = x g(x)$$

$$\frac{d}{dx}\Big(G(x)\beta(x)\Big) = x g(x)$$

$$\beta(x) = \frac{1}{G(x)} \int_{y=0}^{x} y g(y)\, dy$$

$$= E(y_1 \mid y_1 < x).$$

Prop $\beta^I(x) = E(y_1 \mid y_1 < x)$ defines a symmetric equilibrium.

Proof

Assume 2, 3, ..., N use this strategy

Expected payoff to bidder 1 if value to 1 is x and he bids b.

$z = \beta^{-1}(b)$.

$$\Pi(b, x) = \overbrace{G(z)}^{\Pr(b \text{ wins})}(x - \beta(z))$$

$$= G(z)x - G(z)E[y_1 \mid y_1 < z]$$

$$= G(z)x - \int_0^z y\, g(y)\, dy$$

$$= G(z)x - G(z)z + \int_0^y G(y)\, dy.$$

$$\Pi(\beta(x), x) - \Pi(b, x)$$

$$= G(x)x - G(x)x + \int_0^x G(y)\,dy$$

$$-\left(G(z)x - G(z)z + \int_0^z G(y)\,dy\right)$$

$$= G(z)(z - x) - \int_x^z G(y)\,dy$$

$$= \int_x^z (G(z) - G(y))\,dy \geq 0.$$

$\square$

$$\beta^I(x) = \frac{1}{G(x)} \int_0^x y\,g(y)\,dy = \frac{1}{G(x)}\left[\left[yG(y)\right]_0^x - \int_0^x G(y)\,dy\right]$$

$$= x - \int_0^x \frac{G(y)}{G(x)}\,dy$$

$$= x - \int_0^x \left(\frac{F(y)}{F(x)}\right)^{N-1} dy.$$

Examples

(i) $F(x) = x, \quad x \in [0,1]$ Uniform $[0,1]$

$$\beta^{I}(x) = \frac{N-1}{N} x$$

(ii) $F(x) = 1 - e^{-\lambda x}, \quad N = 2$

$$\beta^{I}(x) = x - \int_0^x \frac{F(y)}{F(x)} dy$$

$$= x - \frac{1}{1-e^{-\lambda x}} \left[y + \lambda^{-1} e^{-\lambda y} \right]_0^x$$

$$= \frac{1}{\lambda} - \frac{x e^{-\lambda x}}{1 - e^{-\lambda x}}$$

$\lambda = 2$: Never bids more than $1/2$

[Neither does other bidder]

Revenue

$$m^{I}(x) = G(x)\, E\left[y_1 \mid y_1 < x \right] = m^{II}(x).$$