Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 4: Due Wednesday October 23.

Q1 Show that if $f : \Re^n \to \Re$ is a convex function then its *epigraph* $\text{epi}(f) = \{(\mathbf{x}, t) : t \geq f(\mathbf{x})\}$ is a convex subset of $\Re^{n+1}$.

Solution: Let $(\mathbf{x}, s), (\mathbf{y}, t) \in \text{epi}(f)$ and $0 < \lambda < 1$. Then,

$$f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}) \leq \lambda s + (1 - \lambda)t.$$

This implies that

$$\lambda(\mathbf{x}, s) + (1 - \lambda)(\mathbf{y}, t) = (\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}, \lambda s + (1 - \lambda)t) \in \text{epi}(f).$$

Q2 A *monomial* is a function f of the form

$$f(\mathbf{x}) = c\prod_{i=1}^{n} x_i^{a_i}$$

where $c > 0$.

The sum of monomials is called a *posynomial*. Let $x_i = e^{y_i}$ for $i = 1, 2, \ldots, n$. Show that this transforms the *Geometric Programming* problem

Minimise $f_0(\mathbf{x})$ subject to $f_i(\mathbf{x}) \leq 1$, $i = 1, 2, \ldots, m$, $x_j > 0, j = 1, 2, \ldots, n$

where $f_0, f_1, \ldots, f_m$ are posynomials, into a convex program.

Solution: Let $x_i = e^{y_i}$ for $i = 1, 2, \ldots, n$. Then the problem becomes,

Minimise $g_0(\mathbf{y})$ subject to $g_i(\mathbf{y}) \leq 1$, $i = 1, 2, \ldots, m$, $y_j > 0, j = 1, 2, \ldots, n$

where

$$g_i(\mathbf{y}) = \sum_{k=1}^{m_i} \exp\left\{\log c_{i,k} + \sum_{j=1}^{n} a_{i,j,k}y_j\right\}, \quad i = 0, 1, \ldots, m,$$

and for some coefficents $c_i, a_{i,j}$.

Finally, note that $g(\mathbf{x}) = \exp\left\{\log c + \sum_{j=1}^n a_i y_j\right\} = ce^{\mathbf{a}^T\mathbf{y}}$ is convex. Indeed,

$$g(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = ce^{\mathbf{a}^T(\lambda\mathbf{x}+(1-\lambda)\mathbf{y})} = ce^{\lambda\mathbf{a}^T\mathbf{x}+(1-\lambda)\mathbf{a}^T\mathbf{y}} \leq \lambda ce^{\mathbf{a}^T\mathbf{x}} + (1-\lambda)ce^{\mathbf{a}^T\mathbf{y}},$$

where the last inequality follows from $c > 0$ and the convexity of the exponential function.

Q3 Use the KKT conditions to solve

Minimise $(x_1 - 5)^2 + (x_2 - 4)^2$ subject to $x_1 + x_2 \leq 1, 2x_1 + 3x_2 \leq 2$.

The KKT conditions for this problem are:

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ 2x_1 + 3x_2 &\leq 2 \\ 2(x_1 - 5) + \lambda_1 + 2\lambda_2 &= 0 \\ 2(x_2 - 4) + \lambda_1 + 3\lambda_2 &= 0 \\ \lambda_1(x_1 + x_2 - 1) = \lambda_2(2x_1 + 3x_2 - 2) &= 0 \\ \lambda_1, \lambda_2 &\geq 0. \end{aligned}$$

This is a convex problem and so any solution to the above serves as a global optimum. There are 4 possibilities to check: $\lambda_i = 0, > 0, i = 1, 2$.

Case 1: $\lambda_1 = 0, \lambda_2 = 0$**:** This gives $x_1 = 5, x_2 = 4$ which is infeasible.

Case 2: $\lambda_1 = 0, \lambda_2 > 0$**:** This gives $2x_1 + 3x_2 = 2, x_1 - 5 = -\lambda_2, x_2 - 4 = -3\lambda_2/2$ which implies $x_1 = 45/17, x_2 = -8/17, \lambda_2 = 40/17 > 0$ which is infeasible.

Case 3: $\lambda_1 > 0, \lambda_2 = 0$ This gives $x_1 + x_2 = 1, x_1 - 5 = x_2 - 4 = -\lambda_1/2$ which implies that $x_1 = 1, x_2 = 0, \lambda_1 = 8$ which satisfies the KKT conditions.

Thus the solution is $x_1 = 1, x_2 = 0$. (Note that we did not impose $x_1, x_2 \geq 0$ and that this is a convex program.)